PDE-CDT Core Course

Analysis of Partial Differential Equations-Part III

Lecture 4

EPSRC Centre for Doctoral Training in
Partial Differential Equations
Trinity Term
25 April – 15 June 2018 (16 hours; Wednesdays)
Final Exam: 22 June 2018 (Friday)
Course format: Teaching Course (TT)

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Hyperbolic Systems of Conservation Laws

$$\begin{cases} \frac{\partial}{\partial t} u_1 + \frac{\partial}{\partial x} f_1(u_1, \cdots, u_m) = 0, \\ \qquad\qquad \cdots\cdots \\ \frac{\partial}{\partial t} u_m + \frac{\partial}{\partial x} f_m(u_1, \cdots, u_m) = 0, \end{cases}$$

$$\mathbf{u}_t + \mathbf{f}(\mathbf{u})_x = 0$$

$\mathbf{u} = (u_1, \cdots, u_m)^\top \in \mathbb{R}^m$ conserved quantities

$\mathbf{f}(\mathbf{u}) = (f_1(\mathbf{u}), \cdots, f_m(\mathbf{u}))^\top$ fluxes

Hyperbolic Systems

$$\mathbf{u}_t + \mathbf{f}(\mathbf{u})_x = 0 \qquad \mathbf{u} = \mathbf{u}(t,x) \in \mathbb{R}^m$$

$$\mathbf{u}_t + \mathbf{A}(\mathbf{u})\mathbf{u}_x = 0 \qquad \mathbf{A}(\mathbf{u}) = \nabla \mathbf{f}(\mathbf{u})$$

The system is **strictly hyperbolic** if each $m \times m$ matrix $\mathbf{A}(\mathbf{u})$ has real distinct eigenvalues

$$\lambda_1(\mathbf{u}) < \lambda_2(\mathbf{u}) < \cdots < \lambda_m(\mathbf{u})$$

Right eigenvectors $\mathbf{r}_1(\mathbf{u}), \cdots, \mathbf{r}_m(\mathbf{u})$ (column vectors)
Left eigenvectors $\mathbf{l}_1(\mathbf{u}), \cdots, \mathbf{l}_m(\mathbf{u})$ (row vectors)

$$\mathbf{A}\mathbf{r}_i = \lambda_i \mathbf{r}_i \qquad \mathbf{l}_i\mathbf{A} = \lambda_i \mathbf{l}_i$$

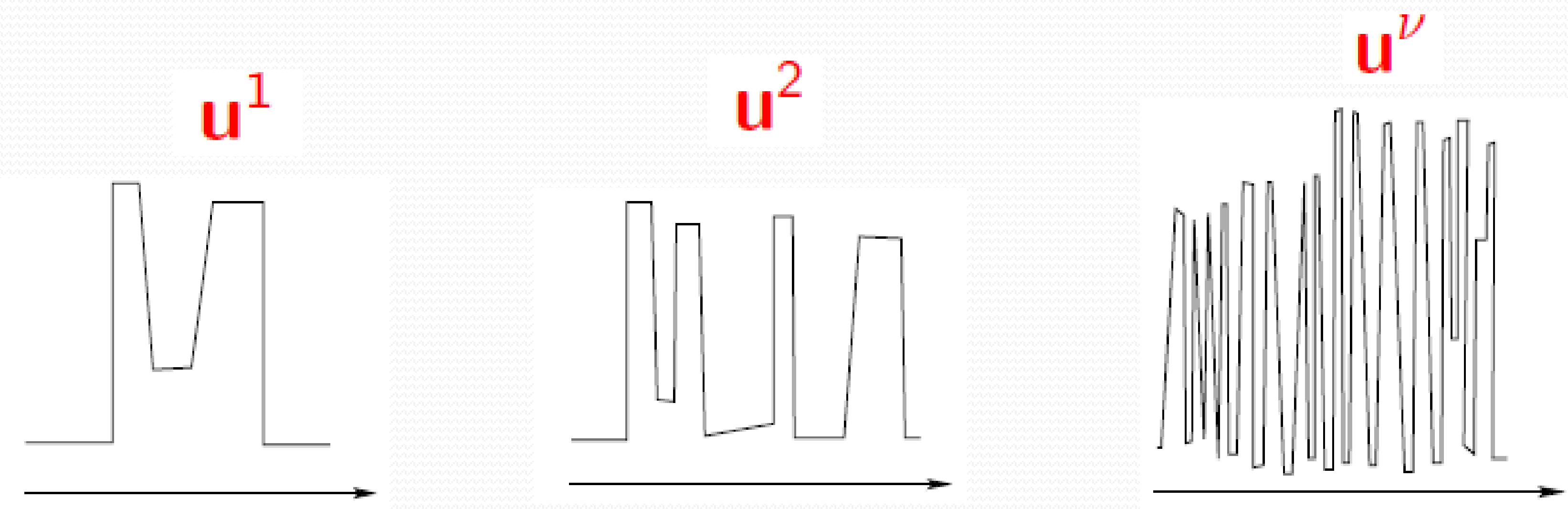
Choose the bases so that

$$\mathbf{l}_i \cdot \mathbf{r}_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Global in Time Solutions to the Cauchy Problem

$$\mathbf{u}_t + \mathbf{f}(\mathbf{u})_x = 0, \qquad \mathbf{u}(0,x) = \mathbf{u}(x)$$

- Construct a sequence of approximate solutions $\{\mathbf{u}^\nu\}_{\nu \geq 1}$
- Show that (a subsequence) converges: $\mathbf{u}^\nu \to \mathbf{u}$ in L^1_{loc}
- Show that the limit $\mathbf{u}$ is an entropy solution.



Need: a-priori bound on the total variation (J. Glimm, 1965)

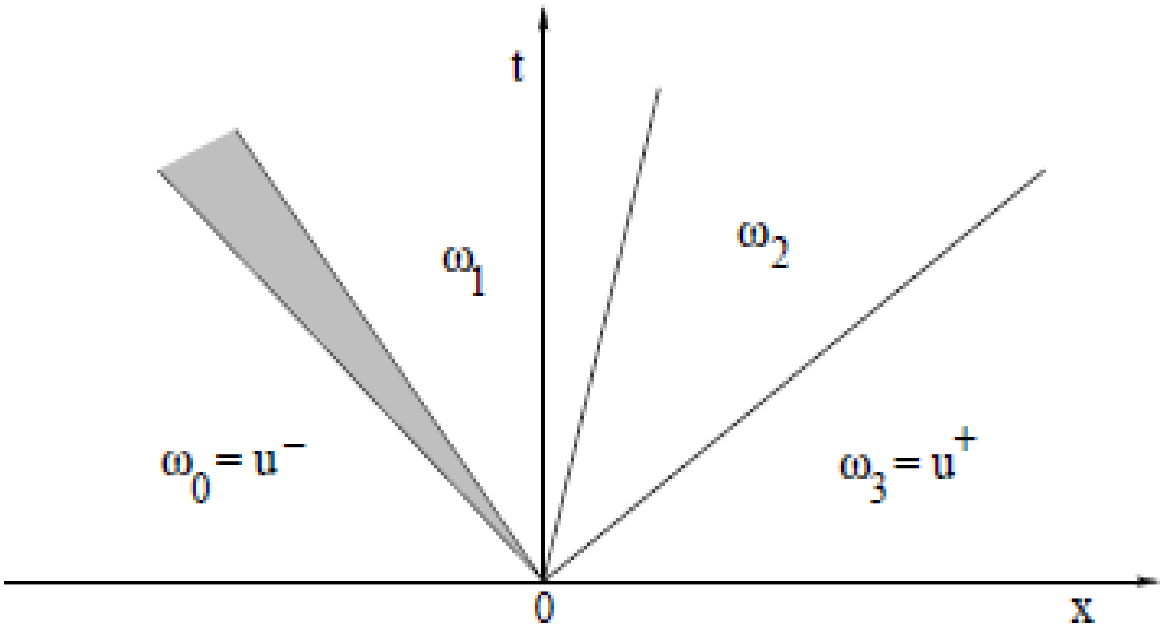
Building Block: The Riemann Problem

$$\mathbf{u}_t + \mathbf{f}(\mathbf{u})_x = 0, \qquad \mathbf{u}(0,x) = \begin{cases} \mathbf{u}^- & x < 0 \\ \mathbf{u}^+ & x > 0 \end{cases}$$

