

PDE-CDT Core Course

Analysis of Partial Differential Equations-Part III

Lecture 3

**EPSRC Centre for Doctoral Training in
Partial Differential Equations
Trinity Term**

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Course format: Teaching Course (TT)

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Hyperbolic Systems of Conservation Laws

$$\begin{cases} \frac{\partial}{\partial t} u_1 + \frac{\partial}{\partial x} f_1(u_1, \dots, u_m) = 0, \\ \dots\dots\dots \\ \frac{\partial}{\partial t} u_m + \frac{\partial}{\partial x} f_m(u_1, \dots, u_m) = 0, \end{cases}$$

$$\mathbf{u}_t + \mathbf{f}(\mathbf{u})_x = 0$$

$\mathbf{u} = (u_1, \dots, u_m)^\top \in \mathbb{R}^m$ conserved quantities

$\mathbf{f}(\mathbf{u}) = (f_1(\mathbf{u}), \dots, f_m(\mathbf{u}))^\top$ fluxes

Euler equations of gas dynamics (1755)

$$\left\{ \begin{array}{ll} \rho_t + (\rho v)_x = 0 & \text{(conservation of mass)} \\ (\rho v)_t + (\rho v^2 + p)_x = 0 & \text{(conservation of momentum)} \\ (\rho E)_t + (\rho E v + p v)_x = 0 & \text{(conservation of energy)} \end{array} \right.$$

ρ = mass density v = velocity

$E = e + v^2/2$ = energy density per unit mass (internal + kinetic)

$p = p(\rho, e)$ constitutive relation

Hyperbolic Systems

$$\mathbf{u}_t + \mathbf{f}(\mathbf{u})_x = 0$$

$$\mathbf{u} = \mathbf{u}(t, x) \in \mathbb{R}^m$$

$$\mathbf{u}_t + \mathbf{A}(\mathbf{u})\mathbf{u}_x = 0$$

$$\mathbf{A}(\mathbf{u}) = \nabla \mathbf{f}(\mathbf{u})$$

The system is **strictly hyperbolic** if each $m \times m$ matrix $\mathbf{A}(\mathbf{u})$ has real distinct eigenvalues

$$\lambda_1(\mathbf{u}) < \lambda_2(\mathbf{u}) < \cdots < \lambda_m(\mathbf{u})$$

Right eigenvectors $\mathbf{r}_1(\mathbf{u}), \dots, \mathbf{r}_m(\mathbf{u})$ (column vectors)

Left eigenvectors $\mathbf{l}_1(\mathbf{u}), \dots, \mathbf{l}_m(\mathbf{u})$ (row vectors)

$$\mathbf{A}\mathbf{r}_i = \lambda_i\mathbf{r}_i \quad \mathbf{l}_i\mathbf{A} = \lambda_i\mathbf{l}_i$$

Choose the bases so that

$$\mathbf{l}_i \cdot \mathbf{r}_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Invariance of Hyperbolicity under Change of Coordinates

Theorem

- Let \mathbf{u} be a smooth solution of the strictly hyperbolic system

$$\mathbf{u}_t + \mathbf{A}(\mathbf{u})\mathbf{u}_x = 0 \quad \text{in } \mathbb{R}_+ \times \mathbb{R}$$

- Assume $\phi : \mathbb{R}^m \rightarrow \mathbb{R}^m$ is a smooth diffeomorphism, with inverse ψ

Then $\mathbf{w} := \phi(\mathbf{u})$ solves the strictly hyperbolic system

$$\mathbf{w}_t + \mathbf{B}(\mathbf{w})\mathbf{w}_x = 0 \quad \text{in } \mathbb{R}_+ \times \mathbb{R}$$

for $\mathbf{B}(\mathbf{w}) := \nabla\phi(\psi(\mathbf{w}))\mathbf{A}(\psi(\mathbf{w}))\nabla\psi(\mathbf{w}) \quad \mathbf{w} \in \mathbb{R}^m$

Dependence of Eigenvalues and Eigenvectors on \mathbf{u}

Theorem

Assume that the matrix function $\mathbf{A}(\mathbf{u})$ is smooth, strictly hyperbolic. Then

- The eigenvalues $\lambda_k(\mathbf{u})$ depend smoothly on $\mathbf{u} \in \mathbb{R}^m, k = 1, \dots, m$
- We can select the right eigenvectors $\mathbf{r}_k(\mathbf{u})$ and left eigenvector $\mathbf{l}_k(\mathbf{u})$ to depend smoothly on $\mathbf{u} \in \mathbb{R}^m$ and satisfy the normalization

$$|\mathbf{r}_k(\mathbf{u})|, |\mathbf{l}_k(\mathbf{u})| = 1, \quad k = 1, \dots, m.$$

*We are not only globally and smoothly defining the eigenvalues and eigenspaces of $\mathbf{A}(\mathbf{u})$, but also globally providing the eigenspaces of $\mathbf{A}(\mathbf{u})$ with an orientation.

Linear Hyperbolic Systems

$$\mathbf{u}_t + \mathbf{A}\mathbf{u}_x = 0$$

$$\mathbf{u}(0, x) = \phi(x)$$

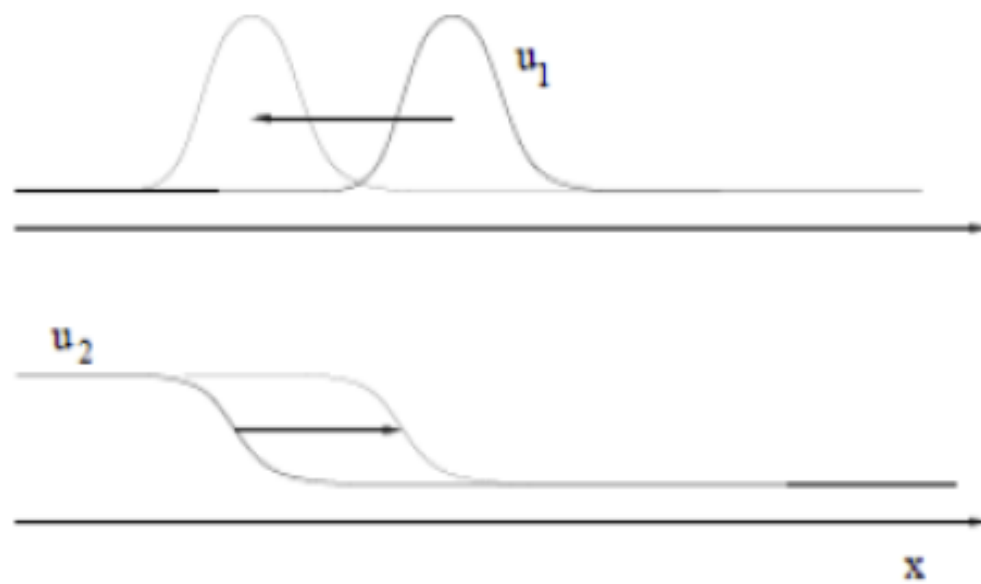
$\lambda_1 < \dots < \lambda_m$ eigenvalues

$\mathbf{r}_1, \dots, \mathbf{r}_m$ eigenvectors

Explicit solutions: **Linear superposition of travelling waves**

$$\mathbf{u}(t, x) = \sum_i \phi_i(x - \lambda_i t) \mathbf{r}_i$$

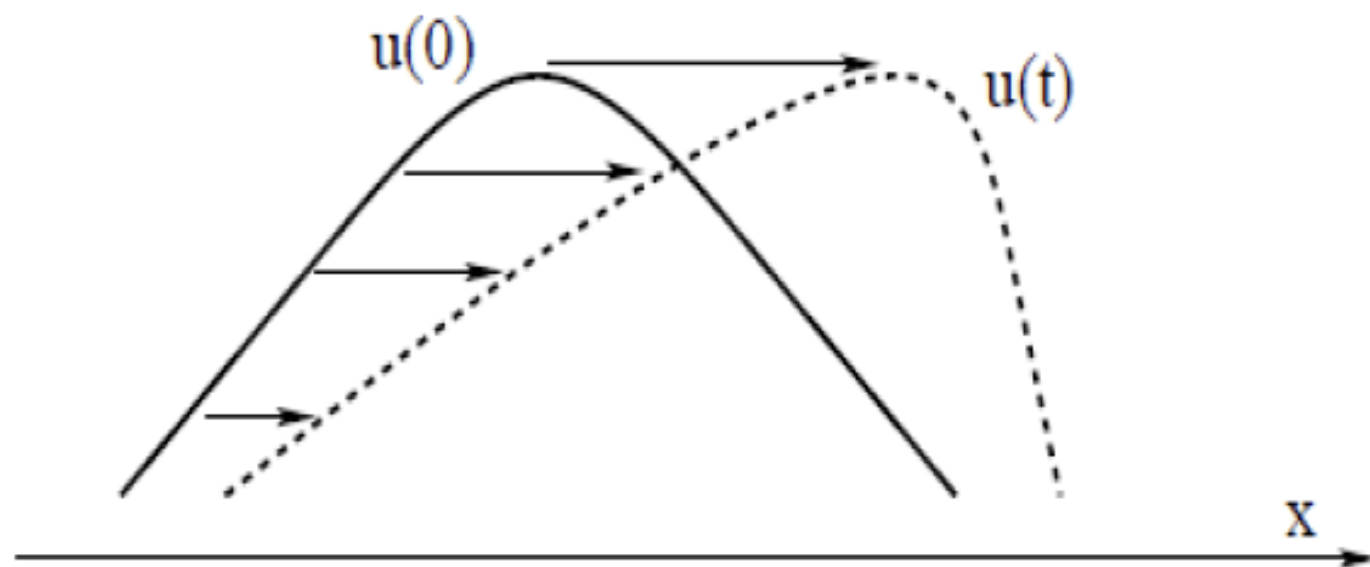
$$\phi_i(s) = \mathbf{l}_i \cdot \phi(s)$$



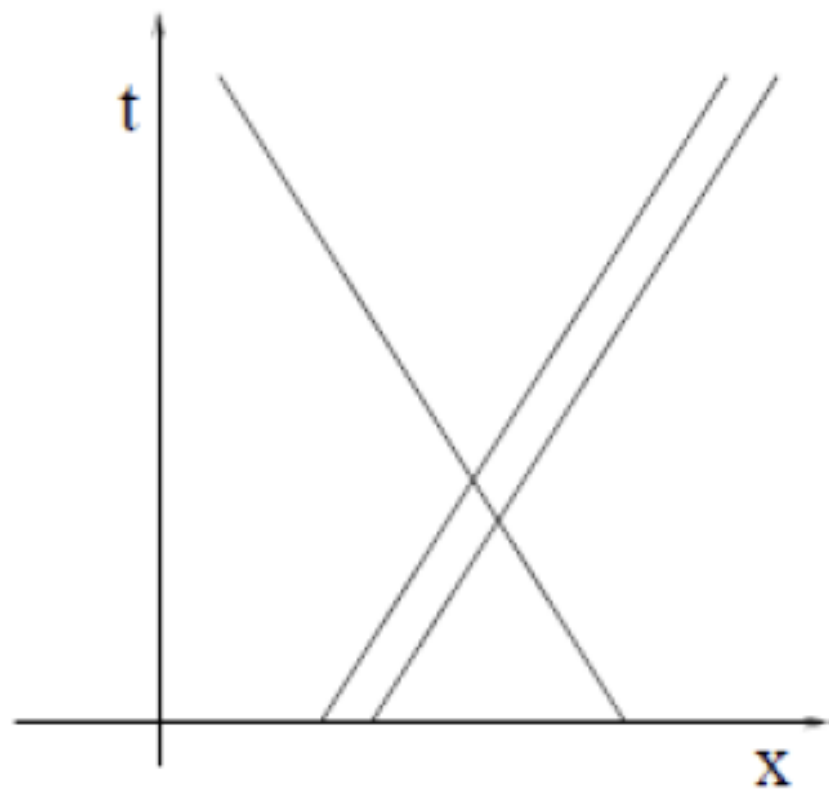
Nonlinear Effects

$$u_t + A(u)u_x = 0$$

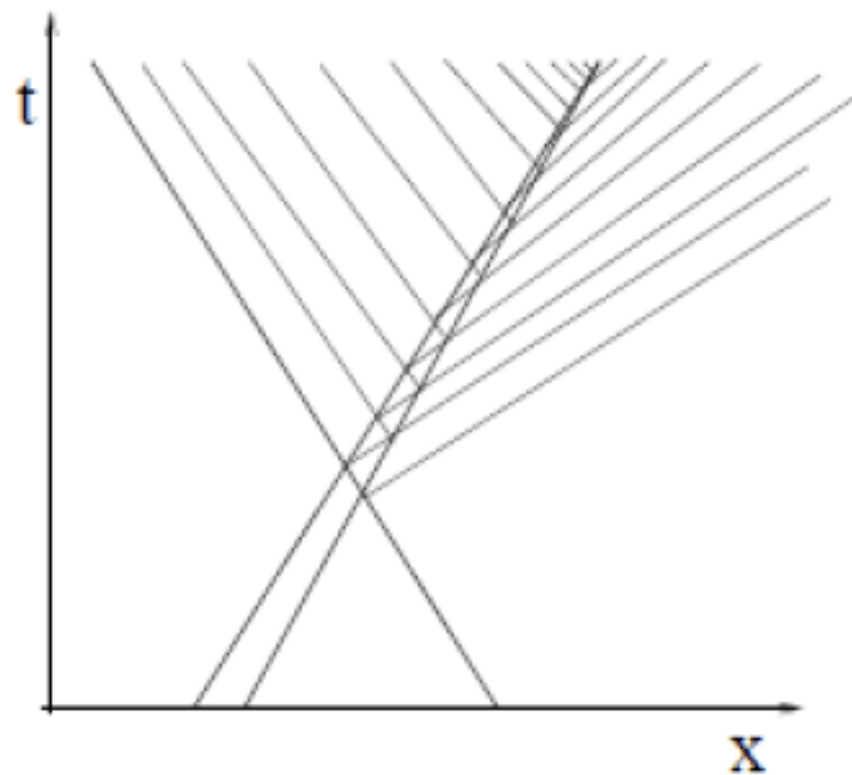
eigenvalues depend on $u \implies$ waves change shape



