II. Transport Equation

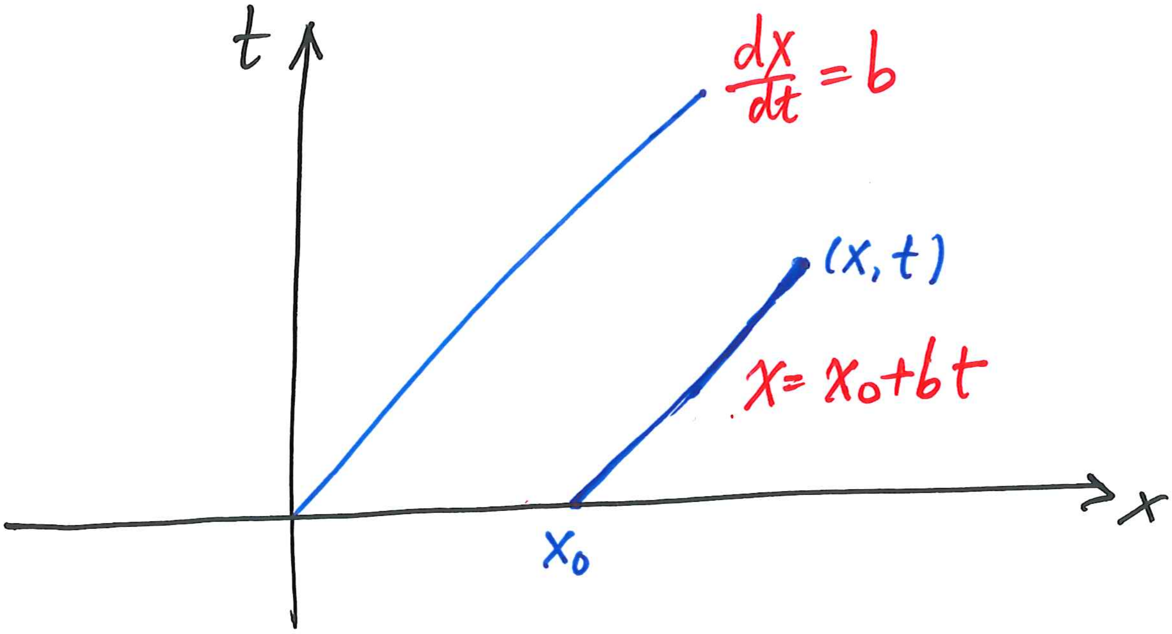
Probably the simplest PDE is

(V) $\quad u_t + b\,u_x = 0. \qquad b$ is a constant.

Along the direction $\frac{dx}{dt} = b$, for any solution $u = u(x,t)$ of (V), we have

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{dx}{dt} = u_t + b\,u_x = 0$$

$\hookrightarrow$ $u \equiv$ const. along the line with direction $(b, 1) \in \mathbb{R}^{1+1}$.



General

(*) $\quad u_t + b\cdot Du = 0 \qquad x\in\mathbb{R}^n,\ t\in\mathbb{R}_+$

$b=(b_1,\cdots,b_n)\in\mathbb{R}^n$

$Du = D_x u = (u_{x_1},\cdots,u_{x_n})$

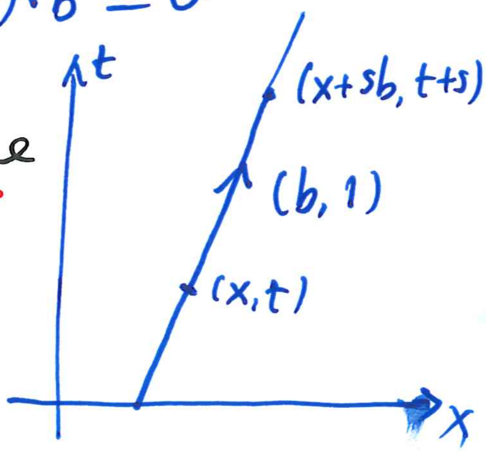
Analysis Assume $u\in C^1(\mathbb{R}^n\times\mathbb{R}_+)$ is a soln of (*)

$\forall (x,t)\in\mathbb{R}^n\times(0,\infty)$, define

$$z(s) = u(x+sb,\ t+s) = u((x,t)+s(b,1))$$

$$\frac{dz}{ds} = \frac{\partial u}{\partial t}(x+sb,\ t+s) + Du(x+sb,\ t+s)\cdot b \equiv 0.$$

↳ u is const. along the line through (x,t) with direction $(b,1)\in\mathbb{R}^{n+1}$



↳ If we know the value of u at any point on each such line, we know its value everywhere in $\mathbb{R}^n\times(0,\infty)$.