- **B. Riemann 1860:** 2×2 **Isentropic Euler equations**
- **P. Lax 1957:** $m \times m$ **systems (+ special assumptions)**
- **T.-P. Liu 1975:** $m \times m$ **systems (generic case)**

***The Riemann solutions are also the vanishing viscosity limits for general hyperbolic systems, possibly non-conservative**

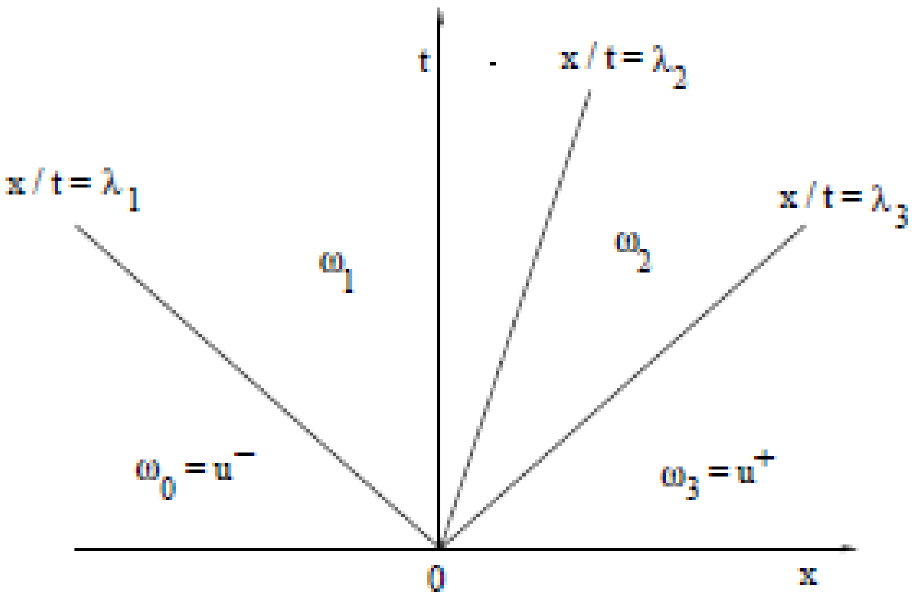
Solution to the Riemann problem



- is invariant w.r.t. rescaling symmetry: $u^\theta(t,x) \doteq u(\theta t, \theta x) \qquad \theta > 0$
- describes local behavior of BV solutions near each point (t_0, x_0)
- describes large-time asymptotics as $t \to +\infty$ (for small total variation)

Riemann Problem for Linear Systems

$$u_t + Au_x = 0 \qquad u(0,x) = \begin{cases} u^- & \text{if} \quad x < 0 \\ u^+ & \text{if} \quad x > 0 \end{cases}$$



$$u^+ - u^- = \sum_{j=1}^{n} c_j r_j \qquad \text{(sum of eigenvectors of } A\text{)}$$

intermediate states : $\omega_i \doteq u^- + \sum_{j \le i} c_j r_j$

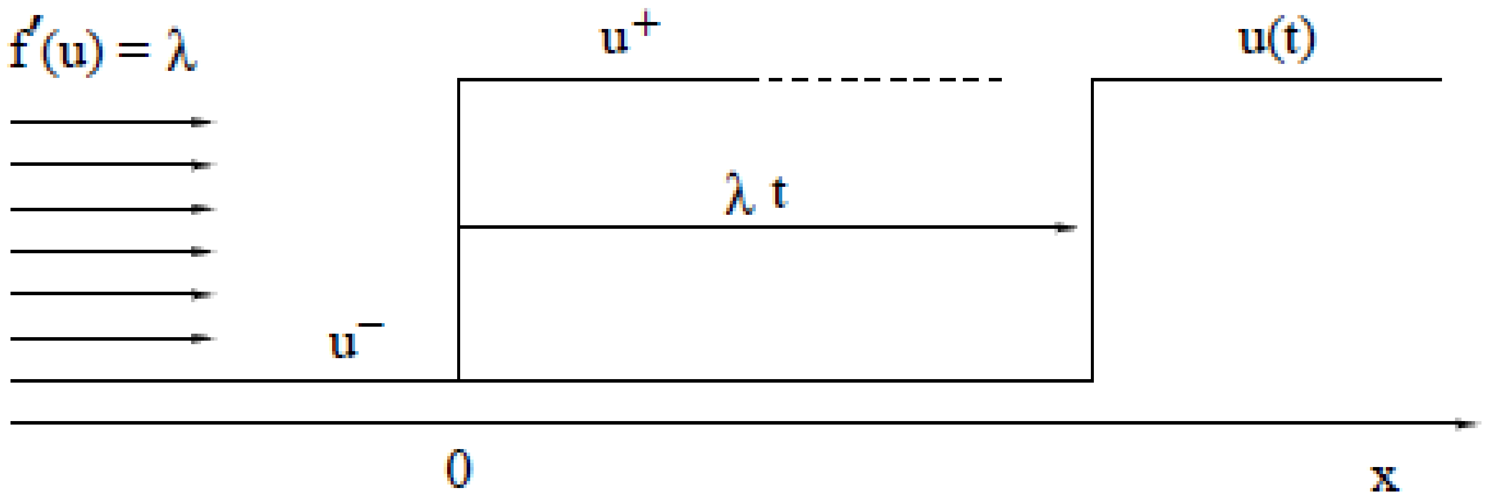
i-th jump: $\omega_i - \omega_{i-1} = c_i r_i$ travels with speed λ_i