eigenvectors depend on u \implies nontrivial wave interactions



linear



nonlinear

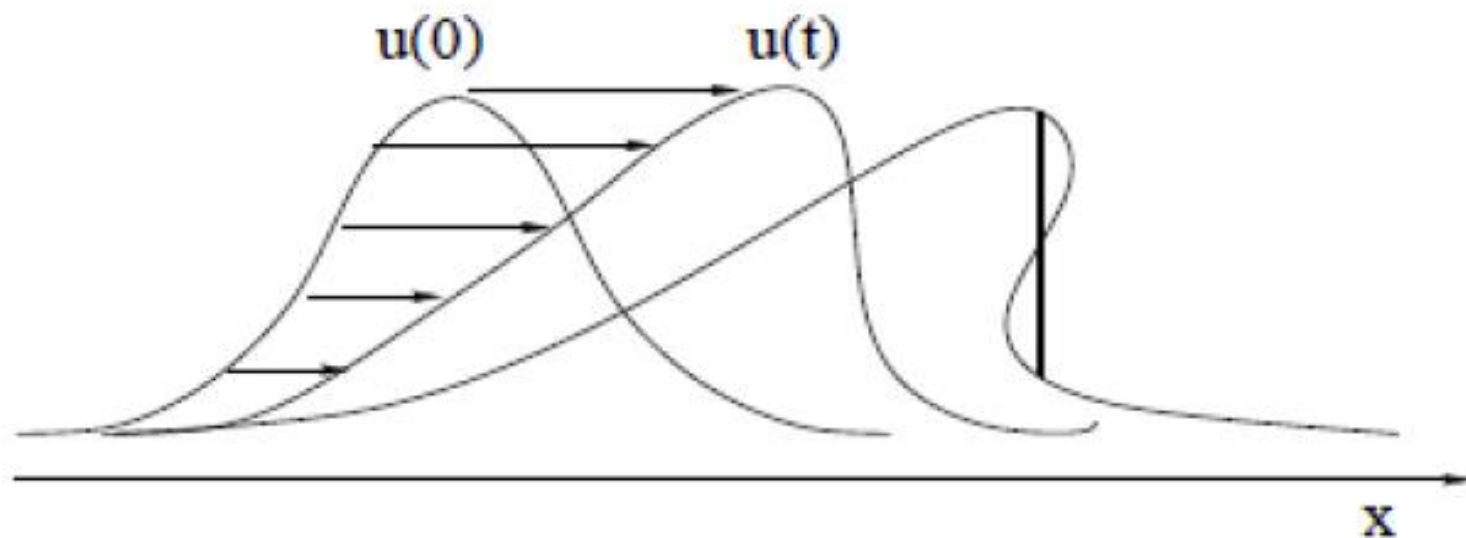
Loss of Regularity

$$u_t + (u^2/2)_x = 0$$

$$u_t + uu_x = 0$$

$$f(u) = u^2/2$$

characteristic speed: $f'(u) = u$

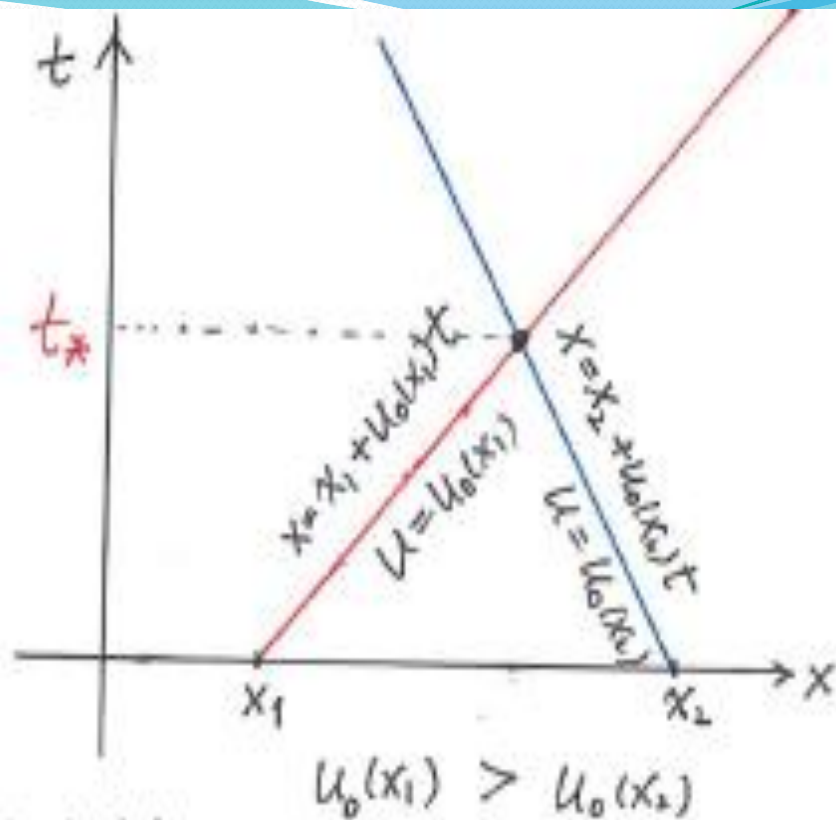


Global solutions only in a space of discontinuous functions

1-D Example

$$\begin{cases} U_t + \left(\frac{U^2}{2}\right)_x = 0 \\ U|_{t=0} = U_0(x) \end{cases}$$

$$U_t + U U_x = 0$$



$$\begin{cases} \frac{dx}{dt} = u. \rightarrow x = x_j + U_0(x_j)t \\ \frac{du}{dt} = 0 \rightarrow u = U_0(x_j), j=1,2 \end{cases}$$

When $t \rightarrow t_* = \frac{x_2 - x_1}{U_0(x_1) - U_0(x_2)} > 0$, $U(t, x)$ is Multi-Valued

Smooth Solutions – Evolution of Wave Components

$$\mathbf{u}_t = -\mathbf{A}(\mathbf{u})\mathbf{u}_x$$

$\lambda_i(\mathbf{u}) = i$ -th eigenvalue, $\mathbf{l}_i(\mathbf{u}), \mathbf{r}_i(\mathbf{u}) = i$ -th eigenvectors

$u_x^i := \mathbf{l}_i \cdot \mathbf{u}_x = [i$ -th component of $\mathbf{u}_x] = [\text{density of } i\text{-waves in } \mathbf{u}]$

$$\mathbf{u}_x = \sum_{i=1}^m u_x^i \mathbf{r}_i(\mathbf{u}) \quad \mathbf{u}_t = - \sum_{i=1}^m \lambda_i(\mathbf{u}) u_x^i \mathbf{r}_i(\mathbf{u})$$

Differentiate the 1st equation w.r.t. t and the 2nd w.r.t. x
 \implies Evolution equation for scalar components u_x^i :

$$(u_x^i)_t + (\lambda_i u_x^i)_x = \sum_{j>k} (\lambda_j - \lambda_k) (\mathbf{l}_i \cdot [\mathbf{r}_j, \mathbf{r}_k]) u_x^j u_x^k$$

Source Terms

$$(\lambda_j - \lambda_k) \left(\mathbf{l}_i \cdot [\mathbf{r}_j, \mathbf{r}_k] \right) u_x^j u_x^k$$

= amount of i -waves produced by the interaction of j -waves with k -waves

$$\lambda_j - \lambda_k = [\text{difference in speed}]$$

= [rate at which j -waves and k -waves cross each other]

$$u_x^j u_x^k = [\text{density of } j\text{-waves}] \times [\text{density of } k\text{-waves}]$$

$$[\mathbf{r}_j, \mathbf{r}_k] = (\nabla \mathbf{r}_k) \mathbf{r}_j - (\nabla \mathbf{r}_j) \mathbf{r}_k \quad (\text{Lie bracket})$$

= [directional derivative of \mathbf{r}_k in the direction of \mathbf{r}_j]

– [directional derivative of \mathbf{r}_j in the direction of \mathbf{r}_k]

$\mathbf{l}_i \cdot [\mathbf{r}_j, \mathbf{r}_k]$ = i -th component of the Lie bracket $[\mathbf{r}_j, \mathbf{r}_k]$ along the basis of eigenvectors $\{\mathbf{r}_1, \dots, \mathbf{r}_m\}$

Shock Solutions

$$\mathbf{u}_t + \mathbf{f}(\mathbf{u})_x = 0$$

$$\mathbf{u}(t, x) = \begin{cases} \mathbf{u}^- & \text{if } x < \lambda t \\ \mathbf{u}^+ & \text{if } x > \lambda t \end{cases} \quad \text{is a weak solution}$$

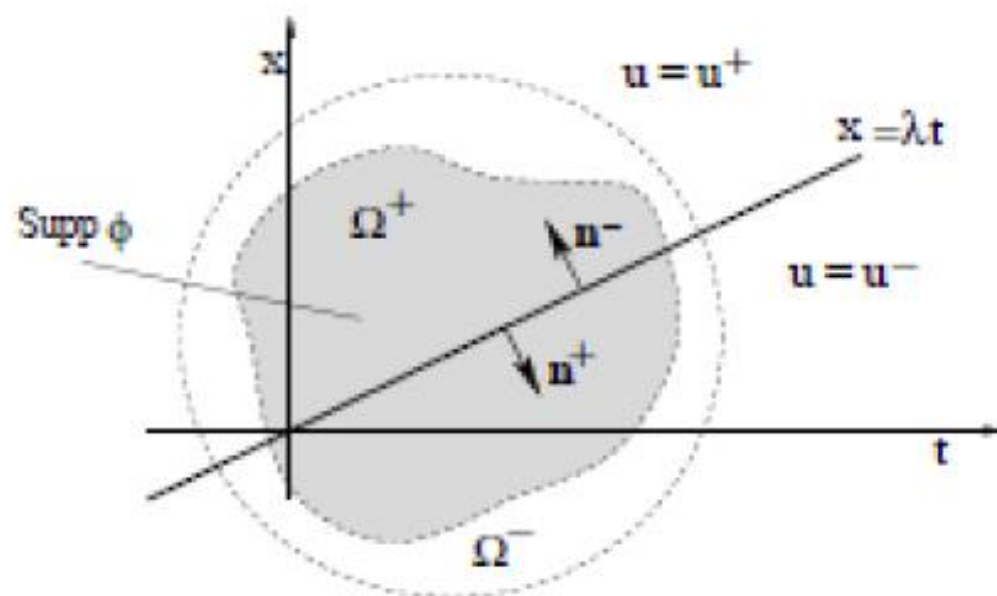
if and only if the **Rankine-Hugoniot Equations** hold:

$$\lambda [\mathbf{u}^+ - \mathbf{u}^-] = \mathbf{f}(\mathbf{u}^+) - \mathbf{f}(\mathbf{u}^-)$$

[Speed of the shock] × [Jump in the state] = [Jump in the flux]

Derivation of the Rankine - Hugoniot Equations

$$\begin{aligned} 0 &= \iint \left\{ u\phi_t + f(u)\phi_x \right\} dxdt = \iint_{\Omega^+ \cup \Omega^-} \operatorname{div} (u\phi, f(u)\phi) dxdt \\ &= \int_{\partial\Omega^+} \mathbf{n}^+ \cdot \mathbf{v} ds + \int_{\partial\Omega^-} \mathbf{n}^- \cdot \mathbf{v} ds \\ &= \int \left[\lambda(u^+ - u^-) - (f(u^+) - f(u^-)) \right] \phi(t, \lambda t) dt. \end{aligned}$$



$$\mathbf{v} \doteq \begin{pmatrix} u\phi \\ f(u)\phi \end{pmatrix}$$

Alternative Formulation

$$\begin{aligned}\lambda(\mathbf{u}^+ - \mathbf{u}^-) &= \mathbf{f}(\mathbf{u}^+) - \mathbf{f}(\mathbf{u}^-) \\ &= \int_0^1 \nabla \mathbf{f}(\theta \mathbf{u}^+ + (1 - \theta) \mathbf{u}^-) \cdot (\mathbf{u}^+ - \mathbf{u}^-) d\theta \\ &= \mathbf{A}(\mathbf{u}^+, \mathbf{u}^-) \cdot (\mathbf{u}^+ - \mathbf{u}^-)\end{aligned}$$

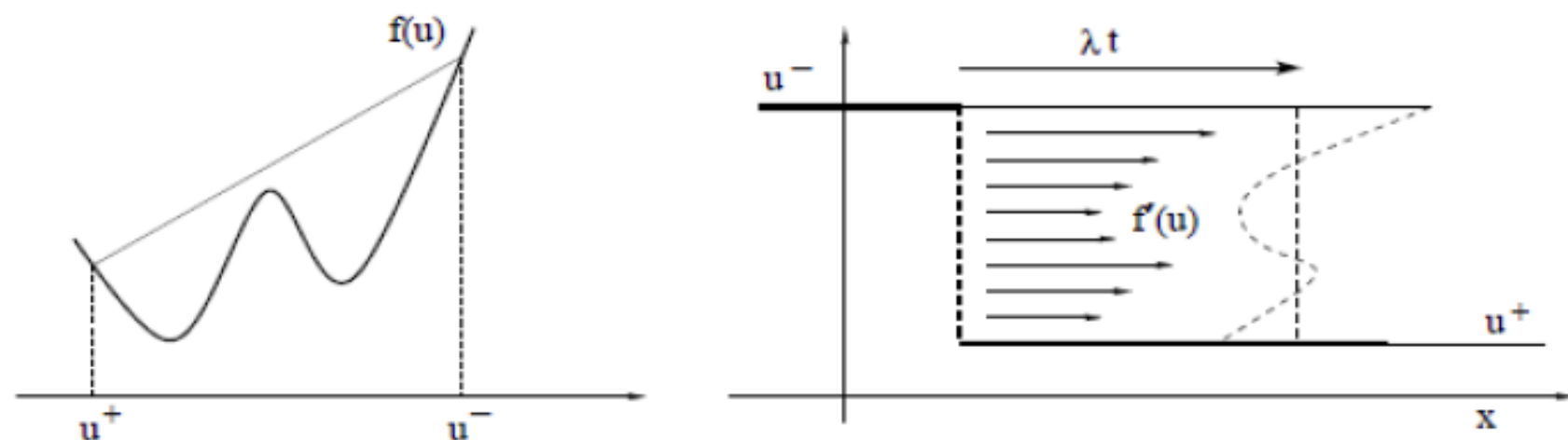
$$\begin{aligned}\mathbf{A}(\mathbf{u}, \mathbf{v}) &:= \int_0^1 \nabla \mathbf{f}(\theta \mathbf{u} + (1 - \theta) \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) d\theta \\ &= [\text{averaged Jacobian matrix}]\end{aligned}$$

The Rankine-Hugoniot conditions hold **if and only if**

$$\lambda(\mathbf{u}^+ - \mathbf{u}^-) = \mathbf{A}(\mathbf{u}^+, \mathbf{u}^-)(\mathbf{u}^+ - \mathbf{u}^-)$$

- The jump $\mathbf{u}^+ - \mathbf{u}^-$ is an eigenvector of the averaged matrix $\mathbf{A}(\mathbf{u}^+, \mathbf{u}^-)$
- The speed λ coincides with the corresponding eigenvalue

The Rankine-Hugoniot condition for the scalar conservation law $u_t + f(u)_x = 0$



$$\lambda = \frac{f(u^+) - f(u^-)}{u^+ - u^-} = \frac{1}{u^+ - u^-} \int_{u^-}^{u^+} f'(s) ds$$

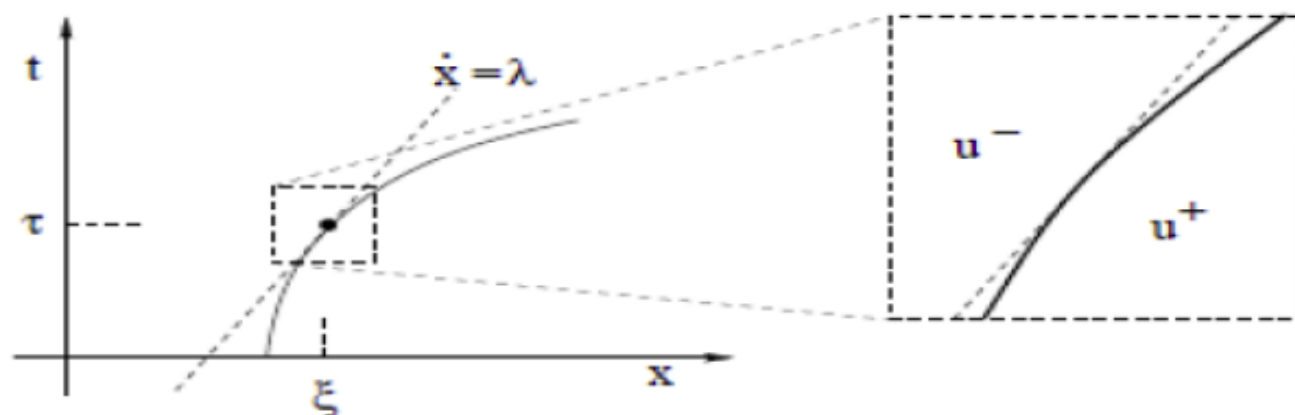
[speed of the shock] = [slope of secant line through u^- , u^+ on the graph of f]
= [average of the characteristic speeds between u^- and u^+]

Points of Approximate Jump

The function $\mathbf{u} = \mathbf{u}(t, x)$ has an approximate jump at a point (τ, ξ) if there exists states $\mathbf{u}^- = \mathbf{u}^+$ and a speed λ such that, setting

$$U(t, x) := \begin{cases} \mathbf{u}^- & \text{if } x < \lambda t \\ \mathbf{u}^+ & \text{if } x > \lambda t \end{cases}$$

there holds: $\lim_{\rho \rightarrow 0^+} \frac{1}{\rho^2} \int_{\tau-\rho}^{\tau+\rho} \int_{\xi-\rho}^{\xi+\rho} |\mathbf{u}(t, x) - U(t - \tau, x - \xi)| dx dt = 0$



Theorem

If \mathbf{u} is a weak solution to the system $\mathbf{u}_t + \mathbf{f}(\mathbf{u})_x = 0$, then the Rankine-Hugoniot equations hold at each point of approximate jump.