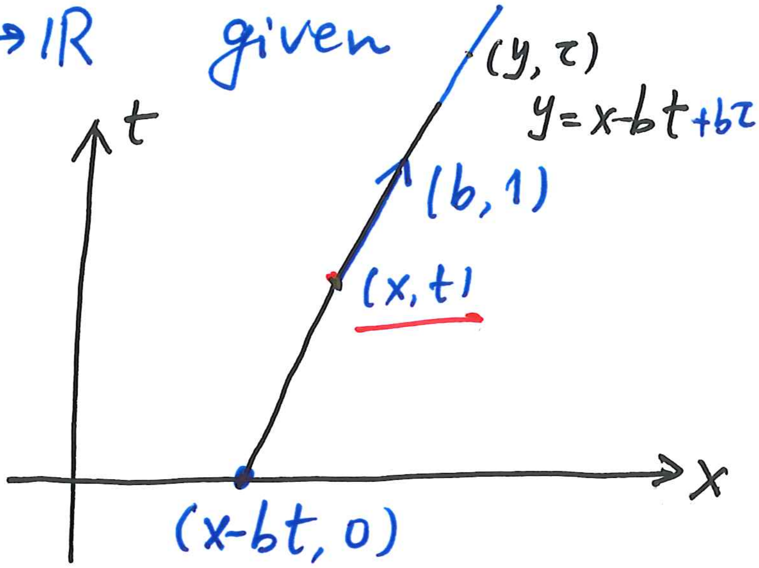
Cauchy Problem

$$(*)\begin{cases} u_t + b\cdot Du = 0 & \mathbb{R}^n\times(0,\infty) \\ u|_{t=0} = g(x) : \mathbb{R}^n \to \mathbb{R} & \text{given} \end{cases}$$

Analysis

$$\boxed{\begin{aligned} u(x,t) &= u(x-bt, 0) \\ &= g(x-bt) \end{aligned}}$$

$\forall x\in\mathbb{R}^n,\ t>0$



$$\begin{cases} y - x = bs \\ \tau - t = s \end{cases}$$

$\hookrightarrow\ y = x + b(\tau - t)$

$\qquad = (x-bt) + b\tau$

$\hookrightarrow$ Uniqueness

If $g\in C^1 \to u(x,t) = g(x-bt)$ satisfies $(*)$

$\hookrightarrow$ Existence of classical Soln.

If $g\notin C^1 \to \nexists\ C^1$ solution.

But we have to accept $u(x,t) = g(x-bt)$ as a solution, which is called a Weak Solution of $(*)$.

Nonhomogeneous Problem

$$(**)\quad \begin{cases} u_t + b\cdot Du = f(x,t) & \mathbb{R}^n\times(0,\infty) \\ u|_{t=0} = g(x), & x\in\mathbb{R}^n \end{cases}$$

Analysis: For $z(s) = u(x+bs,\ t+s)$.

$$\frac{dz(s)}{ds} = u_t(x+bs,\ t+s) + Du(x+bs,\ t+s)\cdot b$$
$$= f(x+bs,\ t+s)$$

$$\int_{-t}^{0} \frac{dz(s)}{ds}\,ds = \int_{-t}^{0} f(x+bs,\ t+s)\,ds$$

$$z(0) - z(-t) = \int_0^t f(x+(s-t)b,\ s)\,ds$$

$$z(0) - z(-t) = u(x,t) - g(x-bt)$$

$$\hookrightarrow \boxed{u(x,t) = g(x-bt) + \int_0^t f(x+(s-t)b,\ s)\,ds}$$

Verification. $u(x,t)$ solves $(**)$ indeed.

* Method of characteristics

$$u_t + b\cdot Du = 0, \quad u|_{t=0} = g(x)$$

$$\begin{cases} \dfrac{dx}{dt} = b \longrightarrow x = x_0 + bt \rightarrow x_0 = x - bt \\ \dfrac{du}{dt} = 0 \longrightarrow u = g(x_0) \\ x|_{t=0} = x_0 \\ u|_{t=0} = g(x_0) \end{cases} \quad \boxed{u = g(x-bt)}$$

PDE $\Longleftrightarrow$ System of ODEs.

* General $b = b(x,t) \in \mathbb{R}^n$

References

1. R. J. DiPerna & P.-L. Lions.
ODEs, Transport Theory and Sobolev spaces.
Invent. Math. 98 (1989), 511-547

2. L. Ambrosio, Transport equation and Cauchy problem for BV vector fields.
Invent. Math. 158 (2004), 227-260

III. Laplace's Equation

$$\Delta u = 0 \qquad x \in \mathbb{R}^n$$

$$\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} \qquad \text{Laplacian}$$

Poisson's Eq.: $\begin{cases} -\Delta u = f \\ f : \mathbb{R}^n \underset{(\Omega)}{} \to \mathbb{R} \text{ is given} \end{cases}$

$u \in C^2$ satisfying $\Delta u = 0$: $\left\{ \begin{array}{l} \text{Harmonic} \\ \text{Potential} \end{array} \right.$ function

* Ubiquity of Laplace's Eq. in Physics.

R. Feynmann, R. Leighton, & M. Sands:

Lectures in Physics, Vol. II.