Scalar Conservation Law

$$u_t + f(u)_x = 0 \qquad u \in \mathbb{R}$$

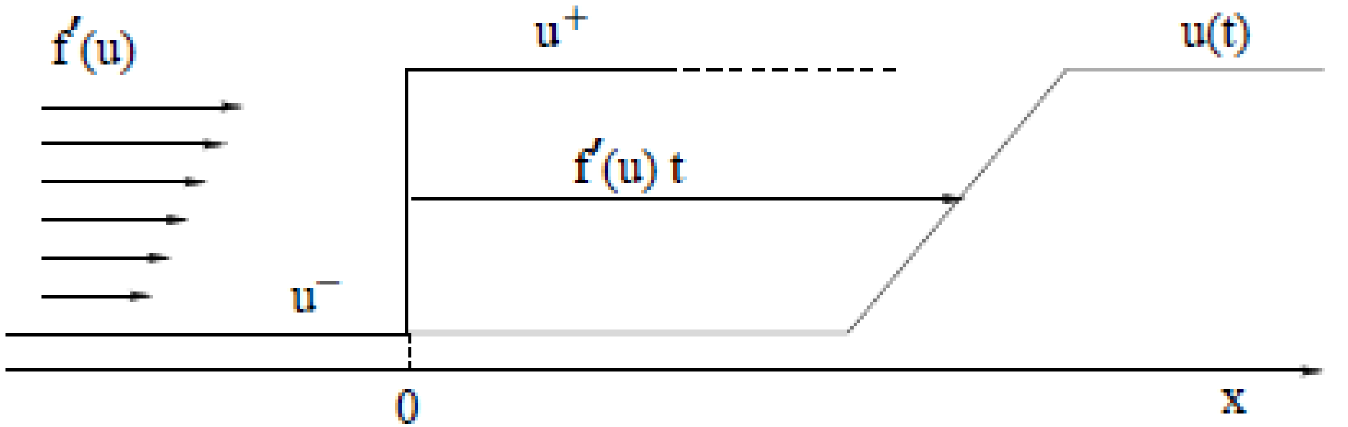
CASE 1: Linear flux: $f(u) = \lambda u$.

Jump travels with speed λ (contact discontinuity)

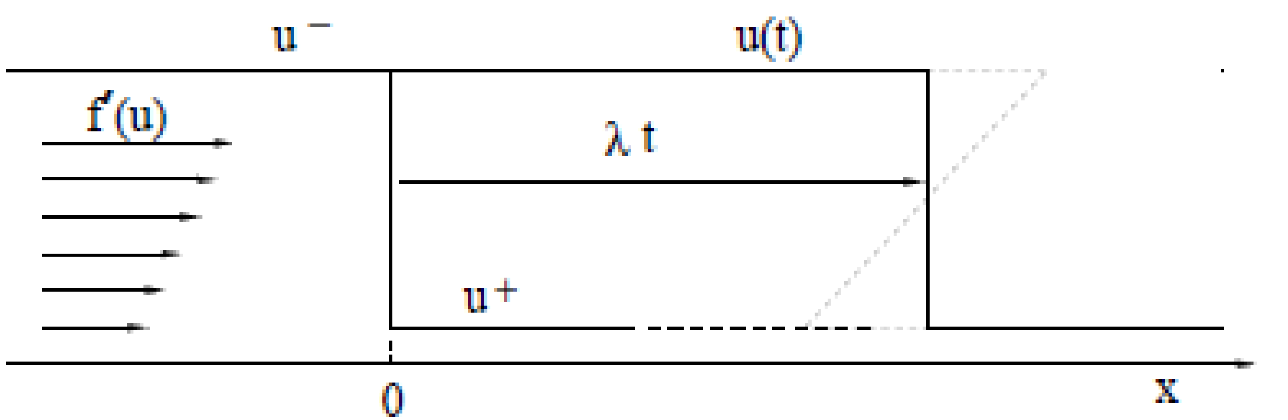


CASE 2: the flux f is convex, so that $u \mapsto f'(u)$ is increasing.

$u^+ > u^- \quad \Longrightarrow \quad$ centered rarefaction wave



$u^+ < u^- \quad \Longrightarrow \quad$ stable shock



$$\lambda = \frac{f(u^+) - f(u^-)}{u^+ - u^-}$$

A class of nonlinear hyperbolic systems

$$u_t + f(u)_x = 0$$

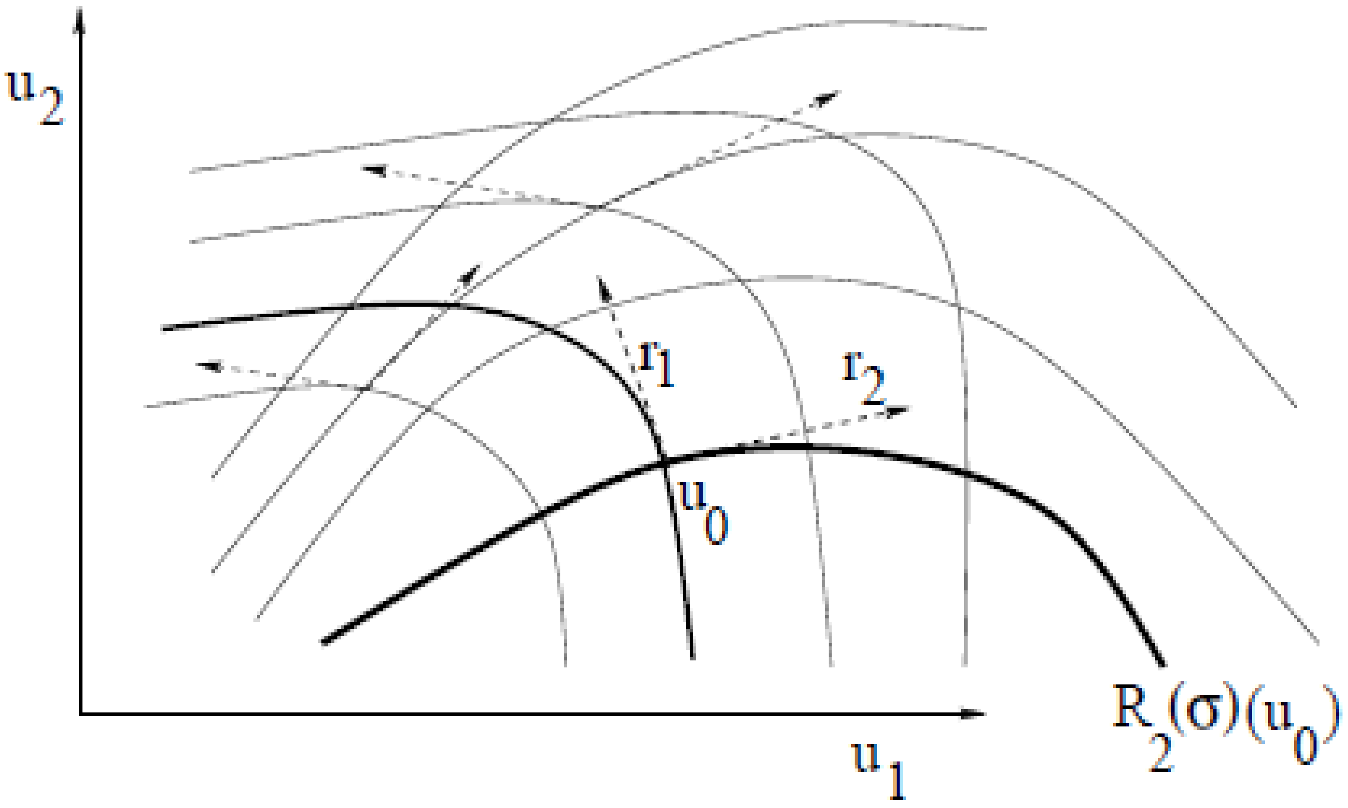
$$A(u) = Df(u) \qquad A(u)r_i(u) = \lambda_i(u)r_i(u)$$