Construction of Shock Waves

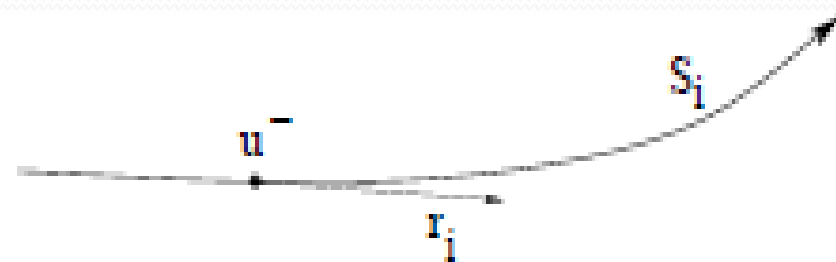
Problem: Given $\mathbf{u}^- \in \mathbb{R}^m$, find the states $\mathbf{u}^+ \in \mathbb{R}^m$ which, for some speed λ , satisfy the Rankine-Hugoniot equations:

$$\lambda(\mathbf{u}^+ - \mathbf{u}^-) = \mathbf{f}(\mathbf{u}^+) - \mathbf{f}(\mathbf{u}^-) = \mathbf{A}(\mathbf{u}^-, \mathbf{u}^+)(\mathbf{u}^+ - \mathbf{u}^-)$$

Alternative Formulation: Fix $i \in \{1, \dots, m\}$. The jump $\mathbf{u}^+ - \mathbf{u}^-$ is a (right) i -eigenvector of the averaged matrix $\mathbf{A}(\mathbf{u}^-, \mathbf{u}^+)$ if and only if it is orthogonal to all (left) eigenvectors $\mathbf{l}_j(\mathbf{u}^+, \mathbf{u}^-)$ of $\mathbf{A}(\mathbf{u}^-, \mathbf{u}^+)$:

$$\mathbf{l}_j(\mathbf{u}^-, \mathbf{u}^+) \cdot (\mathbf{u}^+ - \mathbf{u}^-) = 0 \quad \text{for all } j \neq i$$

Implicit Function Theorem \implies For each i , there exists a curve $s \rightarrow S_i(s)(\mathbf{u}^-)$ of points that satisfy $(RH)_i$.



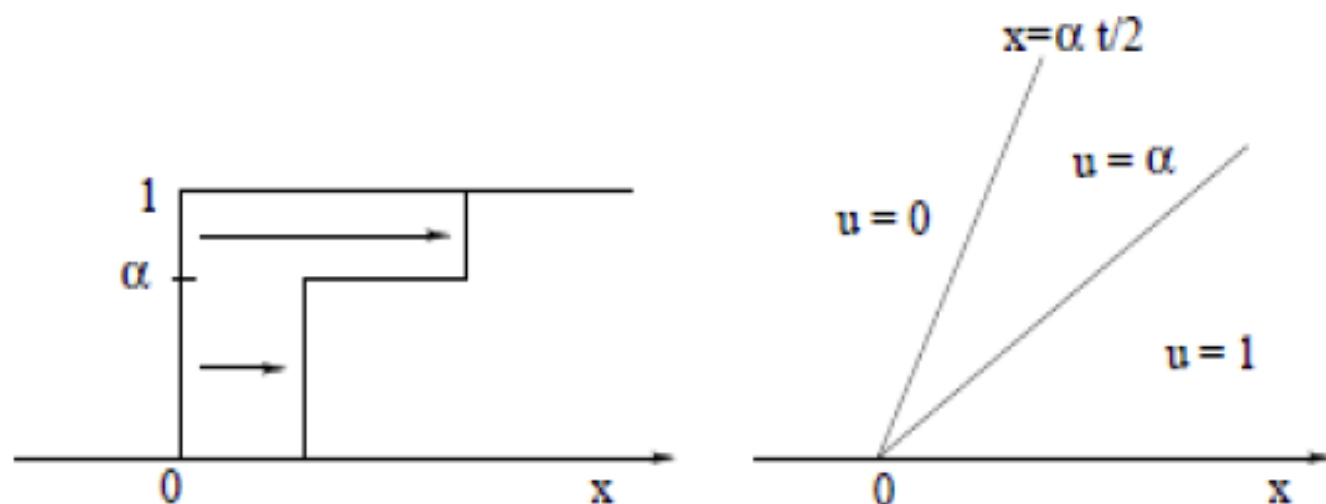
Non-uniqueness of Weak solutions

Example: a Cauchy problem for Burgers' equation

$$u_t + (u^2/2)_x = 0 \quad u(0, x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

Each $\alpha \in [0, 1]$ yields a weak solution

$$u_\alpha(t, x) = \begin{cases} 0 & \text{if } x < \alpha t/2 \\ \alpha & \text{if } \alpha t/2 < x < (1 + \alpha)t/2 \\ 1 & \text{if } x \geq (1 + \alpha)t/2 \end{cases}$$



Admissibility Conditions on Shocks

$$\mathbf{u}_t + \mathbf{f}(\mathbf{u})_x = 0$$

- Solutions should be stable w.r.t. small initial perturbations
- Solutions should be limits of suitable approximations and/or physical regularisations (**Vanishing viscosity, relaxation, ...**)
- Any convex entropy should not increase
-

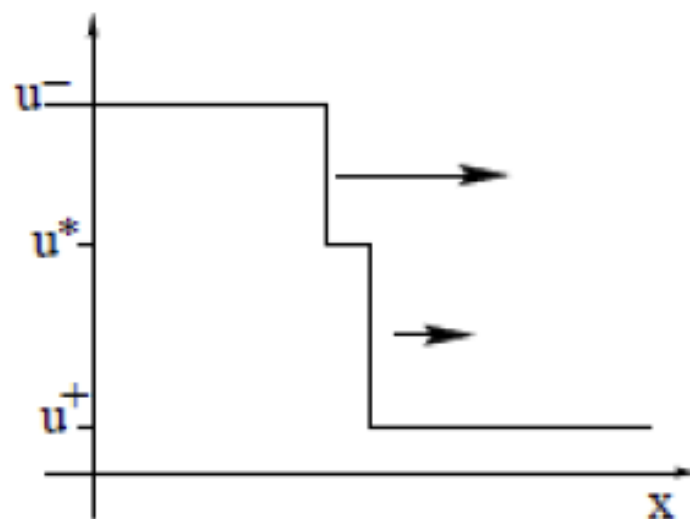
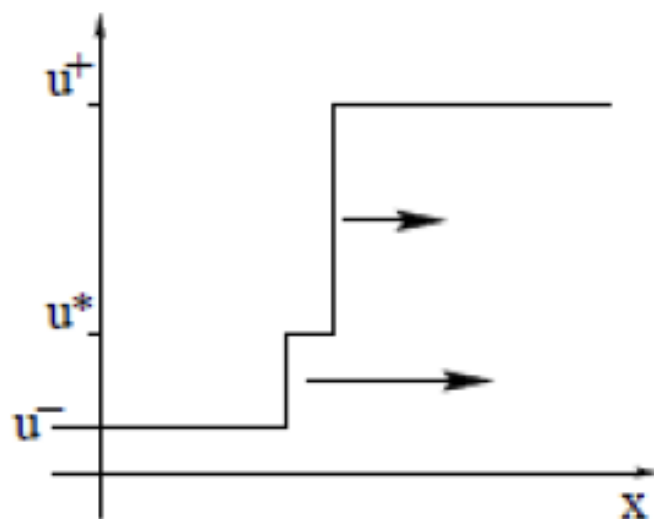
Stability conditions: the scalar case

Perturb the shock with left and right states u^- , u^+ by inserting an intermediate state $u^* \in [u^-, u^+]$

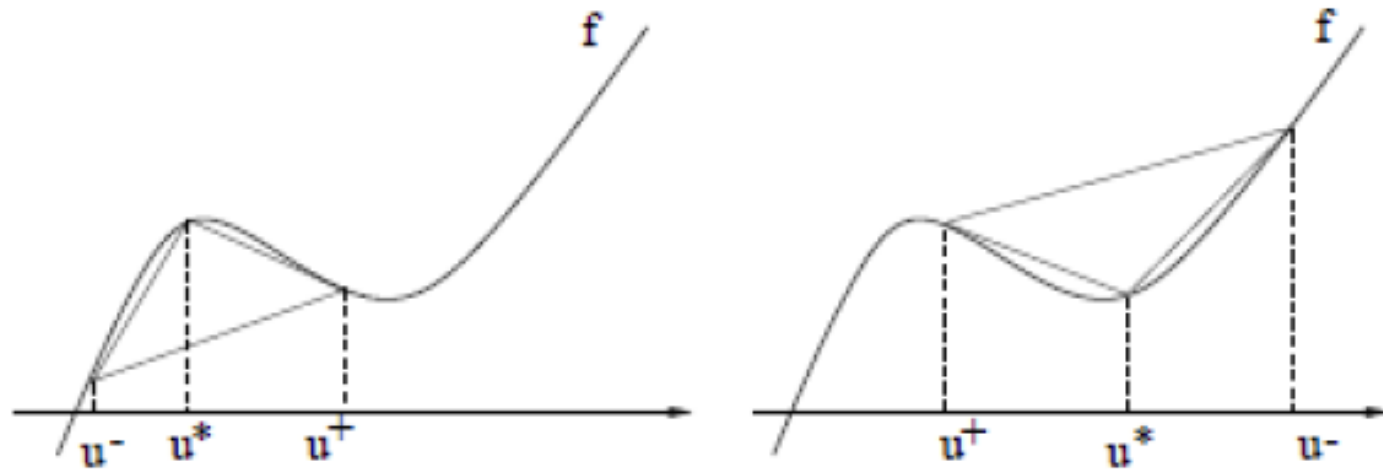
Initial shock is stable \iff

[speed of jump behind] \leq [speed of jump ahead]

$$\frac{f(u^*) - f(u^-)}{u^* - u^-} \geq \frac{f(u^+) - f(u^*)}{u^+ - u^*}$$



speed of a shock = slope of a secant line to the graph of f



Stability conditions:

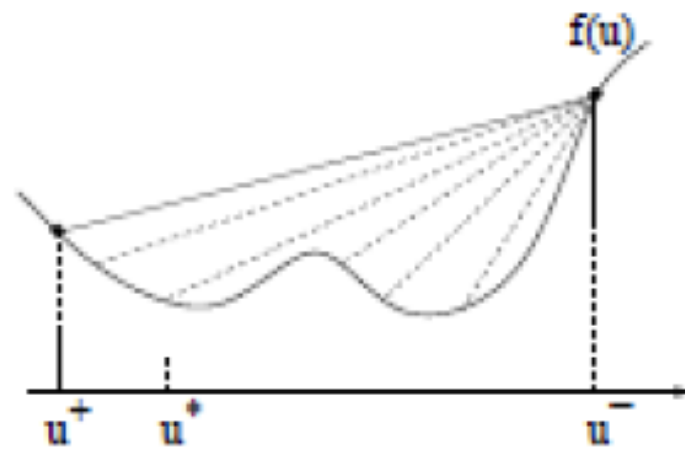
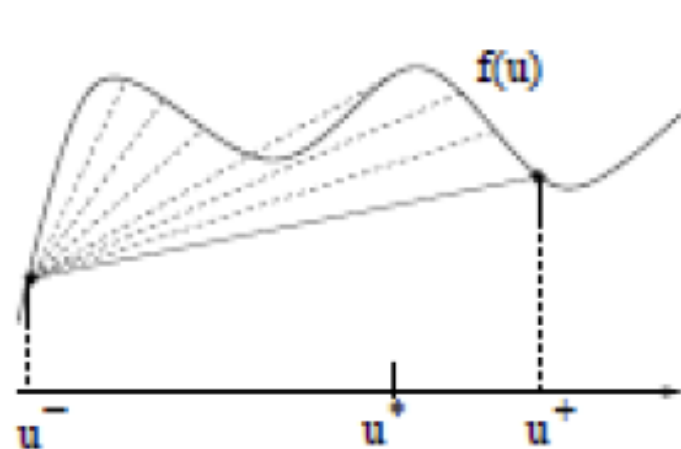
- when $u^- < u^+$ the graph of f should remain above the secant line
- when $u^- > u^+$, the graph of f should remain below the secant line

General stability conditions

Scalar case: stability holds if and only if

$$\frac{f(u^*) - f(u^-)}{u^* - u^-} \geq \frac{f(u^+) - f(u^-)}{u^+ - u^-}$$

for every intermediate state $u^* \in [u^-, u^+]$



Vector Valued Case: $\mathbf{u}^+ = S_i(\sigma)(\mathbf{u}^-)$ for some $\sigma \in \mathbb{R}$

Admissibility Condition (T.-P. Liu)

The speed $\lambda(\sigma)$ of the shock joining \mathbf{u}^- with \mathbf{u}^+ must be less or equal to the speed of every smaller shock, joining \mathbf{u}^- with an intermediate state $\mathbf{u}^* = S_i(s)(\mathbf{u}^-)$, $s \in [0, \sigma]$:

$$\lambda(\mathbf{u}^-, \mathbf{u}^+) \leq \lambda(\mathbf{u}^-, \mathbf{u}^*)$$

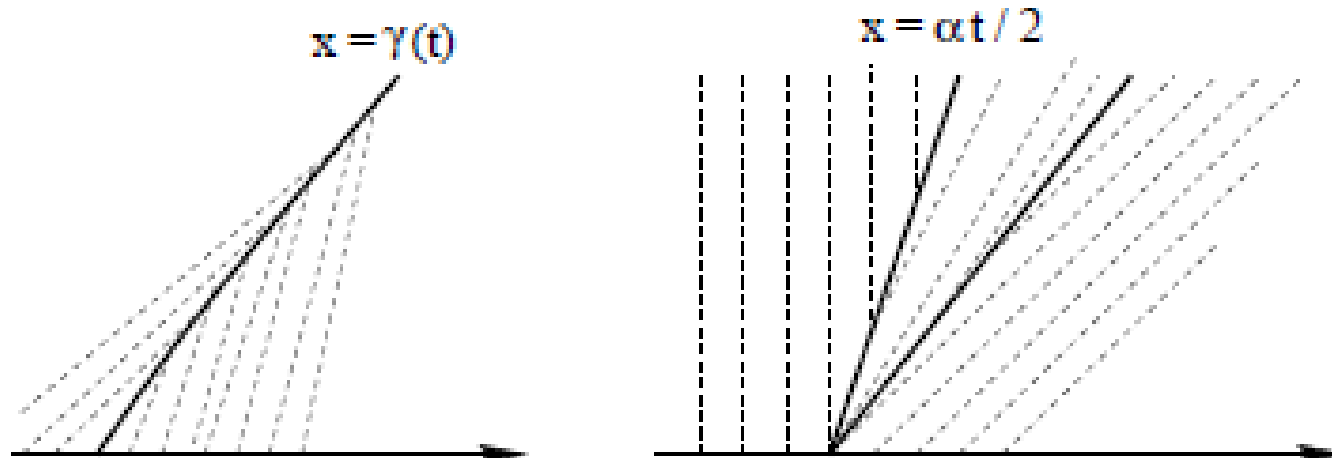
- The Liu condition singles out precisely the solutions which are limits of vanishing viscosity approximations

$$\mathbf{u}_t^\varepsilon + \mathbf{f}(\mathbf{u}^\varepsilon)_x = \varepsilon \mathbf{u}_{xx}^\varepsilon \quad \mathbf{u}^\varepsilon \rightarrow \mathbf{u} \quad \text{as } \varepsilon \rightarrow 0$$

Admissibility Condition (P. Lax)

A shock connecting the states u^- , u^+ , travelling with speed $\lambda = \lambda_i(u^-, u^+)$ is *admissible* if

$$\lambda_i(u^-) \geq \lambda_i(u^-, u^+) \geq \lambda_i(u^+)$$



- Geometric meaning: characteristics flow toward the shock from both sides
- The Liu condition implies the Lax condition