Addison-Wesley, 1966

Examples

1. Maxwell Eq.

$$\begin{cases} \text{Curl } E = 0 \\ \text{div } E = 4\pi\rho \end{cases}$$

ρ — charge density

$$\text{Curl } E = 0 \xrightarrow[\text{electric potential}]{\exists \phi} E = -\nabla\phi$$

$$\hookrightarrow \Delta\phi = \text{div}(\nabla\phi) = -\text{div } E = -4\pi\rho$$

2. Analytic Function

$$f(z) = f(x+iy) = u(x,y) + i\,v(x,y)$$

Cauchy-Riemann Eqs.

$$\begin{cases} u_x = v_y \\ v_x = u_y \end{cases}$$

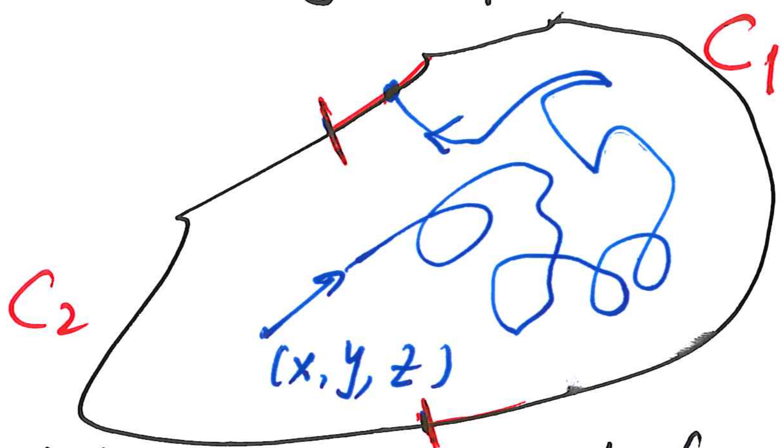
$$\hookrightarrow \begin{cases} u_{xx} + u_{yy} = 0 \\ v_{xx} + v_{yy} = 0 \end{cases}$$

Examples (Conti.)

3. **Brownian Motion** in a Container D:

Particles inside D move completely randomly until they hit the bdry ∂D where they stop.

$$\partial D = C_1 \cup C_2$$

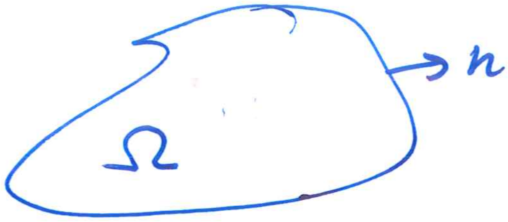


$u(x,y,z)$ – Probability that a particle which begins at the point (x,y,z) and stops at some point on C_1.

$$\begin{cases} \Delta u = 0 & \text{in } D \\ u|_{C_1} = 1 \\ u|_{C_2} = 0 \end{cases}$$

$\boxed{\text{Dirichlet Problem}}$

Basic Problems



$$\Delta u = f \qquad \text{in } \Omega$$

$$u\big|_{\partial\Omega} = h \qquad \underline{\text{Dirichlet}}$$

or

$$\frac{\partial u}{\partial n}\Big|_{\partial\Omega} = h \qquad \underline{\text{Neumann}}$$

or

$$\Big(\frac{\partial u}{\partial n} + \underset{\substack{\vee \\ 0}}{\underline{\alpha}}\, u\Big)\Big|_{\partial\Omega} = h \qquad \underline{\text{Robin}}$$

§1 Mean-Value Property (MVP)

Thm 3.1 If $u \in C^2$, then

$$\boxed{\Delta u = 0} \iff \boxed{u(x) = \fint_{\partial B(x,r)} u \, dS_y = \fint_{B(x,r)} u \, dy \quad \forall x \in \mathbb{R}^n}$$

where

$$\fint_{B(x,r)} u \, dy = \frac{1}{|B(x,r)|} \int_{B(x,r)} u \, dy$$

$$= \frac{1}{\alpha(n) r^n} \int_{B(x,r)} u \, dy$$

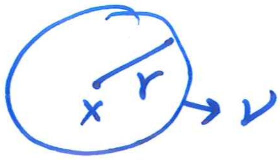
$$\fint_{\partial B(x,r)} u \, dS_y = \frac{1}{|\partial B(x,r)|} \int_{\partial B(x,r)} u \, dS_y$$

$$= \frac{1}{n\alpha(n) r^{n-1}} \int_{\partial B(x,r)} u \, dS_y$$

$\alpha(n) =$ Volume of $B(0,1) \subset \mathbb{R}^n = \dfrac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}$

$\alpha(2) = \pi$, $\alpha(3) = \dfrac{4\pi}{3}$

$\Gamma(s) = \int_0^\infty x^{s-1} e^{-x} \, dx$.



Proof.

