

# Foundation Module: PDE Problems

October 2017

**Instructions:** Work out at least **5** of the following 6 problems. Please specify which are the 5 problems you want to be graded. Please work on these problems only by yourself.

1. Let  $\Omega \subset \mathbb{R}^n$  open and bounded. Consider the initial-boundary value problem:

$$\begin{cases} u_t - \Delta u = f & \text{for } x \in \Omega, t > 0, \\ u|_{\partial\Omega \times \{t>0\}} = g, \\ u|_{t=0} = u_0(x), \end{cases} \quad (1)$$

where  $f, g$ , and  $u_0$  are given smooth functions depending only on  $x \in \mathbb{R}^n$ .

(a). Using the energy method to show the uniqueness of solutions in the class of functions in  $C^{2,1}(\Omega \times (0, \infty)) \cap C^{1,0}(\bar{\Omega} \times [0, \infty))$ .

(b). Consider the corresponding stationary problem:

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u|_{\partial\Omega} = g. \end{cases} \quad (2)$$

Using the energy method to show the uniqueness of solutions in the class of functions in  $C^2(\Omega) \cap C^1(\bar{\Omega})$ .

(c). For the unique solution  $u(x, t)$  of problem (1) and the unique solution  $u_*(x)$  of problem (2), show that there exists a constant  $C > 0$  such that

$$\int_{\Omega} |u(x, t) - u_*(x)|^2 dx \leq \int_{\Omega} |u_0(x) - u_*(x)|^2 dx e^{-Ct} \quad \text{for } t > 0.$$

2. Let  $\Omega \subset \mathbb{R}^n$  be open, bounded, smooth domain, and  $f \in L^2(\Omega)$ . Consider Poisson's problem:

$$(*) \quad \begin{cases} -\Delta u = f & \text{for } x \in \Omega, \\ u|_{\partial\Omega} = g, \end{cases}$$

and the variational problem:

$$(**) \quad I[u] = \min_{\omega \in \mathcal{A}} I[\omega],$$

with

$$I[\omega] \equiv \int_{\Omega} \left( \frac{1}{2} |\nabla \omega|^2 - \omega f \right) dx, \quad \mathcal{A} \equiv \{ \omega \in C^2(\Omega) \cap C^1(\bar{\Omega}) : \omega|_{\partial\Omega} = g \}.$$

Assume  $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ . Show that  $u(x)$  solves Poisson's problem (\*) if and only if  $u(x)$  is a minimizer of the variational problem (\*\*).

3. Let  $M$  be a first-order operator of the form:

$$M = \partial_t + a \cdot \nabla_x + b, \quad x = (x_1, \dots, x_n) \in \mathbb{R}^n,$$

where  $a = (a_1, \dots, a_n) \in \mathbb{R}^n$  and  $b \in \mathbb{R}$  are constants. Consider the following mixed problem:

$$\begin{cases} u_{tt} - \Delta_x u = 0 & \text{for } x_1 > 0, t > 0, \\ u|_{t=0} = f(x), \quad u_t|_{t=0} = g(x) & \text{for } x_1 > 0, \\ Mu|_{x_1=0} = 0, & \text{for } t > 0, \end{cases}$$

where  $f, g$  vanish for all sufficiently small  $x_1 > 0$  and are smooth. Prove that there exists a solution  $u(x, t)$  provided  $a_1 \leq 0$ .

4. Let  $b \in C(\mathbb{R})$ . Show that the solution  $u(x, t)$  of the Cauchy problem of the quasilinear partial differential equation

$$u_t + b(u)u_x = 0$$

with initial condition  $u(x, 0) = u_0(x)$  is given implicitly by

$$u(x, t) = u_0(x - b(u(x, t))t).$$

Show that the solution becomes singular for some positive  $t$ , unless  $b(u_0(s))$  is a nondecreasing function of  $s$ .

5. Derive an explicit formula for the solution of the nonhomogeneous initial-boundary problem for the heat equation:

$$\begin{cases} u_t - u_{xx} = f(x, t) & \text{for } x > 0, \\ u|_{x=0} = g(t) & \text{for } t > 0, \\ u|_{t=0} = u_0(x) & \text{for } x \geq 0, \end{cases}$$

for smooth and compact supported functions  $f, g$ , and  $u_0$ . Hint: Divide the problem into three problems.

6. Consider Burgers' equation

$$u_t + \left(\frac{u^2}{2}\right)_x = 0,$$

with initial data:

$$u|_{t=0} = \begin{cases} 1 & \text{for } x < -1, \\ 0 & \text{for } -1 < x < 0, \\ 2 & \text{for } 0 < x < 1, \\ 0 & \text{for } x > 1. \end{cases}$$

Construct the entropy solution for all  $t > 0$ .