Mathematical Entropy – Entropy Flux

$$\mathbf{u}_t + \mathbf{f}(\mathbf{u})_x = 0$$

Definition: A function $\eta : \mathbb{R}^m \rightarrow \mathbb{R}$ is called an **Entropy**, with **Entropy Flux** $q : \mathbb{R}^m \rightarrow \mathbb{R}$ if

$$\nabla \eta(\mathbf{u}) \nabla \mathbf{f}(\mathbf{u}) = \nabla q(\mathbf{u})$$

For **smooth** solutions $\mathbf{u} = \mathbf{u}(t, x)$, this implies

$$\begin{aligned} \eta(\mathbf{u})_t + q(\mathbf{u})_x &= \nabla \eta(\mathbf{u}) \mathbf{u}_t + \nabla q(\mathbf{u}) \mathbf{u}_x \\ &= -(\nabla \eta(\mathbf{u}) \nabla \mathbf{f}(\mathbf{u})) \mathbf{u}_x + \nabla q(\mathbf{u}) \mathbf{u}_x = 0 \end{aligned}$$

$\implies \eta(\mathbf{u})$ is an additional conserved quantity,
with flux $q(\mathbf{u})$

Existence of Entropy – Entropy Flux Pairs

$$\nabla \eta(\mathbf{u}) \nabla \mathbf{f}(\mathbf{u}) = \nabla q(\mathbf{u}).$$

$$\left(\frac{\partial \eta}{\partial u_1} \cdots \frac{\partial \eta}{\partial u_m} \right) \begin{pmatrix} \frac{\partial f_1}{\partial u_1} & \cdots & \frac{\partial f_1}{\partial u_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial u_1} & \cdots & \frac{\partial f_m}{\partial u_m} \end{pmatrix} = \left(\frac{\partial q}{\partial u_1} \cdots \frac{\partial q}{\partial u_m} \right)$$

- A systems of m equations for 2 unknown functions: $\eta(\mathbf{u})$ and $q(\mathbf{u})$
- Over-determined if $m > 2$
- However, most of physical systems (described by several conservation laws) are endowed with natural entropies

Entropy Admissibility Condition

A weak solution \mathbf{u} of the hyperbolic system $\mathbf{u}_t + \mathbf{f}(\mathbf{u})_x = 0$ is **Entropy Admissible** if

$$\eta(\mathbf{u})_t + q(\mathbf{u})_x \leq 0$$

in the sense of distributions, for every entropy-entropy flux pair (η, q) with $\nabla^2 \eta(\mathbf{u}) \geq$, i.e. convex.

$$\iint \{ \eta(\mathbf{u}) \varphi_t + q(\mathbf{u}) \varphi_x \} dx dt \geq 0 \quad \varphi \in C_c^\infty, \varphi \geq 0$$

- Smooth solutions conserve all entropies
- Solutions with shocks are admissible if they dissipate all convex entropies

Consistency with Vanishing Viscosity Approximations

$$\mathbf{u}_t^\varepsilon + \mathbf{f}(\mathbf{u}^\varepsilon)_x = \varepsilon \mathbf{u}_{xx}^\varepsilon \quad \mathbf{u}^\varepsilon \rightarrow \mathbf{u} \quad \text{as } \varepsilon \rightarrow 0$$

For any entropy-entropy flux pair

$$(\eta(\mathbf{u}), q(\mathbf{u})) \quad \nabla^2 \eta(\mathbf{u}) \geq 0,$$

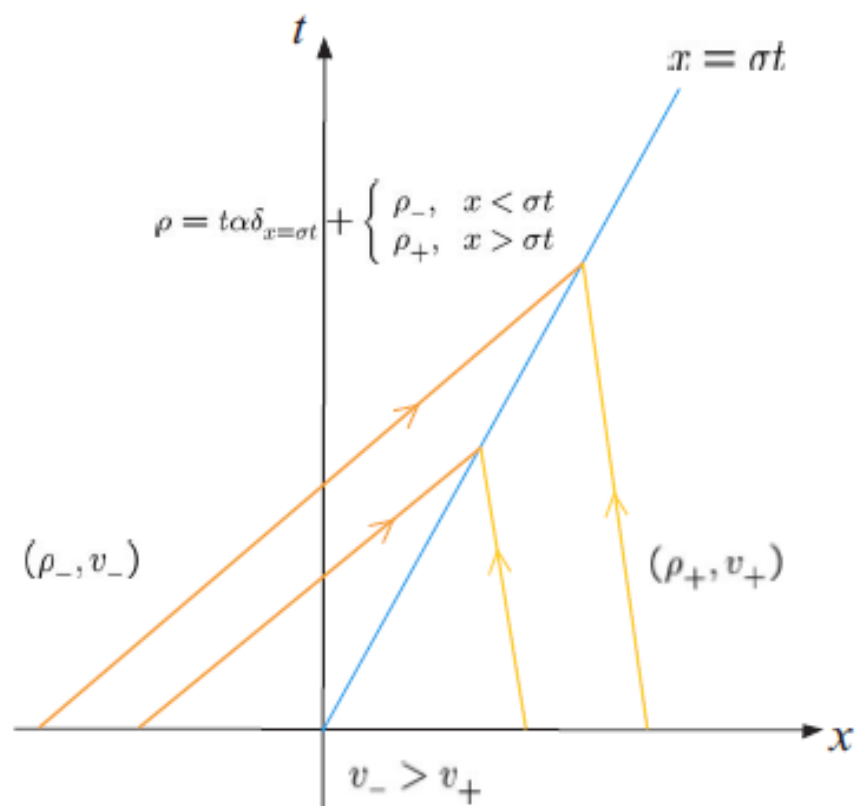
multiply $\nabla \eta(\mathbf{u}^\varepsilon)$ both sides of the system yields

$$\begin{aligned} \eta(\mathbf{u}^\varepsilon)_t + q(\mathbf{u}^\varepsilon)_x &= \varepsilon \eta(\mathbf{u}^\varepsilon)_{xx} - \varepsilon (\mathbf{u}_x)^\top \nabla^2 \eta(\mathbf{u}^\varepsilon) \mathbf{u}_x \\ &\leq \varepsilon \eta(\mathbf{u}^\varepsilon)_{xx} \rightarrow 0 \end{aligned}$$

in the sense of distributions.

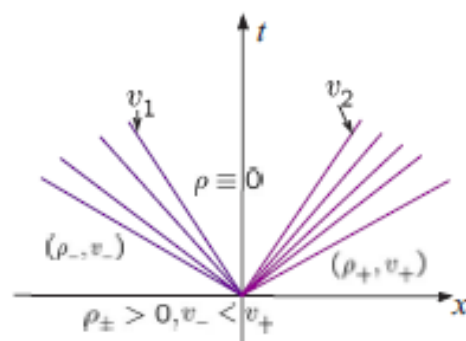
Pressureless Euler Equations

$$\partial_t \rho + \partial_x(\rho v) = 0, \quad \partial_t(\rho v) + \partial_x(\rho v^2) = 0$$

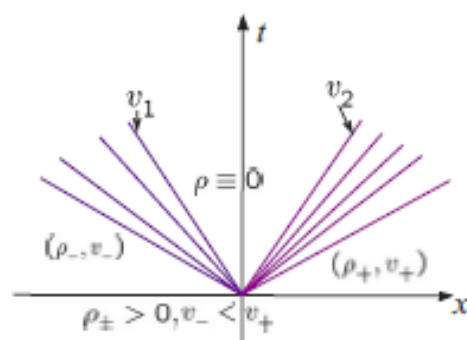


$$\alpha = \frac{1}{\sqrt{1 + \sigma^2}}(\sigma[\rho] - [\rho v]) > 0, \quad \sigma = \frac{\sqrt{\rho_+}v_+ + \sqrt{\rho_-}v_-}{\sqrt{\rho_+} + \sqrt{\rho_-}} \in (v_+, v_-)$$

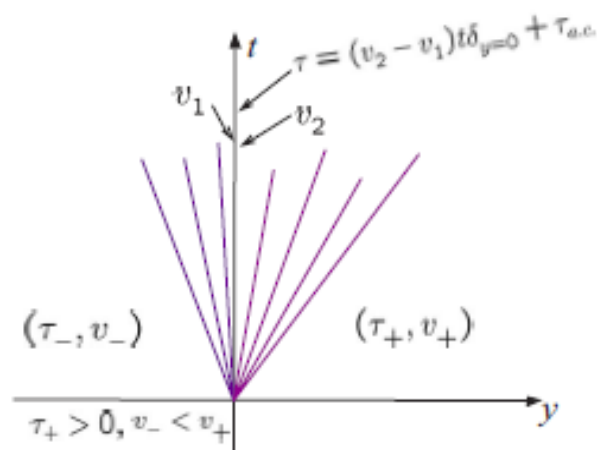
$$\partial_t \rho + \partial_x(\rho v) = 0, \quad \partial_t(\rho v) + \partial_x(\rho v^2 + p(\rho)) = 0$$



$$\partial_t \rho + \partial_x(\rho v) = 0, \quad \partial_t(\rho v) + \partial_x(\rho v^2 + p(\rho)) = 0$$



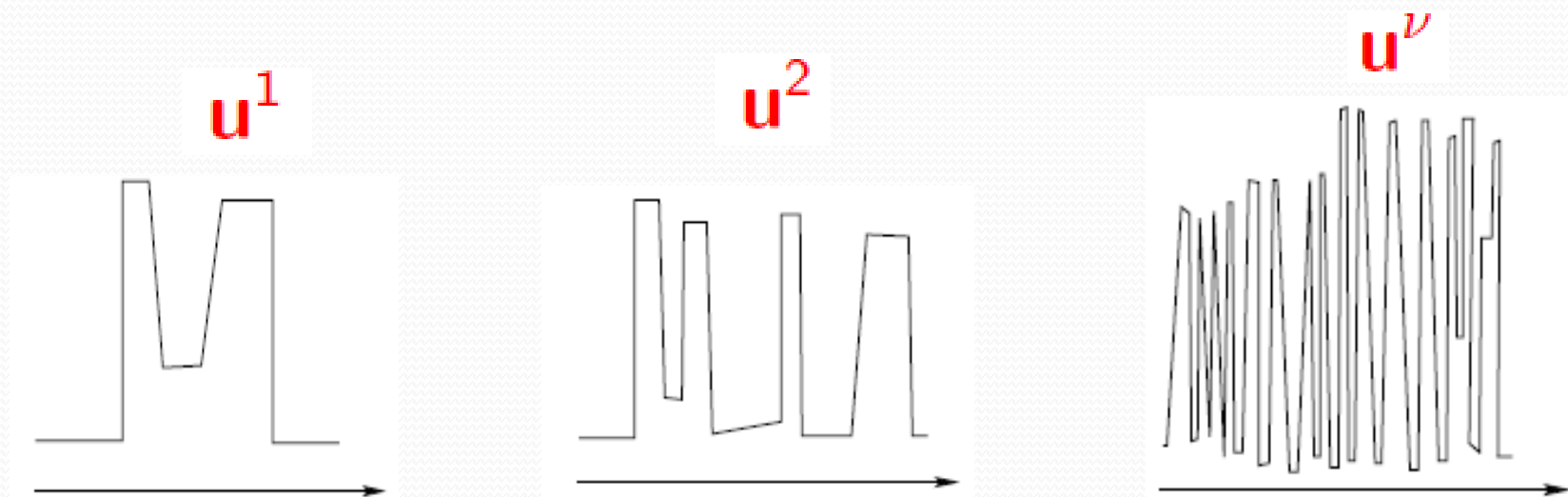
$$(t, x) \rightarrow (t, y) : y_t = \rho(t, x), y_x = -(\rho v)(t, x); \quad \tau(t, y) = 1/\rho(t, x)$$



Global in Time Solutions to the Cauchy Problem

$$\mathbf{u}_t + \mathbf{f}(\mathbf{u})_x = 0, \quad \mathbf{u}(0, x) = \mathbf{u}(x)$$

- Construct a sequence of approximate solutions $\{\mathbf{u}^\nu\}_{\nu \geq 1}$
- Show that (a subsequence) converges: $\mathbf{u}^\nu \rightarrow \mathbf{u}$ in L^1_{loc}
- Show that the limit \mathbf{u} is an entropy solution.



Need: a-priori bound on the total variation (J. Glimm, 1965)

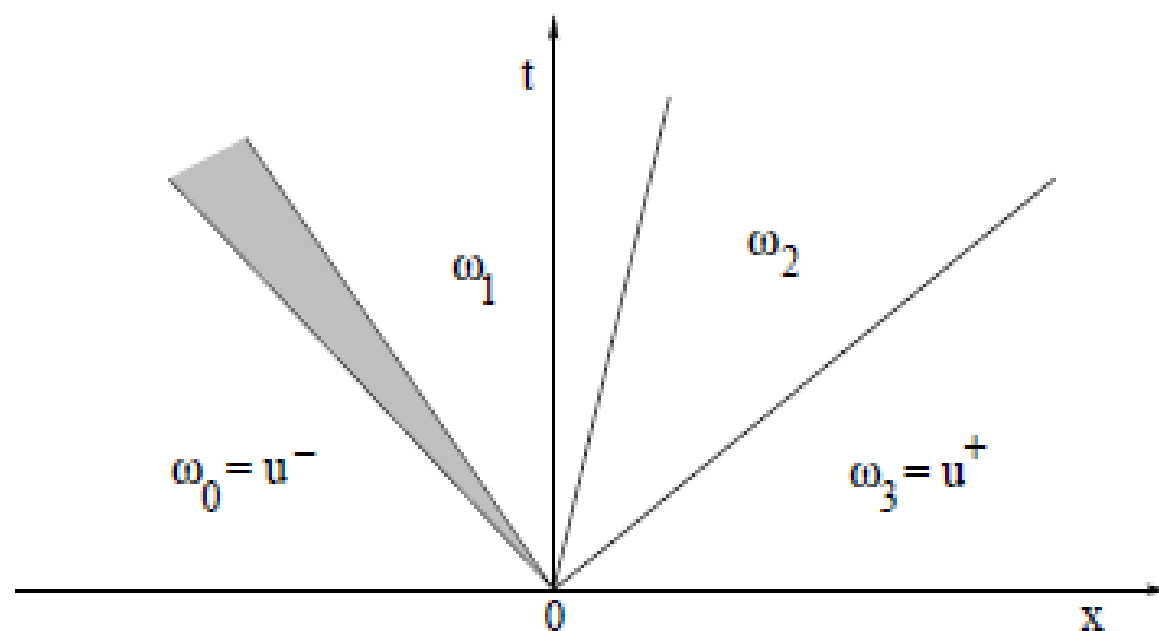
Building Block: The Riemann Problem

$$\mathbf{u}_t + \mathbf{f}(\mathbf{u})_x = 0, \quad \mathbf{u}(0, x) = \begin{cases} \mathbf{u}^- & x < 0 \\ \mathbf{u}^+ & x > 0 \end{cases}$$

- **B. Riemann 1860:** 2×2 isentropic Euler equations
- **P. Lax 1957:** $m \times m$ systems (+ special assumptions)
- **T.-P. Liu 1975:** $m \times m$ systems (generic case)

*The Riemann solutions are also the vanishing viscosity limits for general hyperbolic systems, possibly non-conservative

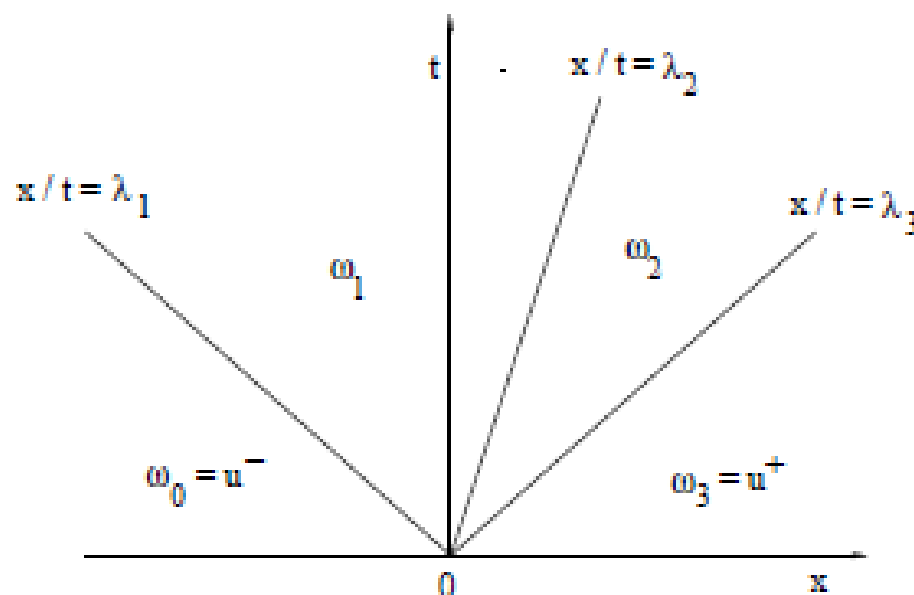
Solution to the Riemann problem



- is invariant w.r.t. rescaling symmetry: $u^\theta(t, x) \doteq u(\theta t, \theta x) \quad \theta > 0$
- describes local behavior of BV solutions near each point (t_0, x_0)
- describes large-time asymptotics as $t \rightarrow +\infty$ (for small total variation)