"⟹" 1. Set

$$\phi(r) = \fint_{\partial B(x,r)} u \, dS_y \overset{y = x + rz}{=\!=} \fint_{\partial B(0,1)} u(x+rz) \, dS_z$$

$$\phi'(r) = \fint_{\partial B(0,1)} Du(x+rz) \cdot z \, dS_z$$

$$\overset{y=x+rz}{=\!=} \fint_{\partial B(x,r)} \frac{\partial u}{\partial \nu} \, dS_y$$

$$= \frac{r}{n} \fint_{B(x,r)} \Delta u(y) \, dy \equiv 0$$

$\hookrightarrow$ $\phi(r) = \text{const.}$

$\hookrightarrow$ $\displaystyle\fint_{\partial B(x,r)} u(y) \, dS_y = \lim_{\varepsilon \to 0} \fint_{\partial B(x,\varepsilon)} u \, dS = u(x)$

2. Method of Shells

$\hookrightarrow$
$$\int_{B(x,r)} u \, dy = \int_0^r \left(\int_{\partial B(x,\rho)} u \, dS \right) d\rho$$

$$= u(x) \int_0^r n \alpha(n) \rho^{n-1} \, d\rho$$

$$= \alpha(n) r^n u(x)$$

$\hookrightarrow$
$$u(x) = \frac{1}{\alpha(n) r^n} \int_{B(x,r)} u \, dy = \fint_{B(x,r)} u \, dy$$

Proof (Conti.)

"$\Leftarrow$" If $\Delta u \not\equiv 0$, then $\exists$ some small ball $B(x,r) \subset \mathbb{R}^n$, s.t. $\Delta u > 0$ in $B(x,r)$
$(<)$

$$0 = \phi'(r) = \frac{r}{n} \fint_{B(x,r)} \Delta u(y)\,dy \underset{(<)}{>} 0$$

* MVP has many important consequences.

Theorem 3.2 (Maximum Principles)

$\Omega \subset \mathbb{R}^n$ open, bounded

$u \in C^2(\Omega) \cap C(\bar{\Omega})$ is harmonic in Ω

$\Rightarrow$

(i) Weak Maximum Principle:

$$\max_{\bar{\Omega}} u = \max_{\partial\Omega} u$$

(ii) Strong Maximum Principle:

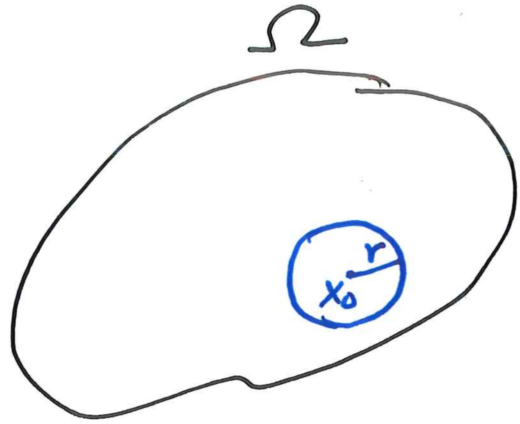
If $\left\{ \begin{array}{l} \Omega \text{ is connected} \\ \exists \text{ a point } x_0 \in \Omega \text{ s.t.} \\ \quad u(x_0) = \max_{\bar{\Omega}} u \end{array} \right.$

$\Rightarrow$ $u = \text{const.}$ in Ω

(iii) Similar assertions with "min".

(By replacing u by $-u$)

Proof.



If $\exists$ a point $x_0 \in \Omega$

s.t.

$$u(x_0) = \max_{\Omega} u \equiv M$$

$\hookrightarrow$ For $0 < r < \operatorname{dist}(x_0, \partial\Omega)$.

$$M = u(x_0) = \fint_{B(x_0,r)} u \, dy \leq M.$$

$\hookrightarrow$ $u \equiv M$ on $B(x_0, r)$

$\hookrightarrow$ The set

$$K \triangleq \{x \in \Omega \mid u(x) = M\}$$

is both open and relatively closed in Ω

$\xrightarrow{\Omega \text{ connected}}$ $K = \Omega$

$\hookrightarrow$ (ii)

$\hookrightarrow$ (i)

Theorem 3.3 (Uniqueness)

Let $f \in C(\Omega)$, $g \in C(\partial\Omega)$.

Consider the BVP $\begin{cases} -\Delta u = f & \text{in } \Omega \\ u|_{\partial\Omega} = g \end{cases}$

$\hookrightarrow$ $\exists$ at most one solution $u \in C^2(\Omega) \cap C(\bar{\Omega})$

<u>Proof.</u> If $\exists$ two solutions u_1, u_2.

$\hookrightarrow$ $u = u_1 - u_2$ satisfies

$$\begin{cases} \Delta u = 0 \\ u|_{\partial\Omega} = 0 \end{cases}$$

$\xrightarrow{\text{Thm 3.2}}$ $\max\limits_{\bar{\Omega}} u = \min\limits_{\bar{\Omega}} u = 0$

$\hookrightarrow$ $u \equiv 0$

$\hookrightarrow$ $u_1 \equiv u_2$

Theorem 3.4 (Smoothness).