Assumption (H) (P.Lax, 1957): Each i-th characteristic field is

- either genuinely nonlinear, so that $\nabla\lambda_i \cdot r_i > 0$ for all u
- or linearly degenerate, so that $\nabla\lambda_i \cdot r_i = 0$ for all u

genuinely nonlinear $\implies$ characteristic speed $\lambda_i(u)$ is strictly increasing along integral curves of the eigenvectors r_i

linearly degenerate $\implies$ characteristic speed $\lambda_i(u)$ is constant along integral curves of the eigenvectors r_i



Shock and Rarefaction curves

$$u_t + f(u)_x \;=\; 0 \qquad\qquad A(u) = Df(u)$$

i-rarefaction curve through u_0: $\quad \sigma \mapsto R_i(\sigma)(u_0)$

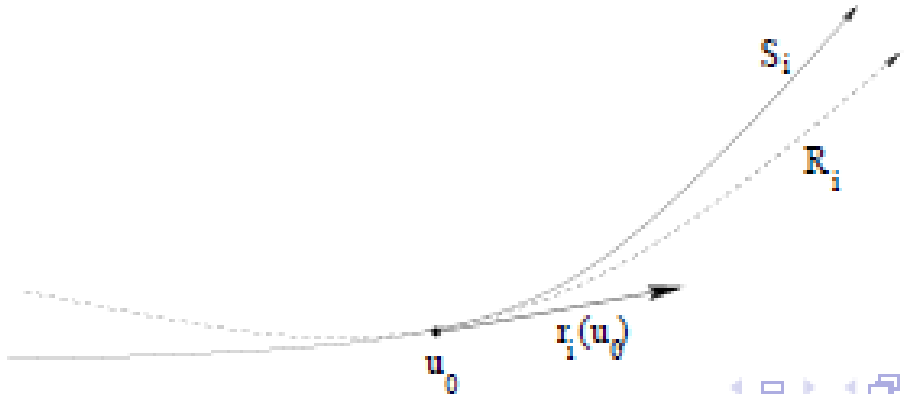
$=$ integral curve of the field of eigenvectors r_i through u_0

$$\frac{du}{d\sigma} = r_i(u), \qquad\qquad u(0) = u_0$$

i-shock curve through u_0: $\quad \sigma \mapsto S_i(\sigma)(u_0)$

$=$ set of points u connected to u_0 by an i-shock, so that

$u - u_0$ is an i-eigenvector of the averaged matrix $A(u, u_0)$



Elementary waves

$$u_t + f(u)_x = 0 \qquad u(0,x) = \begin{cases} u^- & \text{if} \quad x < 0 \\ u^+ & \text{if} \quad x > 0 \end{cases}$$

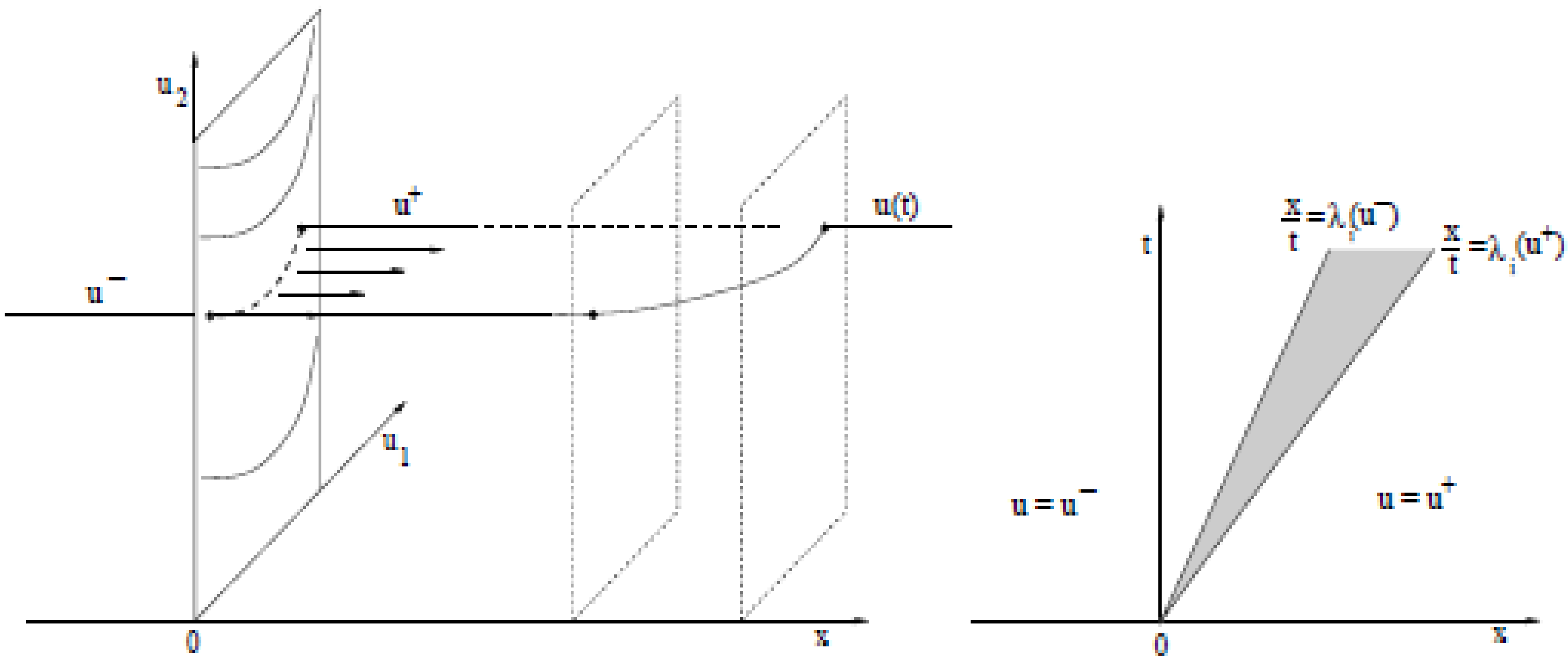
CASE 1 (Centered rarefaction wave). Let the i-th field be genuinely nonlinear.

If $u^+ = R_i(\sigma)(u^-)$ for some $\sigma > 0$, then

$$u(t,x) = \begin{cases} u^- & \text{if} \quad x < t\lambda_i(u^-), \\ R_i(s)(u^-) & \text{if} \quad x = t\lambda_i(s) \quad s \in [0,\sigma] \\ u^+ & \text{if} \quad x > t\lambda_i(u^+) \end{cases}$$