Riemann Problem for Linear Systems

$$u_t + Au_x = 0 \quad u(0, x) = \begin{cases} u^- & \text{if } x < 0 \\ u^+ & \text{if } x > 0 \end{cases}$$



$$u^+ - u^- = \sum_{j=1}^n c_j r_j \quad (\text{sum of eigenvectors of } A)$$

$$\text{intermediate states : } \omega_i \doteq u^- + \sum_{j \leq i} c_j r_j$$

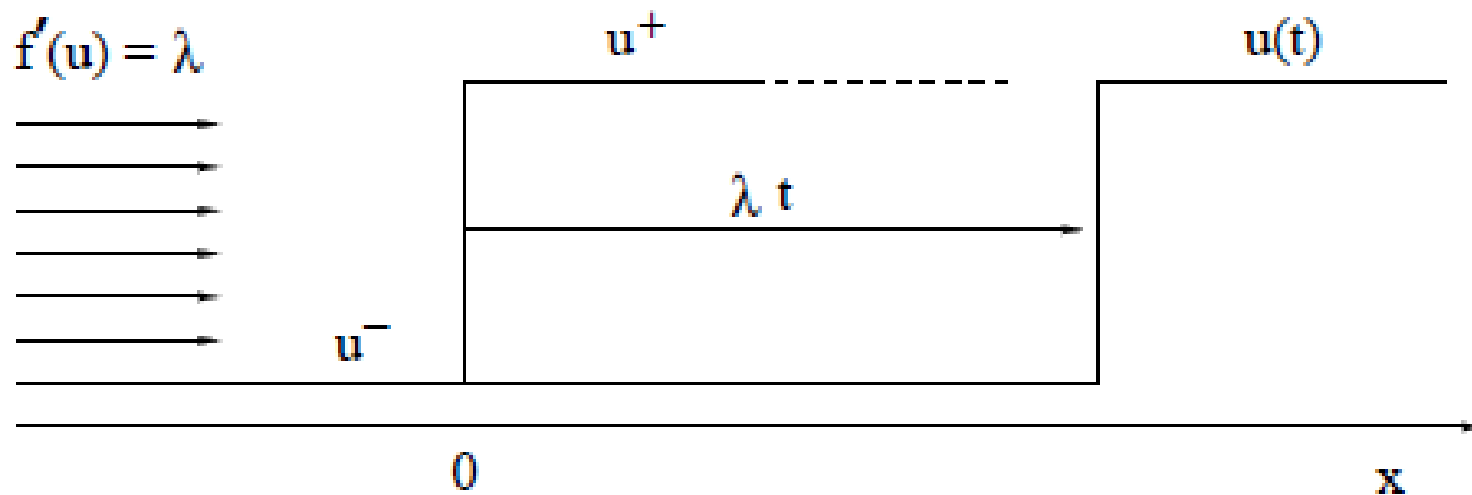
i -th jump: $\omega_i - \omega_{i-1} = c_i r_i$ travels with speed λ_i

Scalar Conservation Law

$$u_t + f(u)_x = 0 \quad u \in \mathbb{R}$$

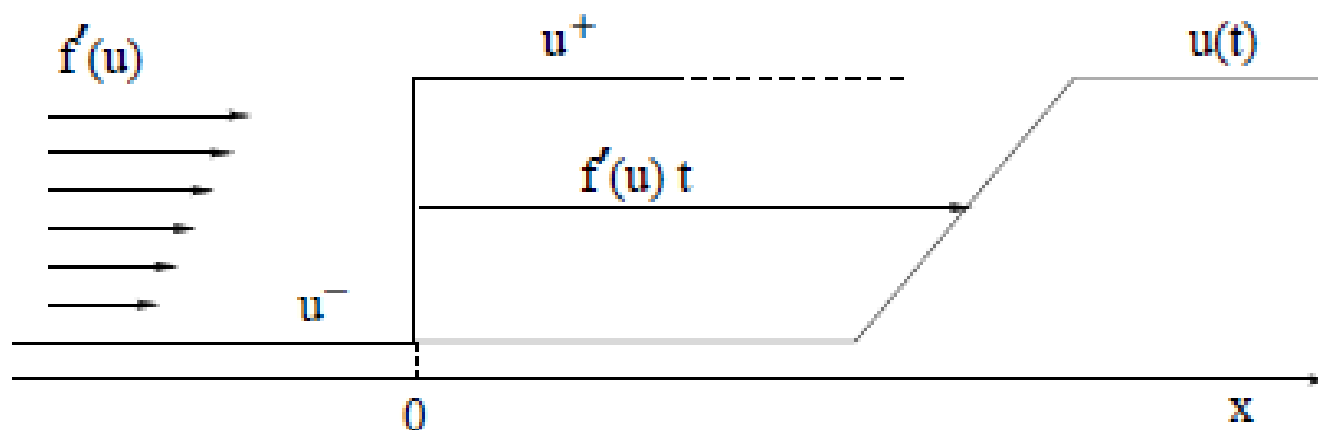
CASE 1: Linear flux: $f(u) = \lambda u$.

Jump travels with speed λ (contact discontinuity)

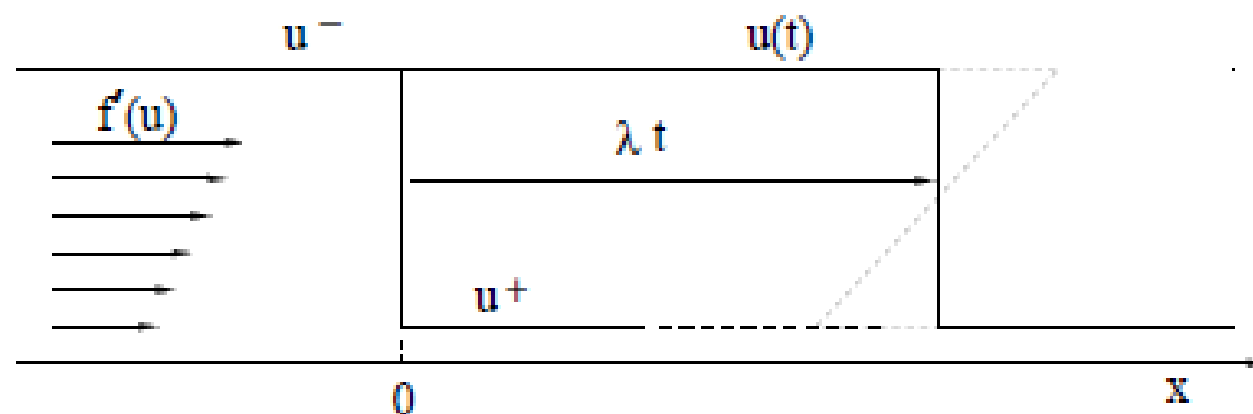


CASE 2: the flux f is convex, so that $u \mapsto f'(u)$ is increasing.

$u^+ > u^- \implies$ centered rarefaction wave



$u^+ < u^- \implies$ stable shock



$$\lambda = \frac{f(u^+) - f(u^-)}{u^+ - u^-}$$

A class of nonlinear hyperbolic systems

$$u_t + f(u)_x = 0$$

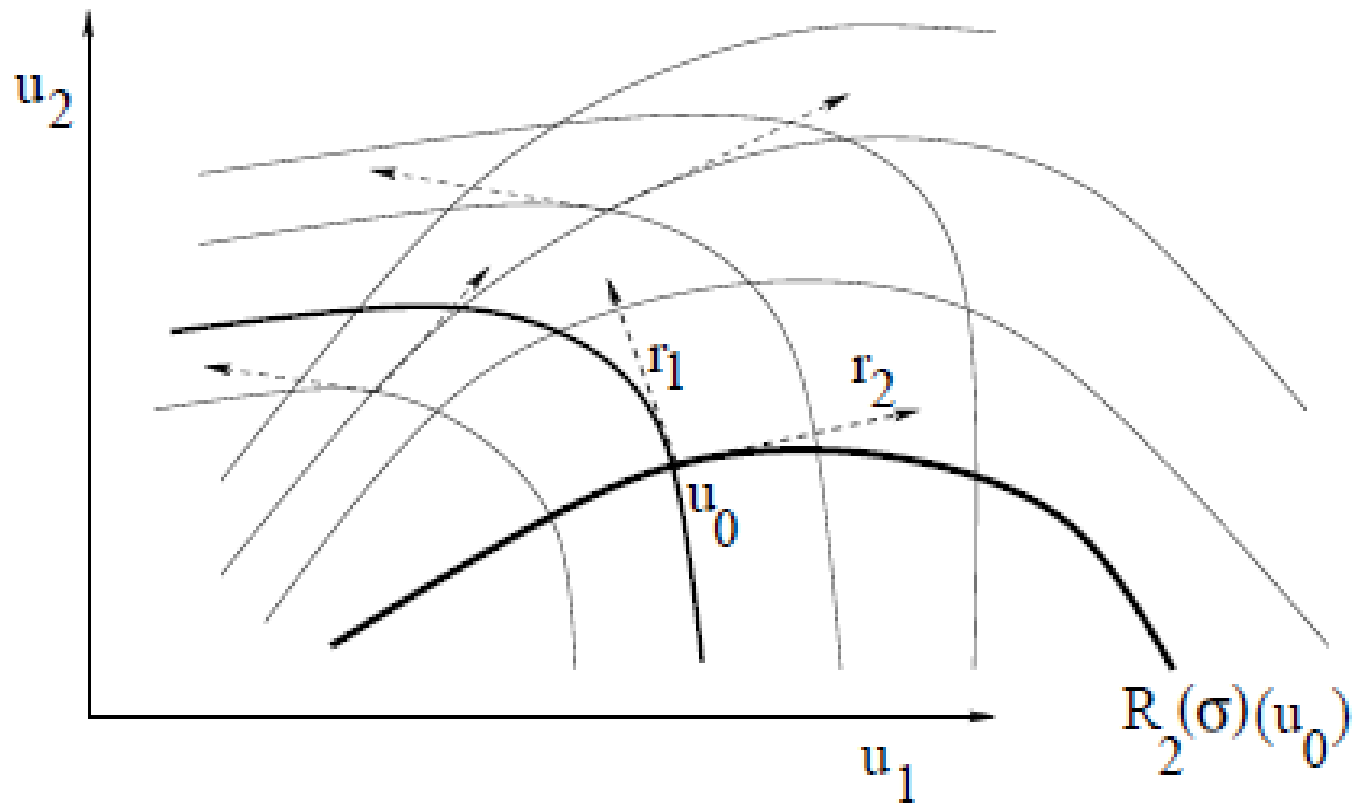
$$A(u) = Df(u) \quad A(u)r_i(u) = \lambda_i(u)r_i(u)$$

Assumption (H) (P.Lax, 1957): Each i -th characteristic field is

- either genuinely nonlinear, so that $\nabla \lambda_i \cdot r_i > 0$ for all u
- or linearly degenerate, so that $\nabla \lambda_i \cdot r_i = 0$ for all u

genuinely nonlinear \implies characteristic speed $\lambda_i(u)$ is strictly increasing along integral curves of the eigenvectors r_i

linearly degenerate \implies characteristic speed $\lambda_i(u)$ is constant along integral curves of the eigenvectors r_i



Shock and Rarefaction curves

$$u_t + f(u)_x = 0 \quad A(u) = Df(u)$$

i-rarefaction curve through u_0 : $\sigma \mapsto R_i(\sigma)(u_0)$

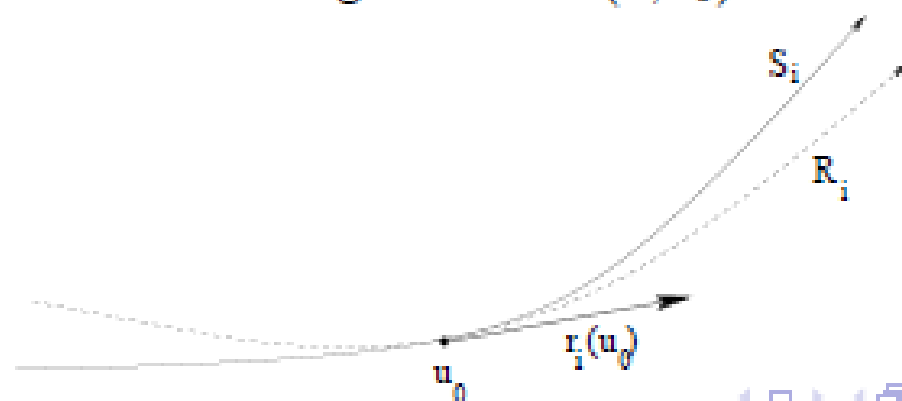
= integral curve of the field of eigenvectors r_i through u_0

$$\frac{du}{d\sigma} = r_i(u), \quad u(0) = u_0$$

i-shock curve through u_0 : $\sigma \mapsto S_i(\sigma)(u_0)$

= set of points u connected to u_0 by an i -shock, so that

$u - u_0$ is an i -eigenvector of the averaged matrix $A(u, u_0)$



Elementary waves

$$u_t + f(u)_x = 0 \quad u(0, x) = \begin{cases} u^- & \text{if } x < 0 \\ u^+ & \text{if } x > 0 \end{cases}$$

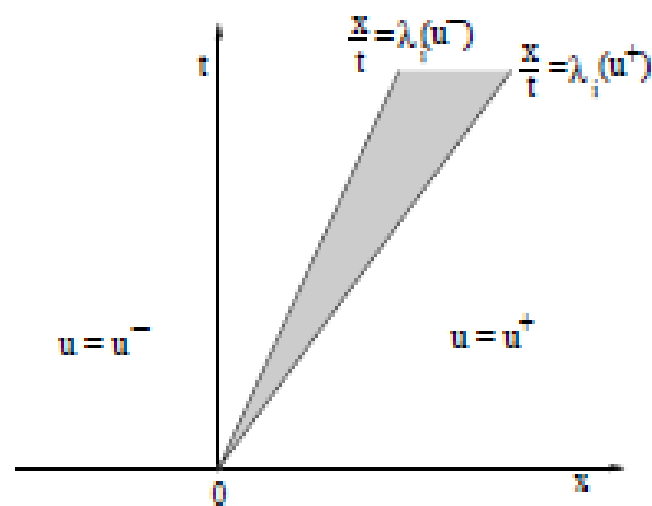
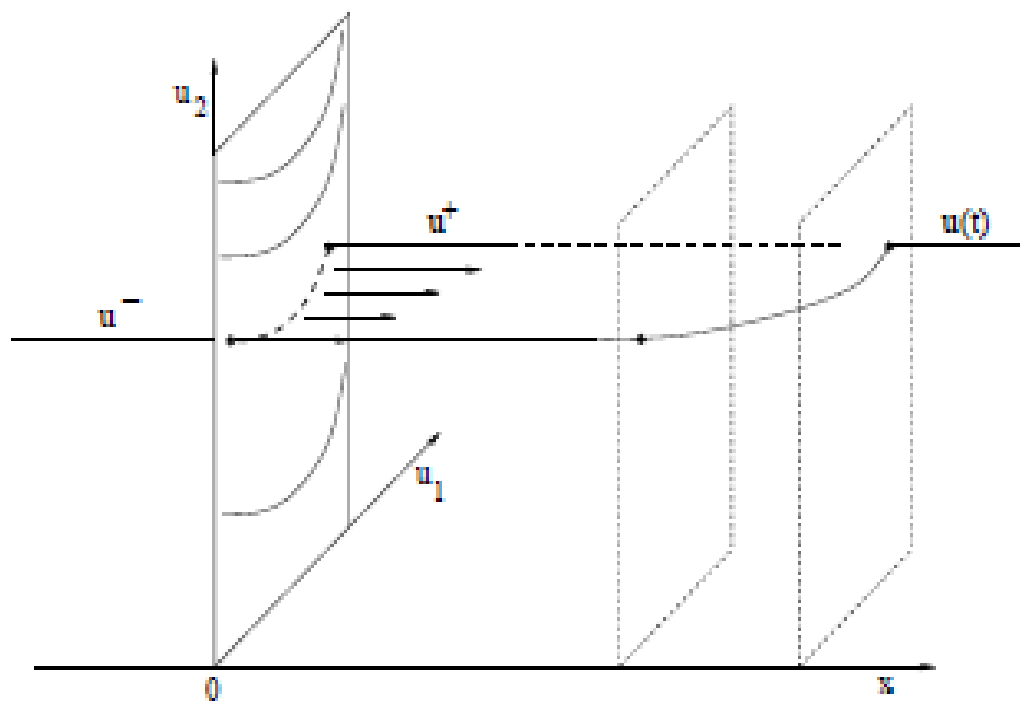
CASE 1 (Centered rarefaction wave). Let the i -th field be genuinely nonlinear.