If $u \in C(\Omega)$ satisfies the MVP for each ball $B(x,r) \in \Omega$

$\hookrightarrow$ $u \in C^\infty(\Omega)$.

Proof. Let $\eta(x)$ be a standard mollifier

$$\eta_\varepsilon(x) = \frac{1}{\varepsilon^n}\eta\left(\frac{x}{\varepsilon}\right) \qquad \begin{cases} \eta \in C^\infty \\ \eta(x) = 0 \quad |x| \geq 1 \\ \int \eta(x)\,dx = 1 \end{cases}$$

$$\hookrightarrow u^\varepsilon = \eta_\varepsilon * u = \underbrace{\int \eta_\varepsilon(y)\,u(x-y)\,dy}_{} \; \boxed{\in C^\infty(\underbrace{\Omega_\varepsilon}_{=\{x\in\Omega \mid \operatorname{dist}(x,\partial\Omega)>\varepsilon\}})}$$

$$= \frac{1}{\varepsilon^n}\int_{B(x,\varepsilon)} \eta\left(\frac{|x-y|}{\varepsilon}\right) u(y)\,dy$$

$$= \frac{1}{\varepsilon^n}\int_0^\varepsilon \eta\left(\frac{r}{\varepsilon}\right)\left(\underbrace{\int_{\partial B(x,r)} u\,dS_y}_{=n\alpha(n)r^{n-1}u(x)}\right)dr$$

$$= u(x)\int_{B(0,\varepsilon)} \eta_\varepsilon(y)\,dy = u(x)\int_{B(0,1)} \eta(y)\,dy$$

$$= u(x)$$

Theorem 3.5 (Local Estimates on derivatives)

Assume that u is harmonic in Ω.

$$\boxed{|D^\alpha u(x_0)| \le \frac{C_k}{r^{n+k}} \|u\|_{L^1(B(x_0,r))}}$$

$\forall\, B(x_0,r)\subset\Omega$, $\forall$ multi-index α with $|\alpha|=k$

where $C_0=\frac{1}{\alpha(n)}$, $C_k=\frac{(2^{n+1}nk)^k}{\alpha(n)}$, $k=1,2,\dots$

$\underline{k=0}$.

$$u(x_0)=\frac{1}{\alpha(n)r^n}\int_{B(x_0,r)} u(y)\,dy$$

$$|u(x_0)| \le \frac{1}{\alpha(n)r^n}\int_{B(x_0,r)}|u(y)|\,dy$$

$$= \frac{C_0}{r^n}\|u\|_{L^1(B(x_0,r))}. \qquad C_0=\frac{1}{\alpha(n)}$$

By Induction $\Longrightarrow$ Estimates.

$\underline{k=1}$ $\quad u_{x_i}(x_0)=⨍_{B(x_0,\frac{r}{2})} u_{x_i}\,dx = \frac{2^n}{\alpha(n)r^n}\int_{\partial B(x_0,\frac{r}{2})} u\nu_i\,dS$

$$|u_{x_i}(x_0)| \le \frac{2^n}{\alpha(n)r^n}\|u\|_{L^\infty(\partial B(x_0,\frac{r}{2}))}\, n\alpha(n)\left(\frac{r}{2}\right)^{n-1} = \frac{2n}{r}\|u\|_{L^\infty(\partial B(x_0,\frac{r}{2}))}$$

$$\le \frac{2n}{r} C_0\left(\frac{2}{r}\right)^n \|u\|_{L^1(B(x_0,r))} = \frac{2^{n+1}n}{\alpha(n)}\frac{1}{r^{n+1}}\|u\|_{L^1(B(x_0,r))}$$

Theorem 3.6 (Liouville Theorem)

$u: \mathbb{R}^n \to \mathbb{R}$ is harmonic and bounded.

$\Rightarrow$ u is constant.

Proof. $\forall x_0 \in \mathbb{R}^n$, $r > 0$

$$|Du(x_0)| \le \frac{C_1}{r^{n+1}} \|u\|_{L^1(B(x_0,r))} \le \frac{C_1}{r^{n+1}} \|u\|_{L^\infty(\mathbb{R}^n)} \alpha(n) r^n$$

$$= \frac{C_1 \alpha(n)}{r} \|u\|_{L^\infty(\mathbb{R}^n)} \to 0, \quad r \to \infty$$

$\Rightarrow$ $Du \equiv 0 \rightarrow u = \text{Const.}$

Remark (Stronger Result): If

$$\frac{|u(x)|}{|x|} \to 0 \quad \text{as} \quad |x| \to \infty$$

$\Rightarrow$ $u \equiv \text{Const.}$

$\Rightarrow$ A nonconstant harmonic function in all $\mathbb{R}^n$ must grow at least linearly at ∞.

Ex. Linear functions are harmonic

Theorem 3.7 (Analyticity)

If u is harmonic in Ω

$\hookrightarrow$ u is analytic in Ω

(See Evans's book, pages 31-32)

Theorem 3.8 (Harnack's Ineq. – Simplest version).