is a weak solution of the Riemann problem

A centered rarefaction wave



CASE 2 (**Shock or contact discontinuity**). Assume that

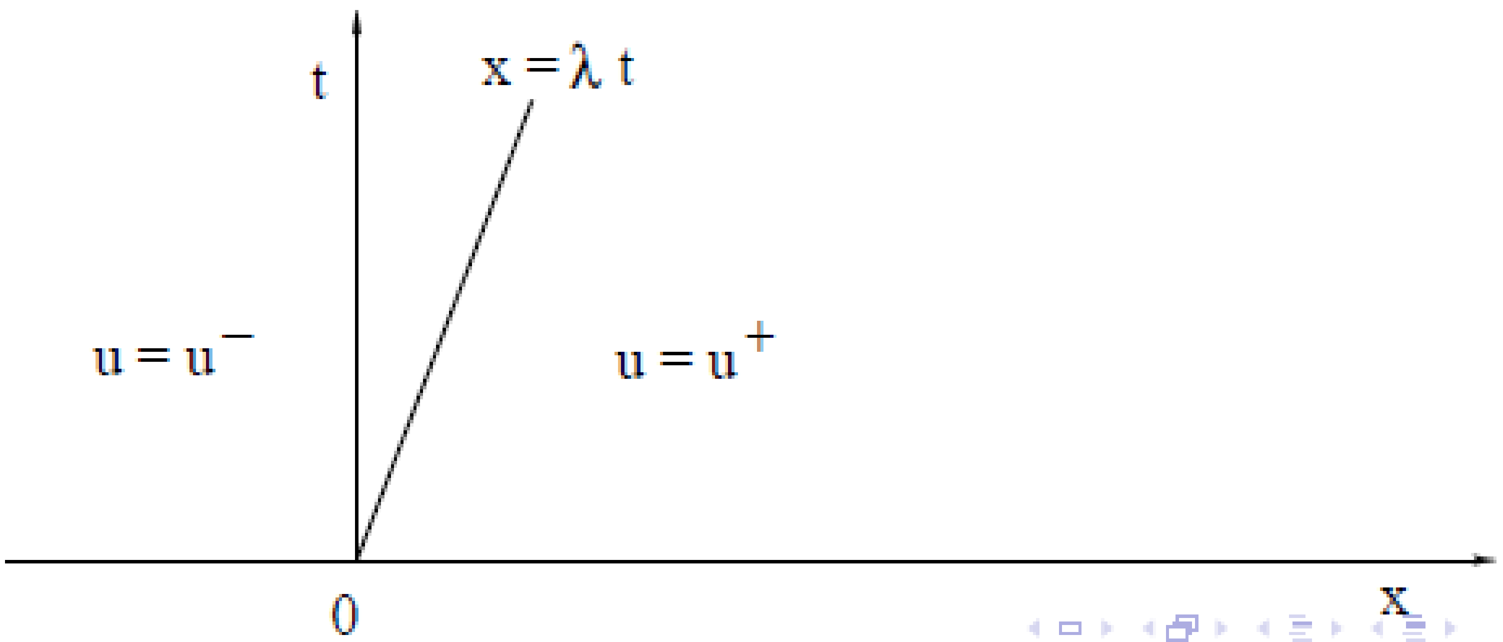
$u^+ = S_i(\sigma)(u^-)$ for some i, σ. Let $\lambda = \lambda_i(u^-, u^+)$ be the shock speed.

Then the function

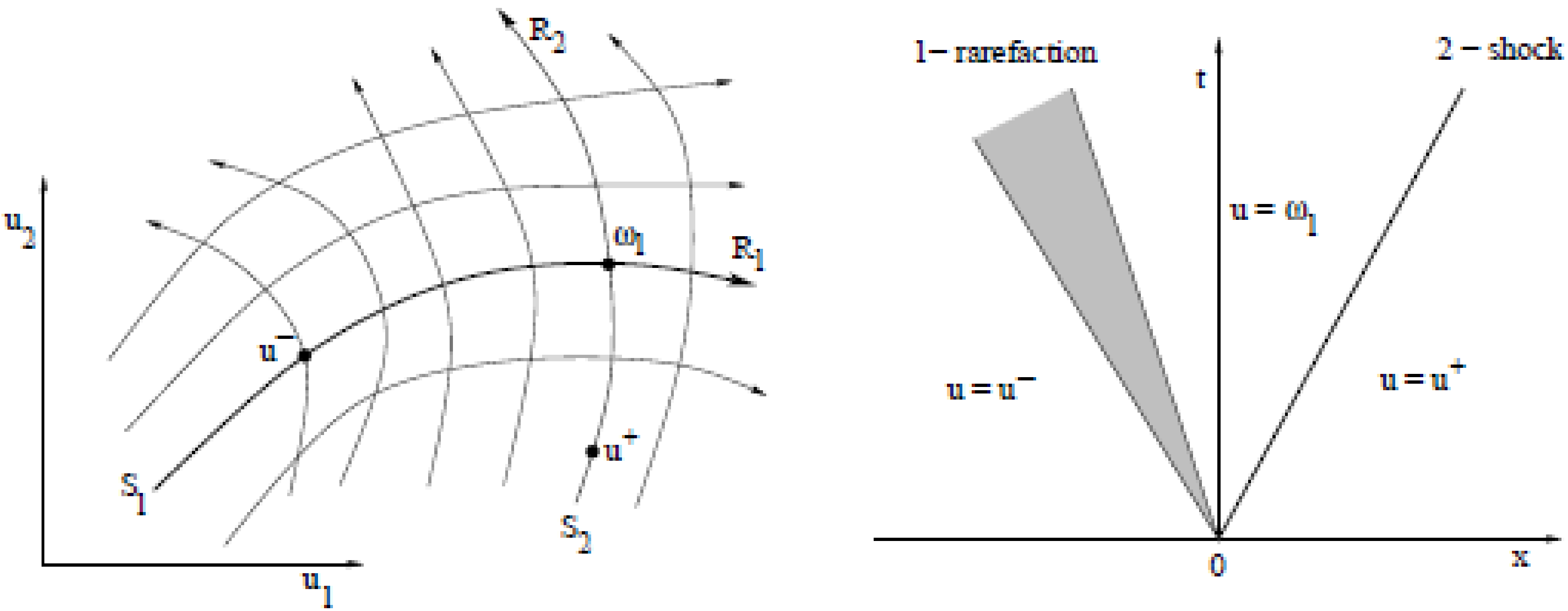
$$u(t,x) = \begin{cases} u^- & \text{if} \quad x < \lambda t, \\ u^+ & \text{if} \quad x > \lambda t, \end{cases}$$

is a weak solution to the Riemann problem.

In the genuinely nonlinear case, this shock is admissible (i.e., it satisfies the Lax condition and the Liu condition) iff $\sigma < 0$.



Solution to a 2 x 2 Riemann problem



Solution of the general Riemann problem (P. Lax, 1957)

$$u_t + f(u)_x = 0 \qquad u(0,x) = \begin{cases} u^- & \text{if} \quad x < 0 \\ u^+ & \text{if} \quad x > 0 \end{cases}$$

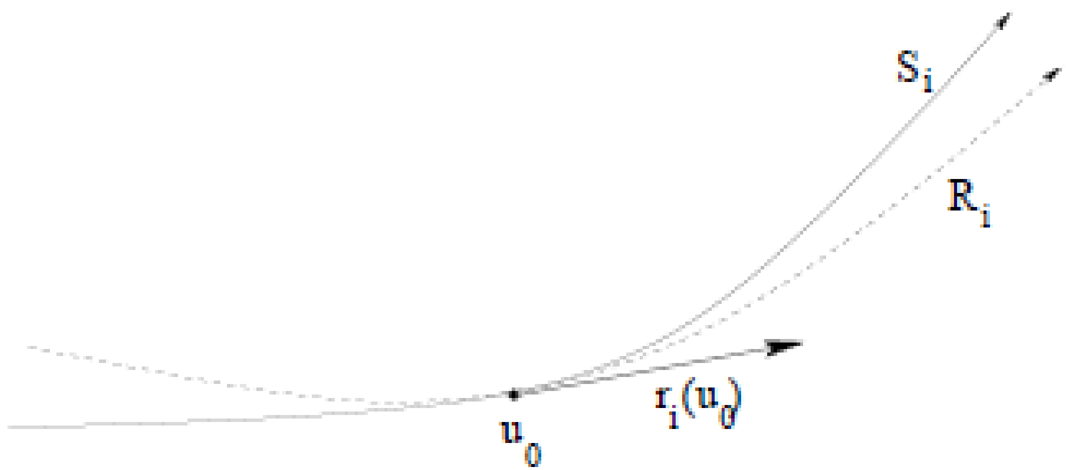
Problem: Find states $\omega_0, \omega_1, \cdots, \omega_m$ such that