If $u^+ = R_i(\sigma)(u^-)$ for some $\sigma > 0$, then

$$u(t, x) = \begin{cases} u^- & \text{if } x < t\lambda_i(u^-), \\ R_i(s)(u^-) & \text{if } x = t\lambda_i(s) \quad s \in [0, \sigma] \\ u^+ & \text{if } x > t\lambda_i(u^+) \end{cases}$$

is a weak solution of the Riemann problem

A centered rarefaction wave



CASE 2 (Shock or contact discontinuity). Assume that

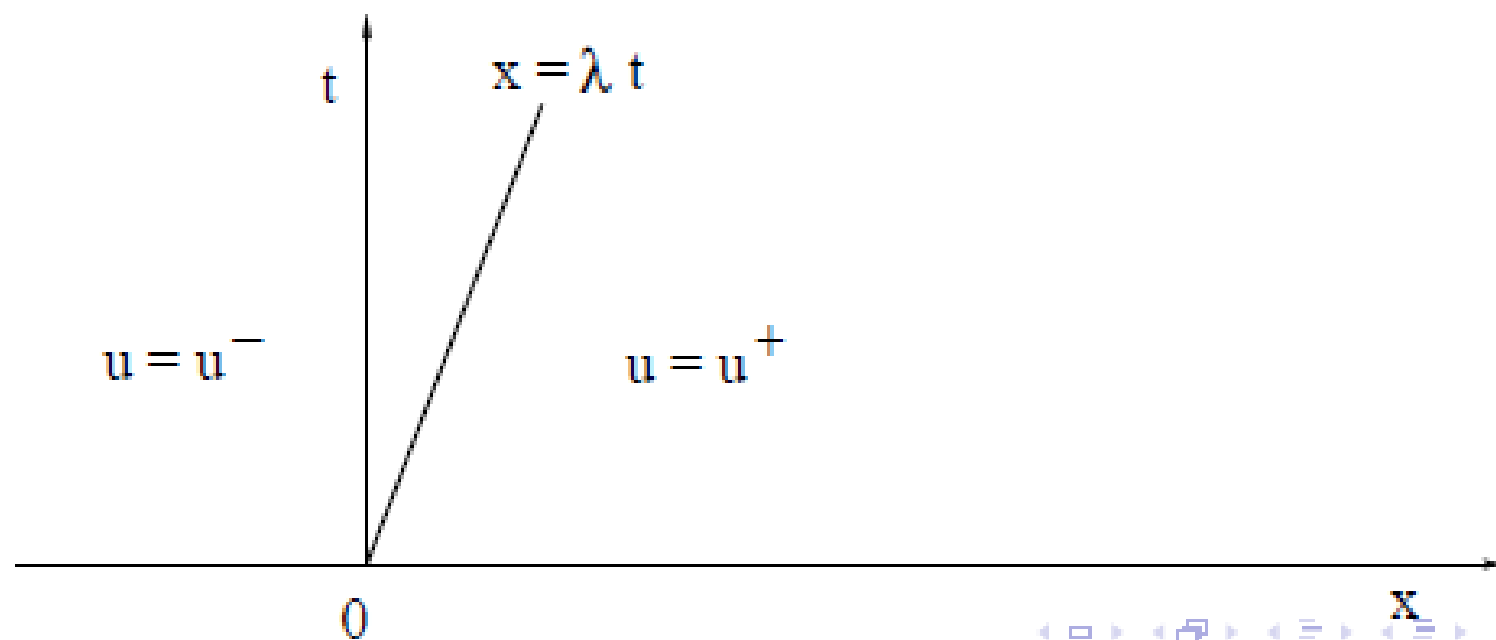
$u^+ = S_i(\sigma)(u^-)$ for some i, σ . Let $\lambda = \lambda_i(u^-, u^+)$ be the shock speed.

Then the function

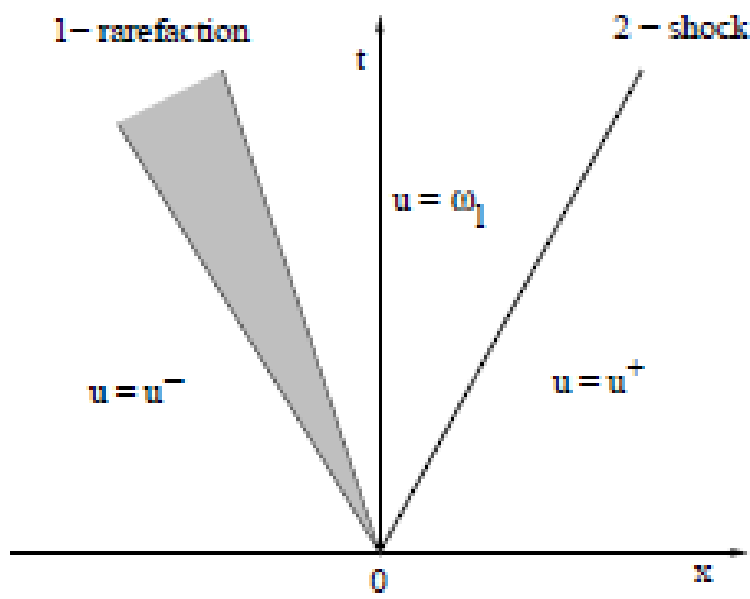
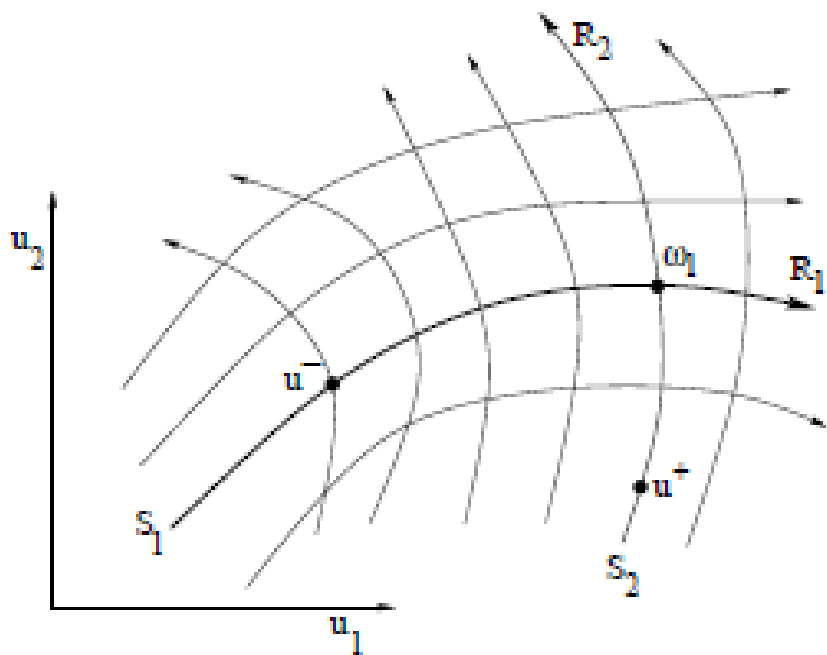
$$u(t, x) = \begin{cases} u^- & \text{if } x < \lambda t, \\ u^+ & \text{if } x > \lambda t, \end{cases}$$

is a weak solution to the Riemann problem.

In the genuinely nonlinear case, this shock is admissible (i.e., it satisfies the Lax condition and the Liu condition) iff $\sigma < 0$.



Solution to a 2 x 2 Riemann problem



Solution of the general Riemann problem (P. Lax, 1957)

$$u_t + f(u)_x = 0 \quad u(0, x) = \begin{cases} u^- & \text{if } x < 0 \\ u^+ & \text{if } x > 0 \end{cases}$$

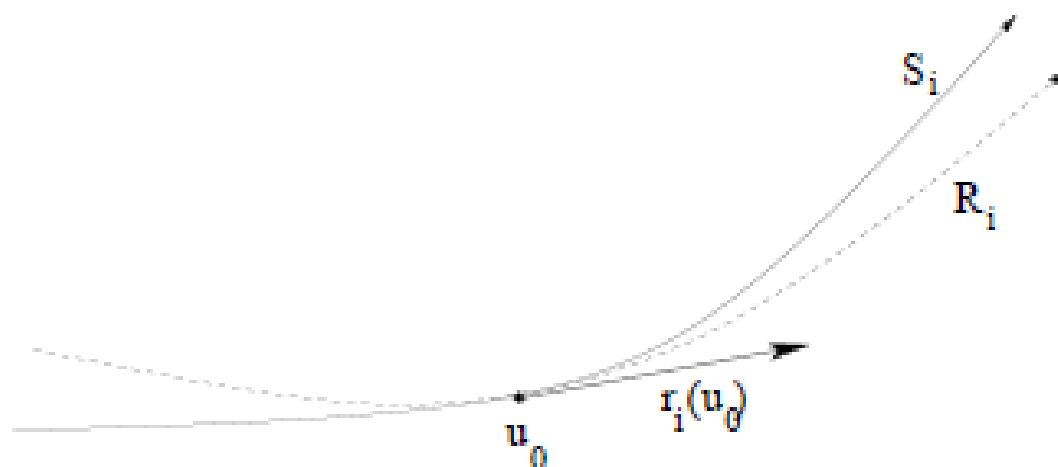
Problem: Find states $\omega_0, \omega_1, \dots, \omega_m$ such that

$$\omega_0 = \mathbf{u}^- \quad \omega_m = \mathbf{u}^+$$

and every couple ω_{i-1}, ω_i are connected by an elementary wave (shock or rarefaction)

$$\begin{cases} \text{either } \omega_i = R_i(\sigma_i)(\omega_{i-1}) & \sigma_i \geq 0 \\ \text{or } \omega_i = S_i(\sigma_i)(\omega_{i-1}) & \sigma_i < 0 \end{cases}$$

define: $\Psi_i(\sigma)(u) = \begin{cases} R_i(\sigma)(u) & \text{if } \sigma \geq 0 \\ S_i(\sigma)(u) & \text{if } \sigma < 0 \end{cases}$



$$(\sigma_1, \sigma_2, \dots, \sigma_n) \mapsto \Psi_n(\sigma_n) \circ \dots \circ \Psi_2(\sigma_2) \circ \Psi_1(\sigma_1)(u^-)$$

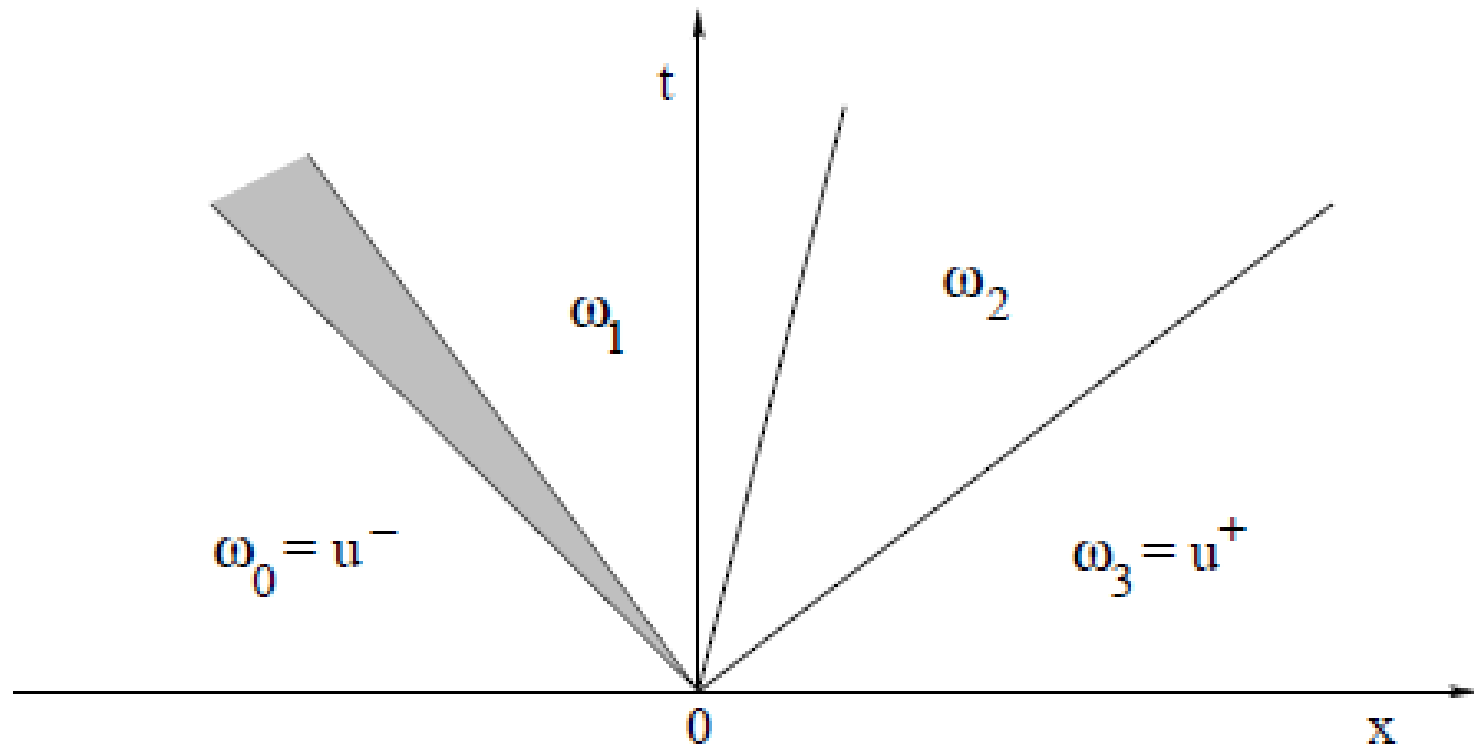
Jacobian matrix at the origin: $J \doteq \left(\begin{array}{c|c|c|c} r_1(u^-) & r_2(u^-) & \cdots & r_n(u^-) \end{array} \right)$

always has full rank

If $|u^+ - u^-|$ is small, then the implicit function theorem yields existence and uniqueness of the intermediate states $\omega_0, \omega_1, \dots, \omega_n$

General solution of the Riemann problem

Concatenation of elementary waves (shocks, rarefactions, or contact discontinuities)



Global solution to the Cauchy problem

$$u_t + f(u)_x = 0, \quad u(0, x) = \bar{u}(x)$$

Theorem (Glimm, 1965).