If u is harmonic and nonnegative in B_r

$$\hookrightarrow \left\{ \begin{array}{l} \boxed{\max_{B_{\frac{r}{2}}} u \le C \min_{B_{\frac{r}{2}}} u} \\ C \not\sim r \end{array} \right.$$

$\Longleftrightarrow$ The values of u are all comparable in the concentric half-ball.

Proof (from Han, §4.2).

$\because$ $u \ge 0$.

$$\hookrightarrow \frac{\partial u}{\partial x_i}(x_0) \overset{MVP}{=} \frac{c}{\rho} \fint_{\partial B_\rho(x_0)} u \cdot \nu_i \, dS \le \frac{c}{\rho} \fint_{\partial B_\rho(x_0)} |u|$$

$$= \frac{c}{\rho} \fint_{\partial B_\rho(x_0)} u \, dS \overset{MVP}{=\!=} \frac{c}{\rho} u(x_0)$$

Taking $\rho = \frac{r}{2}$.

$$\hookrightarrow \begin{cases} |\nabla u(x)| \le \frac{C_1}{r} u(x) & \forall\, x \in B_{\frac{r}{2}} \\ C_1 = C_1(n) \;\; \cancel{\sim}\; r \end{cases}$$

W.O.L.G. we may assume $u > 0$

(Otherwise, consider $u+\varepsilon$ instead of u, passing to limit $\varepsilon \to 0$ at the end of argt.)

$$\hookrightarrow \quad |\nabla \log u(x)| \le \frac{C_1}{r} \quad \text{in } B_{\frac{r}{2}}$$

$\forall\, x, y \in B_{\frac{r}{2}}$,

$$\hookrightarrow \log \frac{u(x)}{u(y)} = \int_0^1 \frac{d}{ds} \log u(sx+(1-s)y)\, d$$

$$= (x-y) \cdot \int_0^1 \nabla \log u(sx+(1-s)y)\, ds$$

$$\le |x-y| \int_0^1 |\nabla \log u(sx+(1-s)y)|\, ds$$

$$\le \frac{C_1}{r} |x-y| \le C_1$$

$$\hookrightarrow \quad \boxed{\frac{u(x)}{u(y)} \le e^{C_1}} \qquad \square$$

Rm: General Harnack Inequality

For any connected open set $V \subset\subset \Omega$,

$\exists\ C = C(n, V) > 0$ s.t.

$$\boxed{\sup_V u \leq C \inf_V u}$$

for all harmonic function $u \geq 0$ in Ω

$$\boxed{\frac{1}{C} u(y) \leq u(x) \leq C u(y) \quad \forall\ x, y \in V \subset\subset \Omega}$$

* Since V is a positive distance away from $\partial\Omega$, there is "room for the averaging effects of Laplace's Eq. to occur"!

§2 Fundamental Solutions / Green's Function

§2.1. Fundamental Solutions

Obervation $\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ is $\boxed{\text{rotational invariant}}$

$y = Ox$ with $OO^T = O^T O = I$.

$\hookrightarrow$ $$\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial y_i^2}$$

$\hookrightarrow$ $\boxed{\text{Symmetry}}$

$\hookrightarrow$ Seek radial solutions

$$u(x) = v(r), \qquad r = |x| = \sqrt{x_1^2 + \cdots + x_n^2}$$

$$\Delta u = 0$$

$\hookrightarrow$ $$v''(r) + \frac{n-1}{r} v'(r) = 0$$

$\hookrightarrow$ $$v(r) = \begin{cases} b \log r + c & n = 2 \\ \frac{b}{r^{n-1}} + c & n \geq 3. \end{cases}$$

b, c are arbitrary constants

Fundamental Solution

$$\Phi(x) = \begin{cases} -\frac{1}{2\pi}\log|x|, & n=2 \\ \frac{1}{n(n-2)\alpha(n)}\frac{1}{|x|^{n-2}}, & n=3 \end{cases} \qquad \forall x \in \mathbb{R}^n,\ |x| \neq 0$$

$\hookrightarrow$

$$\begin{cases} -\Delta\Phi = \delta_0 & \text{in } \mathbb{R}^n \\ |D\Phi(x)| \le \frac{C}{|x|^{n-1}}, & x \neq 0 \\ |D^2\Phi(x)| \le \frac{C}{|x|^n}, & x \neq 0. \end{cases}$$

$\longleftrightarrow \boxed{\int_{\partial B_r(0)} \frac{\partial \Phi}{\partial r}\, dS = -1}$

Set $u(x) = \int_{\mathbb{R}^n} \Phi(x-y) f(y)\, dy$

$\hookrightarrow$

$$\begin{cases} u \in C^2(\mathbb{R}^n) \\ -\Delta u = f & \text{in } \mathbb{R}^n \end{cases}$$

Theorem 3.9 (Representation Formula).