$$\omega_0 = \mathbf{u}^- \qquad \omega_m = \mathbf{u}^+$$

and every couple ω_{i-1}, ω_i are connected by an elementary wave (shock or rarefaction)

$$\begin{cases} \text{either } \omega_i = R_i(\sigma_i)(\omega_{i-1}) & \sigma_i \geq 0 \\ \text{or } \omega_i = S_i(\sigma_i)(\omega_{i-1}) & \sigma_i < 0 \end{cases}$$

define: $$\Psi_i(\sigma)(u) = \begin{cases} R_i(\sigma)(u) & \text{if } \sigma \geq 0 \\ S_i(\sigma)(u) & \text{if } \sigma < 0 \end{cases}$$



$$(\sigma_1, \sigma_2, \ldots, \sigma_n) \mapsto \Psi_n(\sigma_n) \circ \cdots \circ \Psi_2(\sigma_2) \circ \Psi_1(\sigma_1)(u^-)$$

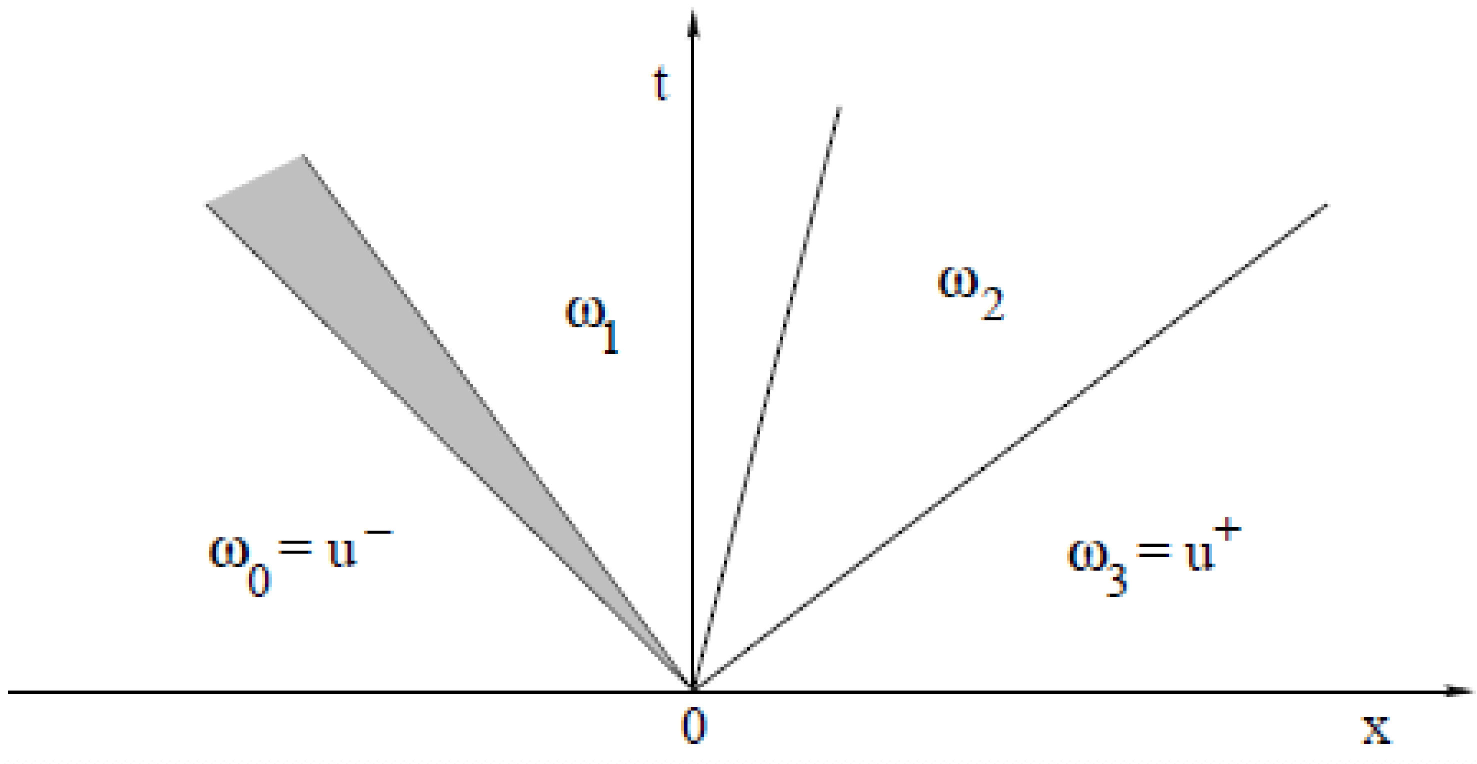
Jacobian matrix at the origin: $$J \doteq \left(r_1(u^-) \middle| r_2(u^-) \middle| \cdots \middle| r_n(u^-) \right)$$

always has full rank

If $|u^+ - u^-|$ is small, then the implicit function theorem yields existence and uniqueness of the intermediate states $\omega_0, \omega_1, \ldots, \omega_n$

General solution of the Riemann problem

Concatenation of elementary waves (shocks, rarefactions, or contact discontinuities)



Global solution to the Cauchy problem

$$u_t + f(u)_x = 0, \qquad u(0,x) = \bar{u}(x)$$

Theorem (Glimm, 1965).

Assume:

- *system is strictly hyperbolic*
- *each characteristic field is either linearly degenerate or genuinely nonlinear*

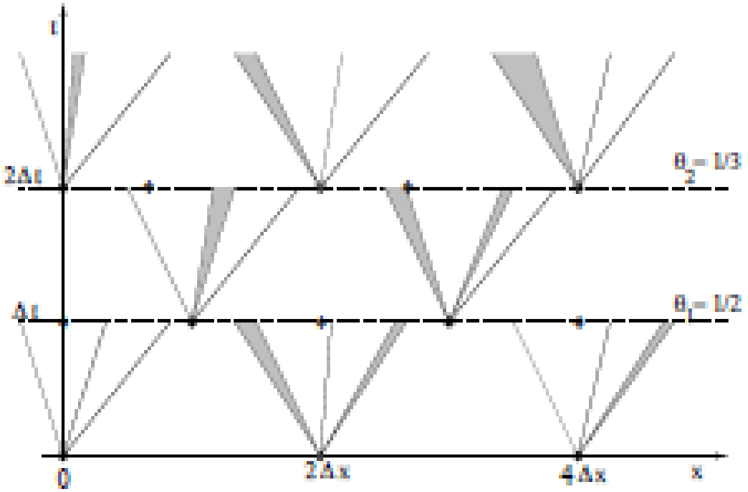
Then there exists a constant $\delta > 0$ such that, for every initial condition $\bar{u} \in \mathbf{L}^1(\mathbb{R};\ \mathbb{R}^n)$ with