Assume:

- *system is strictly hyperbolic*
- *each characteristic field is either linearly degenerate or genuinely nonlinear*

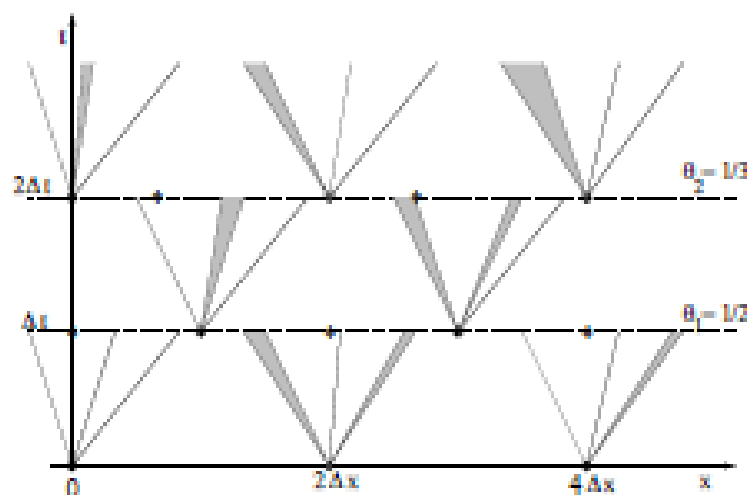
Then there exists a constant $\delta > 0$ such that, for every initial condition $\bar{u} \in L^1(\mathbb{R}; \mathbb{R}^n)$ with

$$\text{Tot. Var.}(\bar{u}) \leq \delta,$$

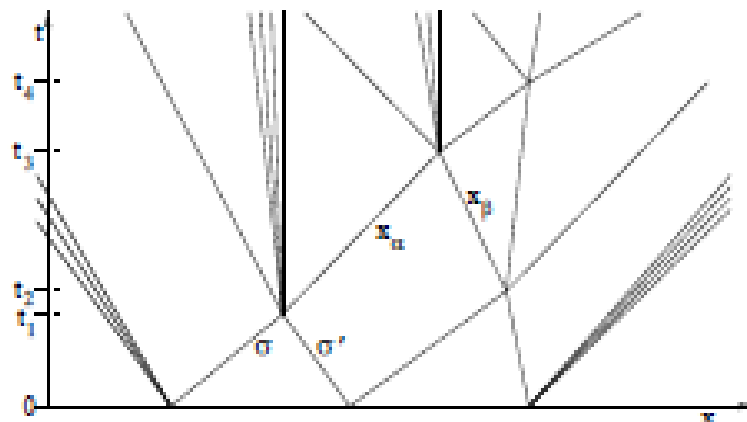
the Cauchy problem has an entropy admissible weak solution $u = u(t, x)$ defined for all $t \geq 0$.

Construction of a sequence of approximate solutions by piecing together solutions of Riemann problems

- on a fixed grid in t - x plane (Glimm scheme)

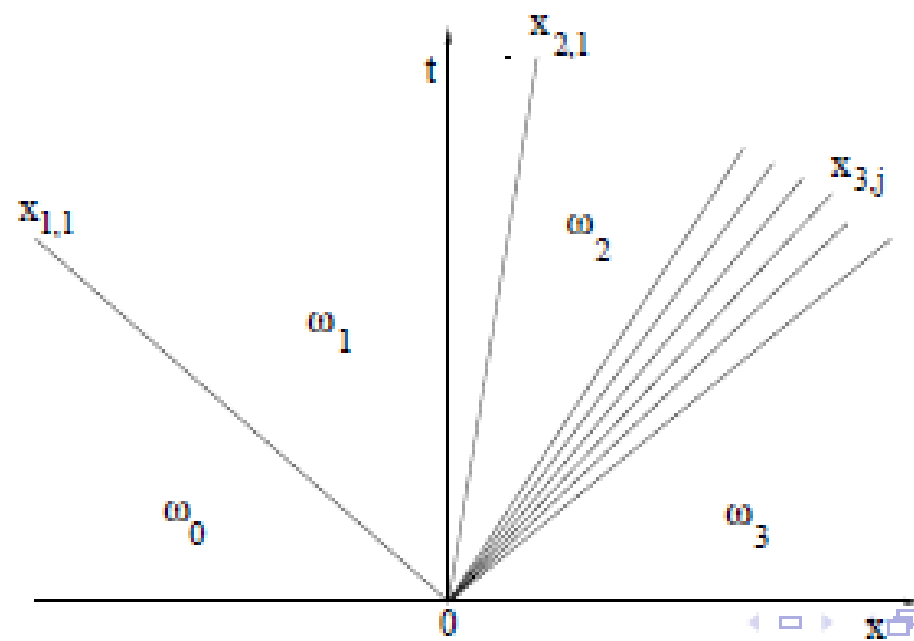
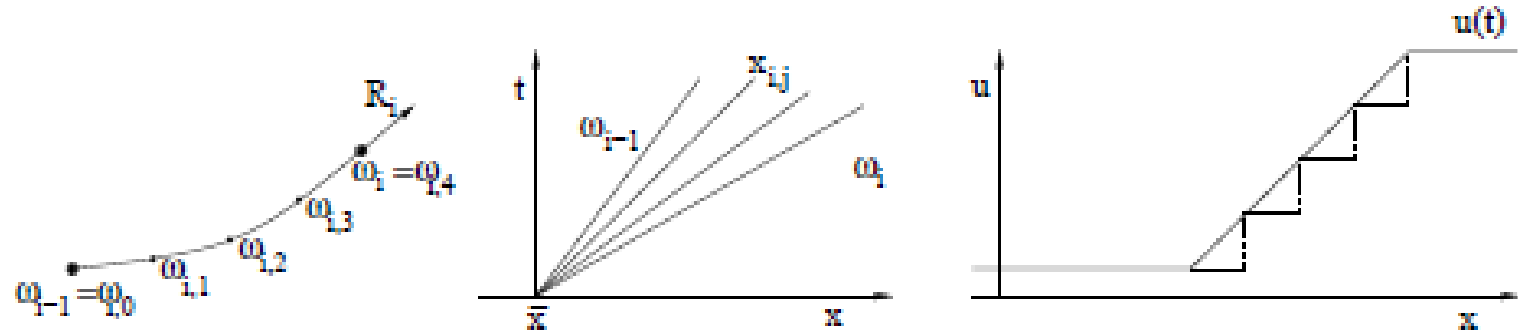


- at points where fronts interact (front tracking)

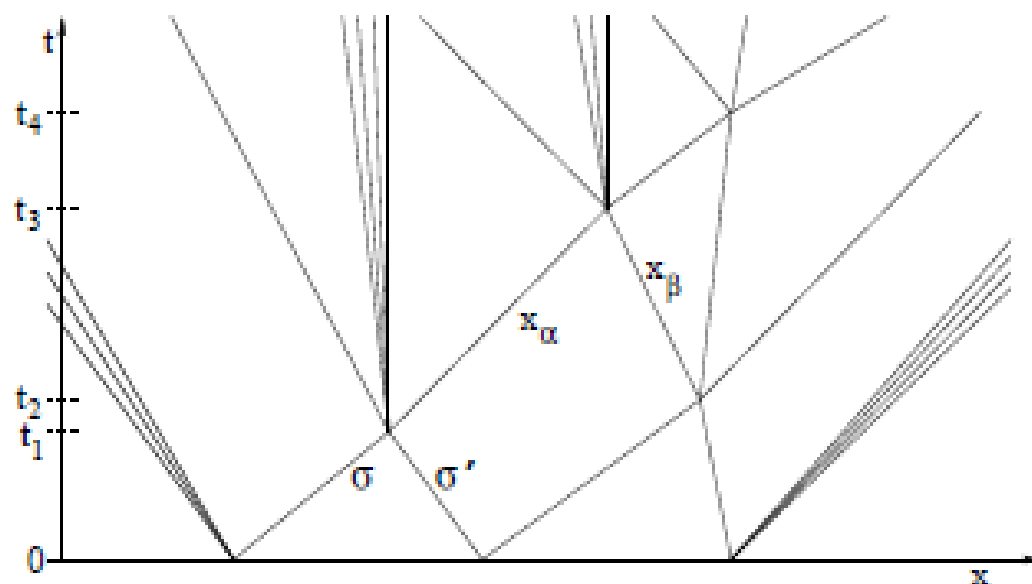


Piecewise constant approximate solution to a Riemann problem

replace centered rarefaction waves with piecewise constant rarefaction fans



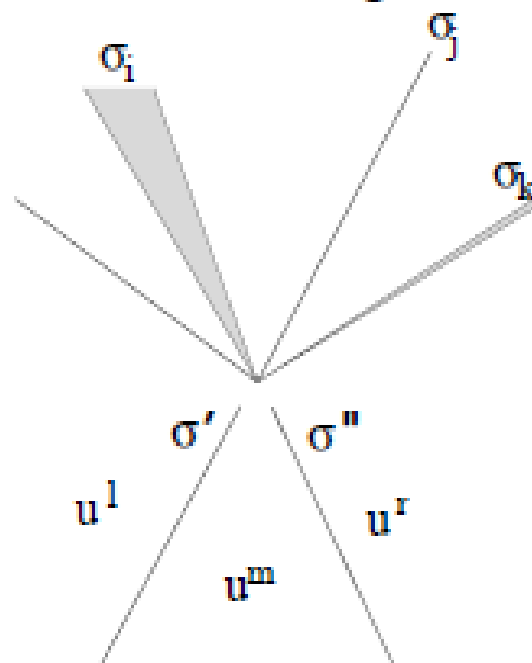
Front Tracking Approximations



- Approximate the initial data \bar{u} with a piecewise constant function
- Construct a piecewise constant approximate solution to each Riemann problem at $t = 0$
- at each time t_j where two fronts interact, construct a piecewise constant approximate solution to the new Riemann problem ...
- NEED TO CHECK: $\left\{ \begin{array}{l} - \text{total variation remains small} \\ - \text{number of wave fronts remains finite} \end{array} \right.$

Interaction estimates

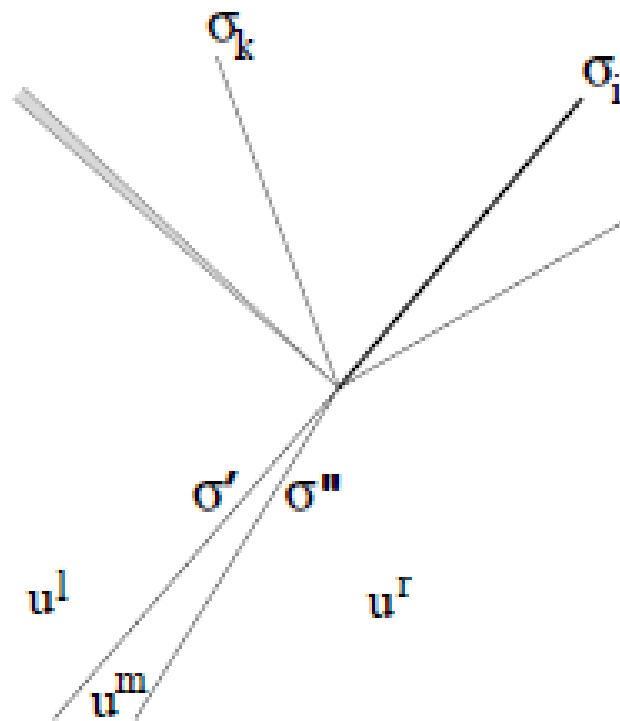
GOAL: estimate the strengths of the waves in the solution of a Riemann problem, depending on the strengths of the two interacting waves σ', σ''



Incoming: a j -wave of strength σ' and an i -wave of strength σ''

Outgoing: waves of strengths $\sigma_1, \dots, \sigma_m$. Then

$$|\sigma_i - \sigma''| + |\sigma_j - \sigma'| + \sum_{k \neq i, j} |\sigma_k| = O(1) \cdot |\sigma' \sigma''|$$



Incoming: two i -waves of strengths σ' and σ''

Outgoing: waves of strengths $\sigma_1, \dots, \sigma_m$. Then

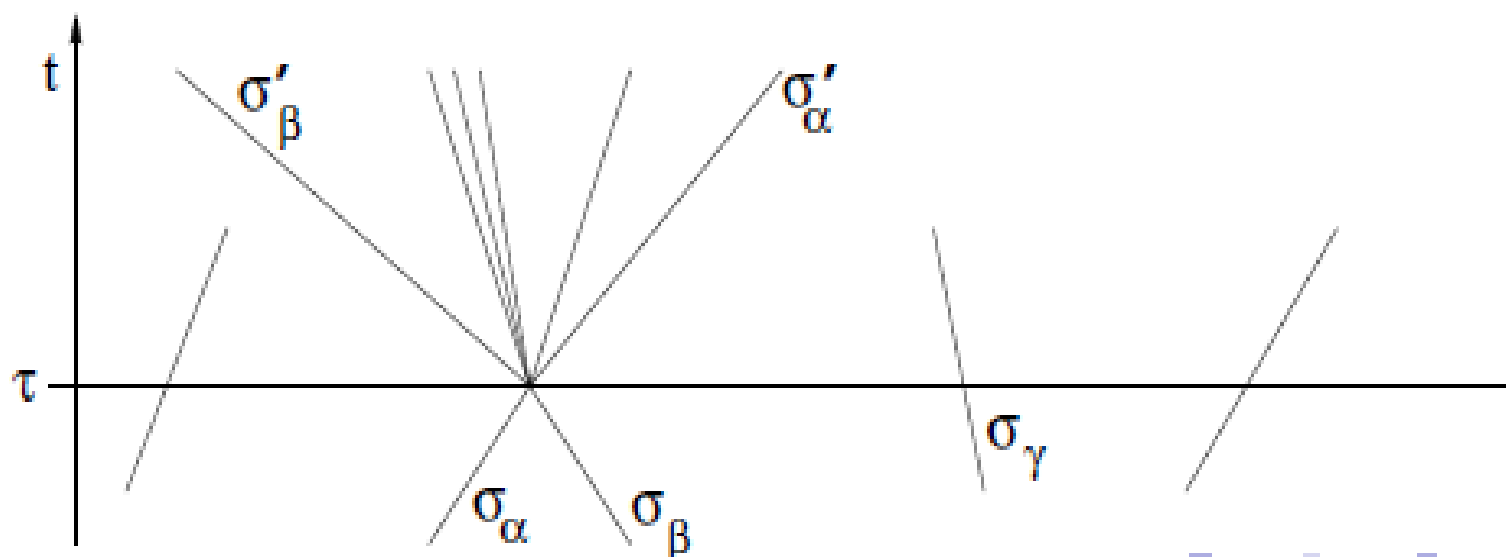
$$|\sigma_i - \sigma' - \sigma''| + \sum_{k \neq i} |\sigma_k| = \mathcal{O}(1) \cdot |\sigma' \sigma''| (|\sigma'| + |\sigma''|)$$

Glimm functionals

Total strength of waves: $V(t) \doteq \sum_{\alpha} |\sigma_{\alpha}|$

Wave interaction potential: $Q(t) \doteq \sum_{(\alpha, \beta) \in \mathcal{A}} |\sigma_{\alpha} \sigma_{\beta}|$

$\mathcal{A} \doteq$ couples of *approaching* wave fronts



Changes in V, Q at time τ when the fronts $\sigma_\alpha, \sigma_\beta$ interact:

$$\Delta V(\tau) = \mathcal{O}(1)|\sigma_\alpha \sigma_\beta|$$

$$\Delta Q(\tau) = -|\sigma_\alpha \sigma_\beta| + \mathcal{O}(1) \cdot V(\tau-) |\sigma_\alpha \sigma_\beta|$$

Choosing a constant C_0 large enough, the map

$$t \mapsto V(t) + C_0 Q(t)$$

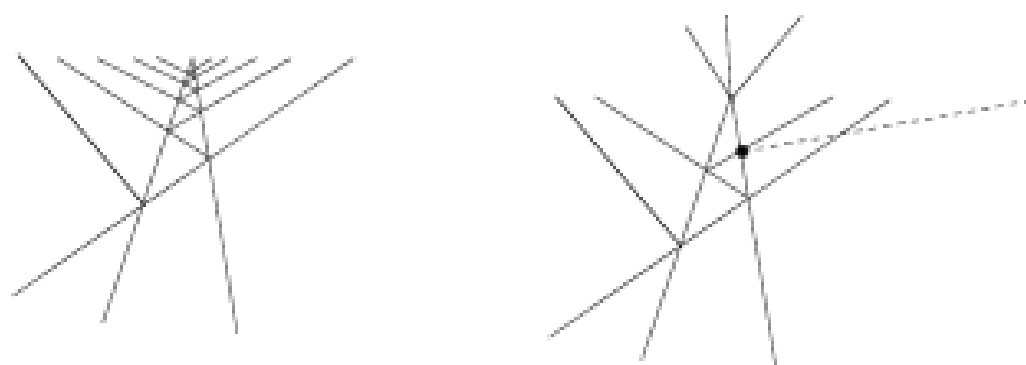
is nonincreasing, as long as V remains small