Let $f \in C_c^2(\mathbb{R}^n)$.

If

$$\begin{cases} \dfrac{|u(x)|}{|x|} \longrightarrow 0 \quad \text{as } |x| \to \infty \\ -\Delta u = f \end{cases}$$

Then u has the form

$$\boxed{u(x) = \int_{\mathbb{R}^n} \Phi(x-y) f(y)\, dy + C \quad \forall x \in \mathbb{R}^n}$$

Proof. Set

$$w(x) = \int_{\mathbb{R}^n} \Phi(x-y) f(y)\, dy$$

$$\frac{|w(x)|}{|x|} \longrightarrow 0 \quad \text{as } |x| \to \infty$$

Define $v = u - w$

$$\begin{cases} \Delta v = 0 \\ \dfrac{|v(x)|}{|x|} \longrightarrow 0 \quad \text{as } |x| \to \infty \end{cases}$$

$v = \text{const.}$ $\square$

§2.2 Green's function

Consider the BVP with Dirichlet BC:

$$(*) \quad \begin{cases} -\Delta u = f & \text{in } \Omega \\ u|_{\partial\Omega} = u_0 \end{cases}$$

Claim: Let $G(x, y)$ be defined for all $x, y \in \Omega$, $x \neq y$, and

$$\begin{cases} -\Delta_y G(x, y) = \delta_x(y) \\ G(x, y) = 0 \quad \text{for } y \in \partial\Omega \end{cases}$$

Then the Solution of $(*)$ is

$$\boxed{u(x) = \int_\Omega G(x, y) f(y)\, dy - \int_{\partial\Omega} u_0(y)\, \nabla_y G(x, y) \cdot \nu \, dS_y}$$

* $G(x, y)$ is called the Green function

Proof In general,

$$\int_\Omega v\Delta u - \int u\Delta v = \int_{\partial\Omega} v\frac{\partial u}{\partial n} - \int_\Omega u\frac{\partial v}{\partial n}$$

Fixing x, take $v(y) = G(x,y)$:

$$\int_\Omega u\Delta v = -\int_\Omega u\,\delta_x = -u(x)$$

$$\int_\Omega v\Delta u = -\int_\Omega G(x,y)f(y)dy$$

$$\int_{\partial\Omega} u\frac{\partial v}{\partial n} = \int_{\partial\Omega} u_0\,\nabla_y G(x,y)\cdot\nu\,dS_y$$

$$\int_{\partial\Omega} v\frac{\partial u}{\partial n}dS = 0 \Longleftarrow v|_{\partial\Omega}=0$$

? Does the Green function $G(x,y)$ exist?

$\boxed{\text{Yes}}$ for any "reasonable" domain.

Construction :

$$G(x,y) = \underbrace{\Phi(x-y)}_{\text{Fundamental solution}} + \varphi^{(x)}(y)$$

Solution of $\begin{cases} \Delta_y \varphi^{(x)} = 0 & \text{in } \Omega \\ \varphi^{(x)}\big|_{\partial\Omega} = -\Phi(x-y), & y \in \partial\Omega \end{cases}$

* For special cases, G can be made very explicit.

Half-Space. $x = (x', x_n) \in \mathbb{R}^n$

$\Omega = \{x_n > 0\}$.

$$\begin{cases} G(x,y) = \Phi(y-x) - \Phi(y-\tilde{x}) \\ \tilde{x} = (x', -x_n) = \text{"reflection of } x\text{"} \end{cases}$$

Ball: $\Omega = B(0,1)$.

$$\begin{cases} G(x,y) = \Phi(y-x) - \Phi(|x|(y-\tilde{x})) \\ \tilde{x} = \dfrac{x}{|x|^2} \end{cases}$$

"reflection" is replaced by inversion

Note: $|x||(y-\tilde{x})| = |y-x|$, $\forall x \in B(0,1)$, $y \in \partial B(0,1)$

The associated Representation Formula for $\begin{cases} \Delta u = 0 \\ u|_{\partial\Omega} = 0 \end{cases}$ for the half-space or ball are "Poisson's Formula" respectively.

see Evans Section 2.2.4

Symmetry of Green's Function

$$G(y,x) = G(x,y), \quad \forall x, y \in \Omega,\ x \neq y$$

"formal Proof": Green's formula

$$(*) \quad \int_\Omega (u\Delta v - v\Delta u)\,d = \int_{\partial\Omega}\left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right)dS$$

Take $u(x) = G(P, x)$, $v(x) = G(Q, x)$, $P \neq Q$

$$\hookrightarrow \quad \begin{cases} -\Delta u = \delta_P(x) \\ u|_{\partial\Omega} = 0 \end{cases} \qquad \begin{cases} -\Delta v = \delta_Q(x) \\ v|_{\partial\Omega} = 0 \end{cases}$$

$$(*) \Rightarrow \quad -\int_\Omega u\,\delta_Q\,dx + \int_\Omega v\,\delta_P = 0$$

$$\Downarrow$$

$$G(P,Q) = G(Q,P)$$

* The rigorous proof can be found in Evans.
§ 2.2.4a

Neumann Function

Consider the Neumann problem:

$$(*) \quad \begin{cases} -\Delta u = f & \text{in } \Omega \\ \left.\dfrac{\partial u}{\partial \nu}\right|_{\partial\Omega} = g \end{cases}$$