$$Tot.Var.(\bar{u}) \leq \delta,$$

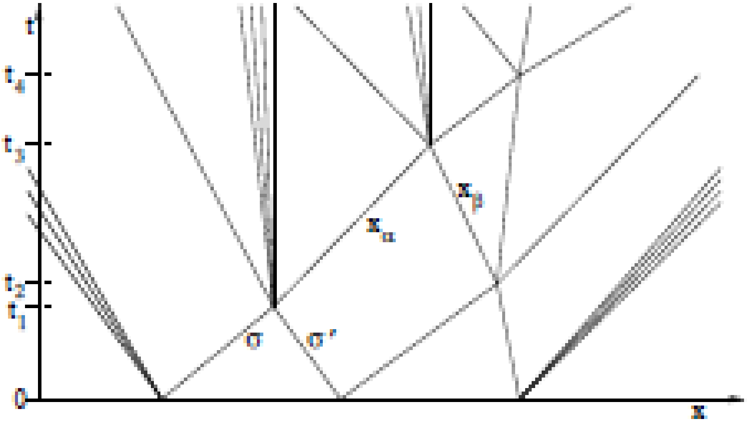
the Cauchy problem has an entropy admissible weak solution $u = u(t,x)$ defined for all $t \geq 0$.

Construction of a sequence of approximate solutions by piecing together solutions of Riemann problems

- on a fixed grid in t-x plane (Glimm scheme)

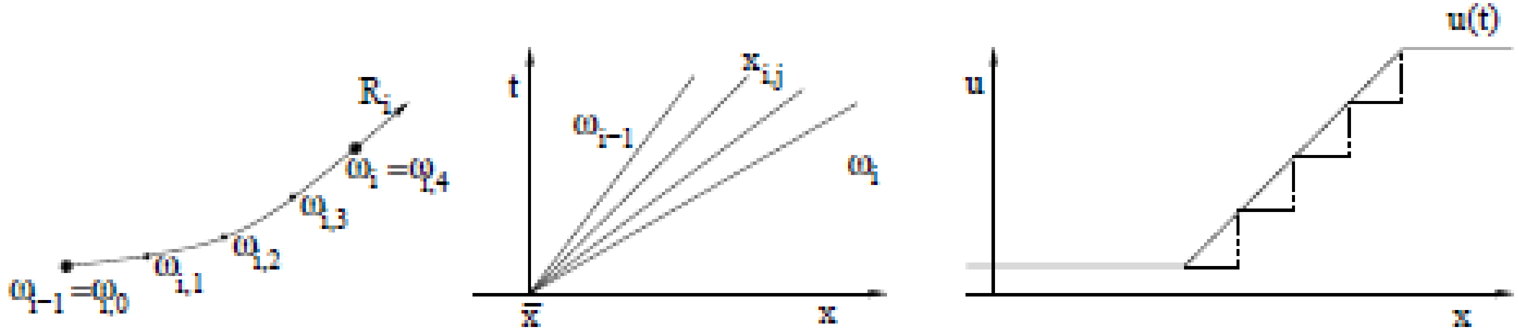


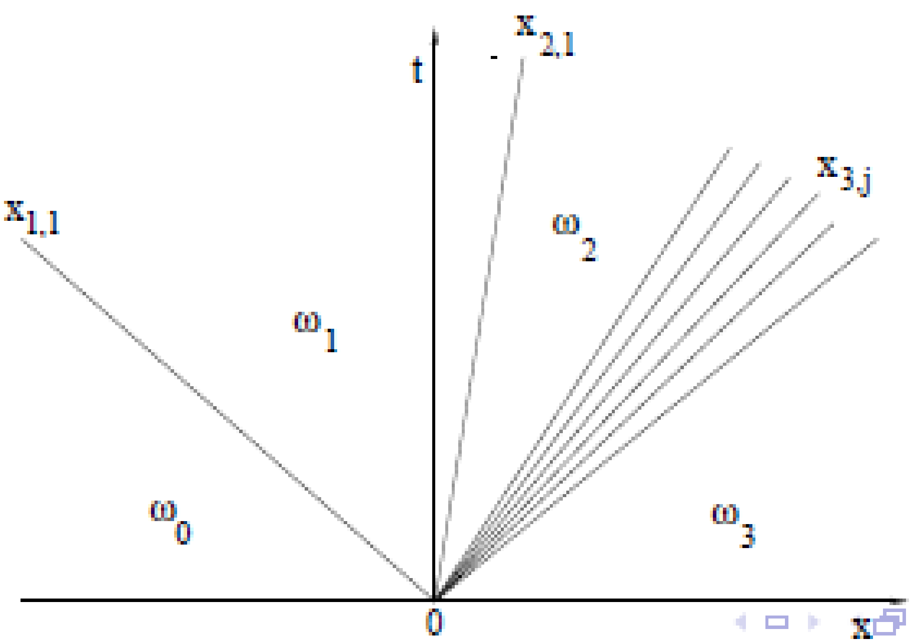
- at points where fronts interact (front tracking)



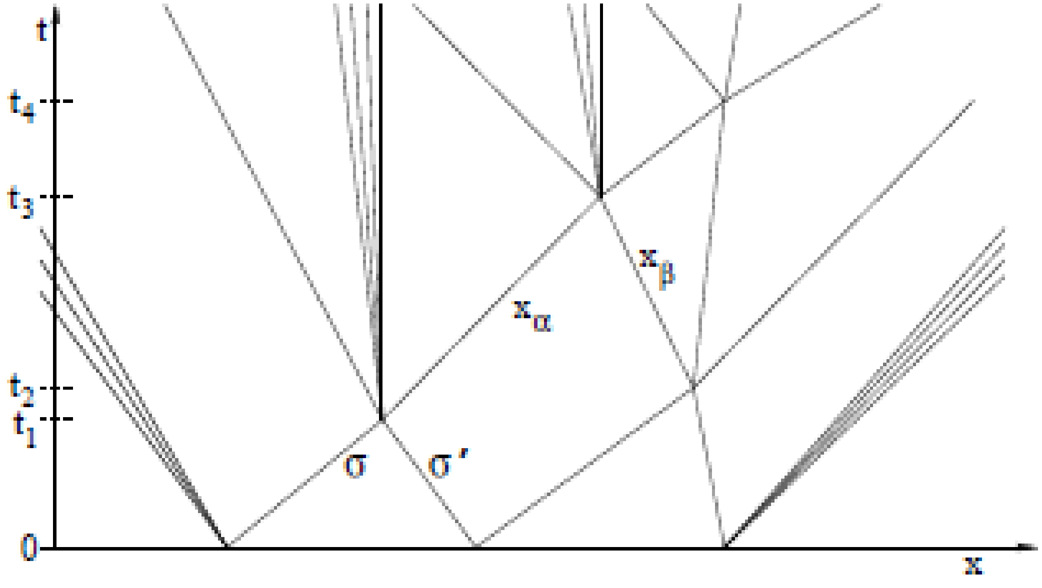
Piecewise constant approximate solution to a Riemann problem