Total variation initially small \implies global BV bounds

$$\text{Tot.Var.}\{u(t, \cdot)\} \leq V(t) \leq V(0) + C_0 Q(0)$$

Front tracking approximations can be constructed for all $t \geq 0$

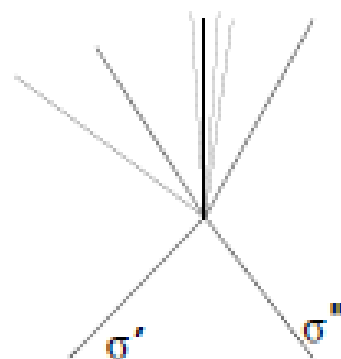
Keeping finite the number of wave fronts



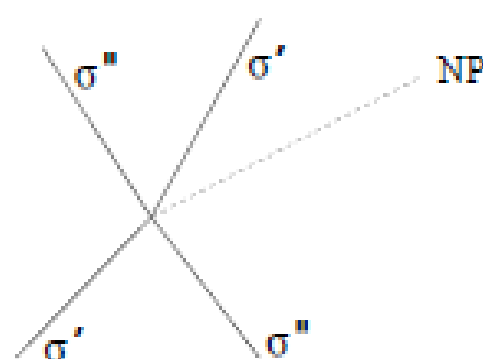
At each interaction point, the **Accurate Riemann Solver** yields a solution, possibly introducing several new fronts

The total number of fronts can become infinite in finite time

accurate Riemann solver



simplified Riemann solver



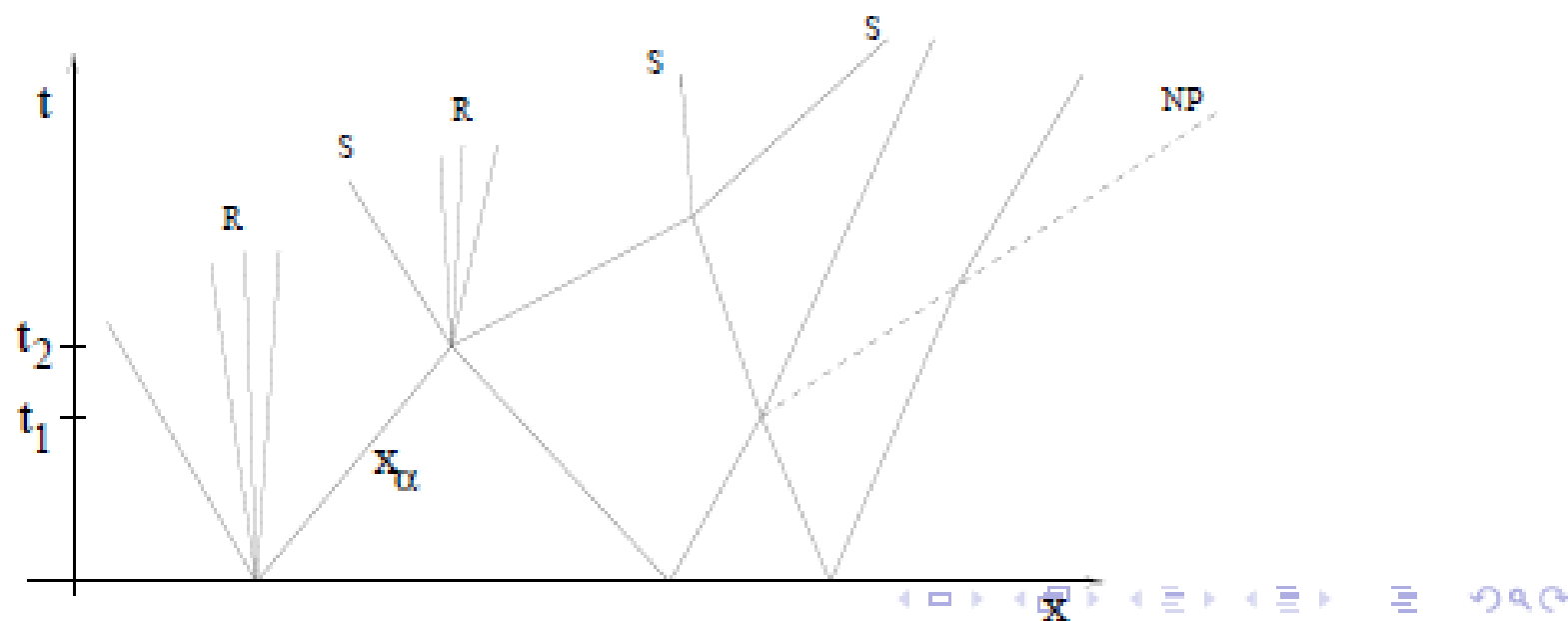
Need: a **Simplified Riemann Solver**, producing only one *"non-physical"* front

A sequence of approximate solutions

$$u_t + f(u)_x = 0 \quad u(0, x) = \bar{u}(x)$$

$(u_\nu)_{\nu \geq 1}$ sequence of approximate front tracking solutions

- initial data satisfy $\|u_\nu(0, \cdot) - \bar{u}\|_{L^1} \leq \varepsilon_\nu \rightarrow 0$
- all shock fronts in u_ν are entropy-admissible
- each rarefaction front in u_ν has strength $\leq \varepsilon_\nu$
- at each time $t \geq 0$, the total strength of all non-physical fronts in $u_\nu(t, \cdot)$ is $\leq \varepsilon_\nu$



Existence of a convergent subsequence

$$\text{Tot.Var.}\{u_\nu(t, \cdot)\} \leq C$$

$$\begin{aligned} \|u_\nu(t) - u_\nu(s)\|_{L^1} &\leq (t - s) \cdot [\text{total strength of all wave fronts}] \cdot [\text{maximum speed}] \\ &\leq L \cdot (t - s) \end{aligned}$$

Helly's compactness theorem \implies a subsequence converges

$$u_\nu \rightarrow u \quad \text{in } \mathbf{L}_{loc}^1$$

Claim: $u = \lim_{\nu \rightarrow \infty} u_\nu$ is a weak solution

$$\iint \left\{ \phi_t u + \phi_x f(u) \right\} dx dt = 0 \quad \phi \in \mathcal{C}_c^1\left(]0, \infty[\times \mathbb{R}\right)$$

Need to show:

$$\lim_{\nu \rightarrow \infty} \iint \left\{ \phi_t u_\nu + \phi_x f(u_\nu) \right\} dx dt = 0$$

$$\int_0^\infty \int_{-\infty}^\infty \left\{ \phi_t(t, x) u_\nu(t, x) + \phi_x(t, x) f(u_\nu(t, x)) \right\} dx dt$$

$$= \sum_j \int_{\partial \Gamma_j} \Phi_\nu \cdot \mathbf{n} \, d\sigma$$

$$\limsup_{\nu \rightarrow \infty} \left| \sum_j \int_{\partial \Gamma_j} \Phi_\nu \cdot \mathbf{n} \, d\sigma \right|$$

$$\leq \limsup_{\nu \rightarrow \infty} \left| \sum_{\alpha \in S \cup \mathcal{R} \cup \mathcal{N} \cup P} \left[\dot{x}_\alpha(t) \cdot \Delta u_\nu(t, x_\alpha) - \Delta f(u_\nu(t, x_\alpha)) \right] \phi(t, x_\alpha(t)) \right|$$

$$\leq \left(\max_{t, x} |\phi(t, x)| \right) \cdot \limsup_{\nu \rightarrow \infty} \left\{ \mathcal{O}(1) \cdot \sum_{\alpha \in \mathcal{R}} \varepsilon_\nu |\sigma_\alpha| + \mathcal{O}(1) \cdot \sum_{\alpha \in \mathcal{N} \cup P} |\sigma_\alpha| \right\}$$

$$= 0$$

The Glimm scheme

$$u_t + f(u)_x = 0 \quad u(0, x) = \bar{u}(x)$$

Assume: all characteristic speeds satisfy $\lambda_i(u) \in [0, 1]$

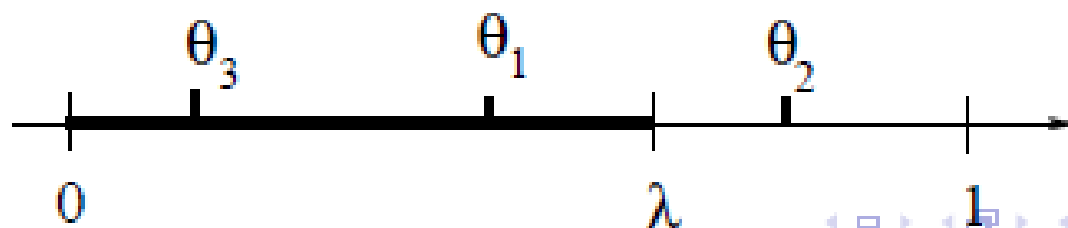
This is not restrictive. If $\lambda_i(u) \in [-M, M]$, simply change coordinates:

$$y = x + Mt, \quad \tau = 2Mt$$

Choose:

- a grid in the t - x plane with step size $\Delta t = \Delta x$
- a sequence of numbers $\theta_1, \theta_2, \theta_3, \dots$ uniformly distributed over $[0, 1]$

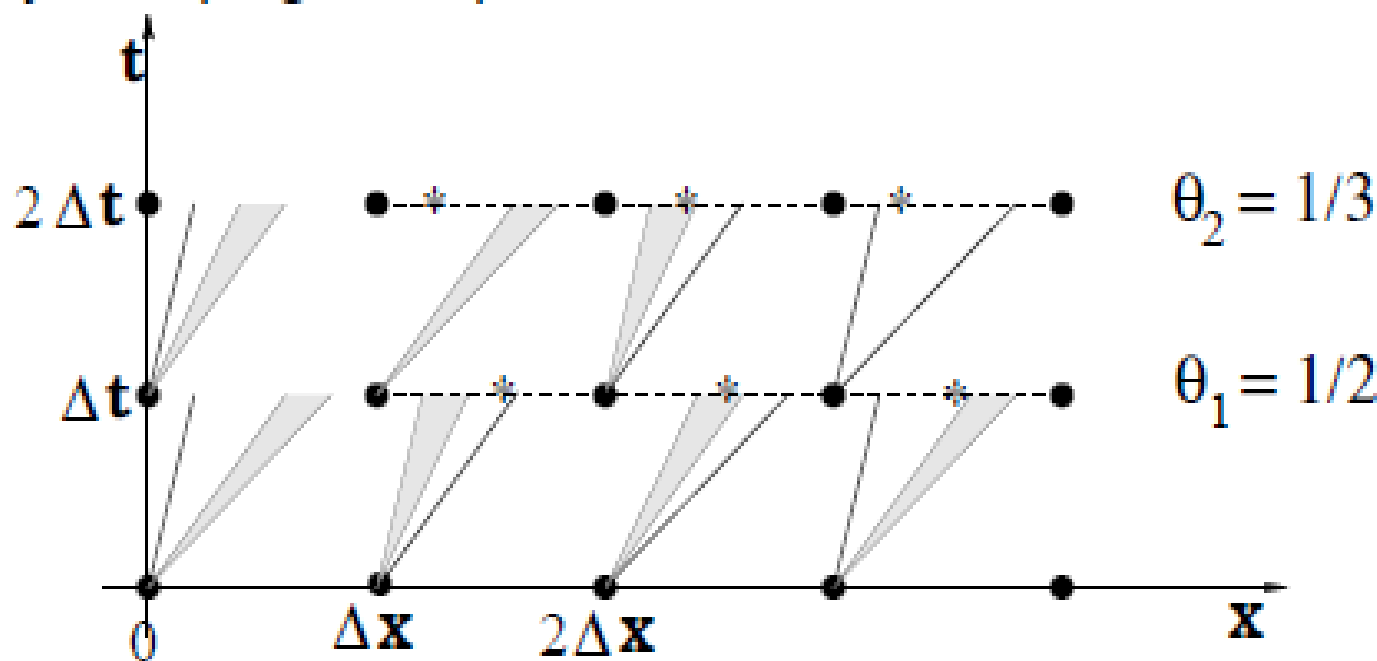
$$\lim_{N \rightarrow \infty} \frac{\#\{j : 1 \leq j \leq N, \theta_j \in [0, \lambda]\}}{N} = \lambda \quad \text{for each } \lambda \in [0, 1].$$



Glimm approximations

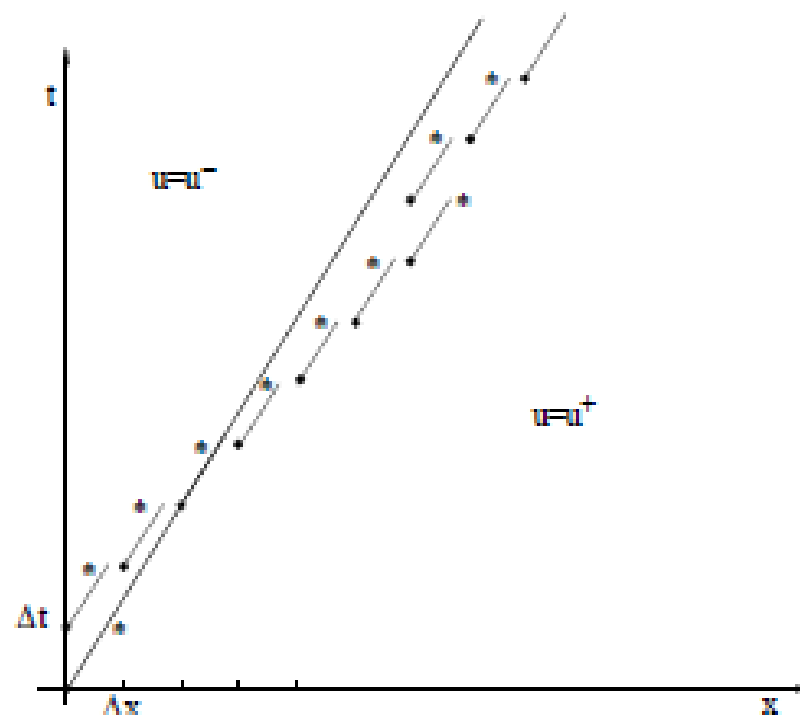
Grid points : $x_j = j \cdot \Delta x$, $t_k = k \cdot \Delta t$

- for each $k \geq 0$, $u(t_k, \cdot)$ is piecewise constant, with jumps at the points x_j . The Riemann problems are solved exactly, for $t_k \leq t < t_{k+1}$
- at time t_{k+1} the solution is again approximated by a piecewise constant function, by a sampling technique



Example: Glimm's scheme applied to a solution containing a single shock

$$U(t, x) = \begin{cases} u^+ & \text{if } x > \lambda t \\ u^- & \text{if } x < \lambda t \end{cases}$$



Fix $T > 0$, take $\Delta x = \Delta t = T/N$

$$x(T) = \#\{j : 1 \leq j \leq N, \theta_j \in [0, \lambda]\} \cdot \Delta t$$

$$= \frac{\#\{j : 1 \leq j \leq N, \theta_j \in [0, \lambda]\}}{N} \cdot T \rightarrow \lambda T \quad \text{as } N \rightarrow \infty$$

Rate of convergence for Glimm approximations

Random sampling at points determined by the equidistributed sequence $(\theta_k)_{k \geq 1}$

$$\lim_{N \rightarrow \infty} \frac{\#\{j ; 1 \leq j \leq N, \theta_j \in [0, \lambda]\}}{N} = \lambda \quad \text{for each } \lambda \in [0, 1].$$

Need fast convergence to uniform distribution. Achieved by choosing:

$$\theta_1 = 0.1, \quad \dots, \quad \theta_{759} = 0.957, \quad \dots, \quad \theta_{39022} = 0.22093, \quad \dots$$

Convergence rate:
$$\lim_{\Delta x \rightarrow 0} \frac{\|u^{\text{Glimm}}(T, \cdot) - u^{\text{exact}}(T, \cdot)\|_{L^1}}{\sqrt{\Delta x} \cdot |\ln \Delta x|} = 0$$

(A. Bressan & A. Marson, 1998)



**Bressan, A.: Hyperbolic Systems of Conservation Laws.
The One-Dimensional Cauchy Problem.
Oxford University Press: Oxford, 2000.**