The Neumann function: $N = N(x, y)$

$$\begin{cases} -\Delta_y N(x, y) = \delta_x(y), & y \in \Omega \\ \left.\dfrac{\partial N}{\partial \nu_y}\right|_{\partial\Omega} = \text{Const.} & \text{for } y \in \partial\Omega \\ \text{Choose "Const." s.t. } \displaystyle\int_{\partial\Omega} \frac{\partial N}{\partial \nu_y} \, dS_y = \int_\Omega \Delta_y N = -1 \end{cases}$$

Note: The Neumann Problem $(*)$ has a solution only if the data f & g are consistent:

$$\int_{\partial\Omega} g \, dS_x + \int_\Omega f \, dx = 0$$

and that the solution is unique only up to an additive constant.

Claim 1: For the solution u of $(*)$, with $\underline{\int_{\partial\Omega} u \, dS_y = 0}$,

$$\boxed{u(x) = \int_{\Omega} N(x,y) f(y)\, dy + \int_{\partial\Omega} N(x,y) g(y)\, dS_y}$$

Proof: $\displaystyle\int_{\Omega} (u \Delta N - N \Delta u)\, dy = \int_{\partial\Omega} \left(u \frac{\partial N}{\partial \nu} - N \frac{\partial u}{\partial \nu}\right) dS_y$

$$\hookrightarrow \quad -u(x) + \int_{\Omega} N f \, dy = 0 - \int_{\partial\Omega} N g \, dS_y$$

Claim 2 $\boxed{N(x,y) = N(y,x).}$

the same proof as for $G(x,y)$

Claim 3. $\boxed{N(x,y) = \Phi(x-y) + \psi^{(x)}(y)}$

$$\begin{cases} \Delta_y \psi^{(x)} = 0 & \text{in } \Omega \\ \left.\dfrac{\partial \psi^{(x)}}{\partial \nu_y}\right|_{\partial\Omega} = -\dfrac{\partial \Phi}{\partial \nu_y} + \text{Const.}, & y \in \partial\Omega \end{cases}$$

§2.3 Energy Method

The "Energy Method" is the most commonly used method in the analysis of PDEs.

↳ Techniques involving the L^2-norm of various expressions.

Example 1. Uniqueness

Consider $\begin{cases} -\Delta u = f & \text{in } \Omega \subset \mathbb{R}^n \text{ open, bdd.} \\ u|_{\partial\Omega} = g & \partial\Omega \in C^1 \end{cases}$

Claim: $\exists$ at most 1 solution $u \in C^2(\Omega) \cap C(\bar{\Omega})$

Proof. If u_1, u_2 are two solutions,

Set $w = u_1 - u_2$: $\begin{cases} \Delta w = 0 & \text{in } \Omega \\ w|_{\partial\Omega} = 0 \end{cases}$

$$0 = \int_\Omega w \Delta w \, dx = \int_{\partial\Omega} w \frac{\partial w}{\partial \nu} dS_x - \int_\Omega |Dw|^2 dx$$

$$= -\int_\Omega |Dw|^2 dx$$

↳ $Dw \equiv 0$ in Ω $\xRightarrow{w|_{\partial\Omega}=0}$ $w \equiv 0$ in Ω.

Example 2 Dirichlet's Principle

$u \in C^2(\bar{\Omega})$ solves $(*)\begin{cases} -\Delta u = f & \text{in } \Omega \\ u|_{\partial\Omega} = g \end{cases}$

$\Updownarrow$ By the energy method

$u \in \mathcal{A} = \{ w \in C^2(\bar{\Omega}) \mid w|_{\partial\Omega} = g \}$ solves

$$I[u] = \min_{w \in \mathcal{A}} \underbrace{I[w]}_{\displaystyle \parallel \atop \displaystyle \int_\Omega \left(\tfrac{1}{2}|Dw|^2 - wf\right) dx.}$$

See Evans, §2.2.5(b)

I. Linear Second-Order Elliptic Eqs

$$-\sum_{i,j=1}^{n} a_{ij}(x)\, u_{x_i x_j} + \sum_{i=1}^{n} b_i(x)\, u_{x_i} + c(x)u$$

$$= f(x)$$

↳ Dr. Yves Capdeboscq's Course

II Nonlinear Second-Order Elliptic Eqs.

$$-\sum_{i,j=1}^{n} a_{ij}(Du, u, x)\, u_{x_i x_j} + B(Du, u, x) = 0$$

$$\operatorname{div} A(Du, u, x) + B(Du, u, x) = 0$$

↳ Further Courses