replace centered rarefaction waves with piecewise constant rarefaction fans





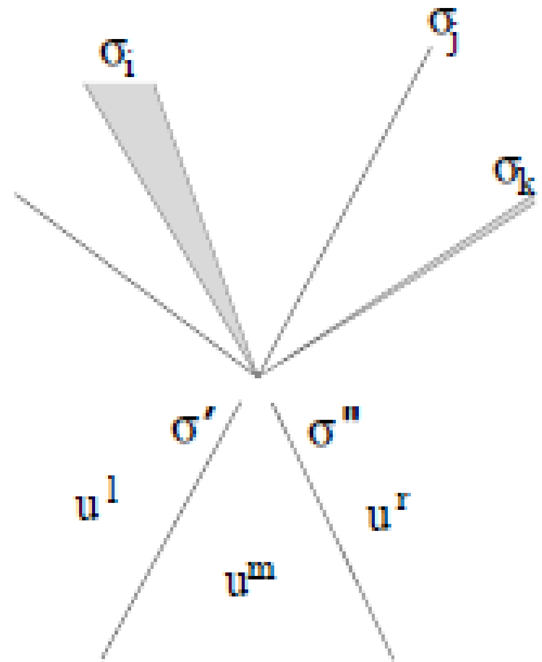
Front Tracking Approximations



- Approximate the initial data $\bar{u}$ with a piecewise constant function
- Construct a piecewise constant approximate solution to each Riemann problem at $t = 0$
- at each time t_j where two fronts interact, construct a piecewise constant approximate solution to the new Riemann problem ...
- NEED TO CHECK: $\begin{cases} \text{- total variation remains small} \\ \text{- number of wave fronts remains finite} \end{cases}$

Interaction estimates

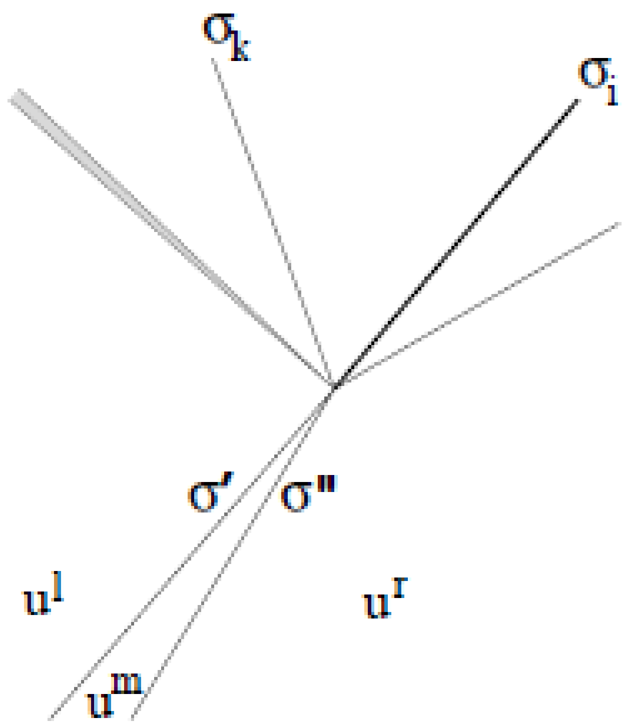
GOAL: estimate the strengths of the waves in the solution of a Riemann problem, depending on the strengths of the two interacting waves σ', σ''



Incoming: a j-wave of strength σ' and an i-wave of strength σ''

Outgoing: waves of strengths $\sigma_1, \cdots, \sigma_m$. Then

$$|\sigma_i - \sigma''| + |\sigma_j - \sigma'| + \sum_{k \neq i,j} |\sigma_k| = O(1) \cdot |\sigma' \sigma''|$$



Incoming: two i-waves of strengths σ' and σ''

Outgoing: waves of strengths $\sigma_1, \cdots, \sigma_m$. Then

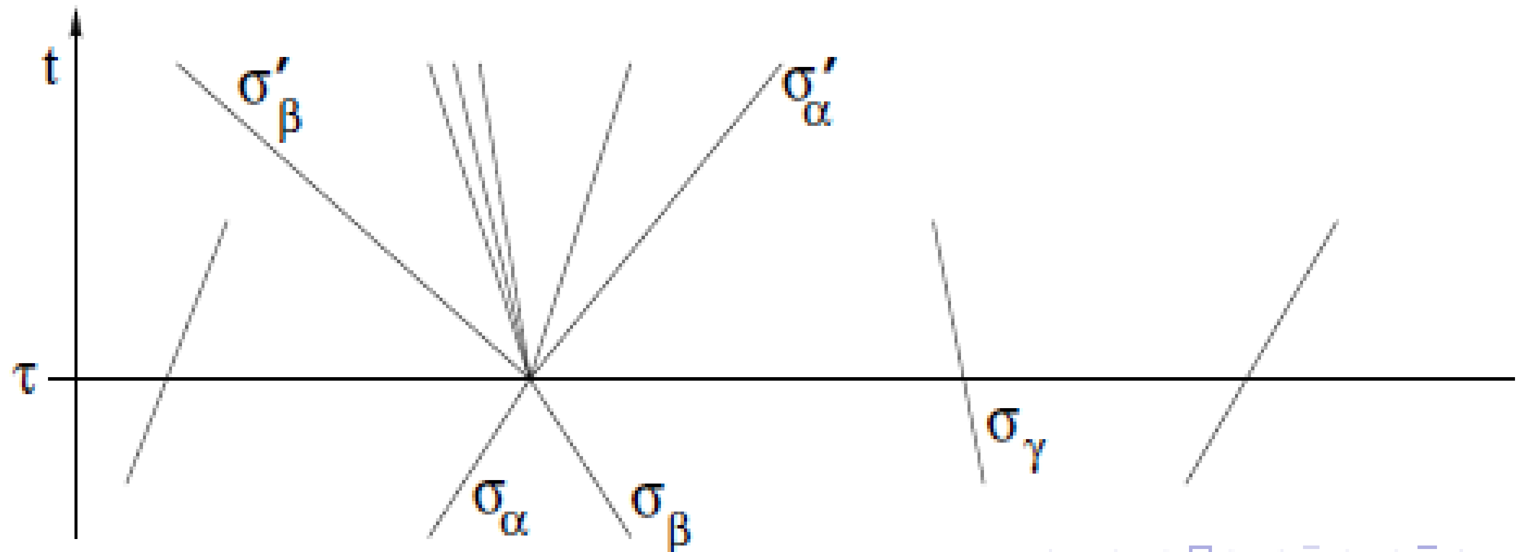
$$|\sigma_i - \sigma' - \sigma''| + \sum_{k \neq i} |\sigma_k| \ = \ \mathcal{O}(1) \cdot |\sigma' \sigma''| \Big(|\sigma'| + |\sigma''| \Big)$$

Glimm functionals

Total strength of waves: $V(t) \doteq \sum_{\alpha} |\sigma_\alpha|$

Wave interaction potential: $Q(t) \doteq \sum_{(\alpha,\beta)\in\mathcal{A}} |\sigma_\alpha \sigma_\beta|$

$\mathcal{A} \doteq$ couples of *approaching* wave fronts



Changes in V, Q at time τ when the fronts $\sigma_\alpha, \sigma_\beta$ interact:

$$\Delta V(\tau) = \mathcal{O}(1)|\sigma_\alpha \sigma_\beta|$$

$$\Delta Q(\tau) = -|\sigma_\alpha \sigma_\beta| + \mathcal{O}(1) \cdot V(\tau-)|\sigma_\alpha \sigma_\beta|$$

Choosing a constant C_0 large enough, the map

$$t \mapsto V(t) + C_0 Q(t)$$

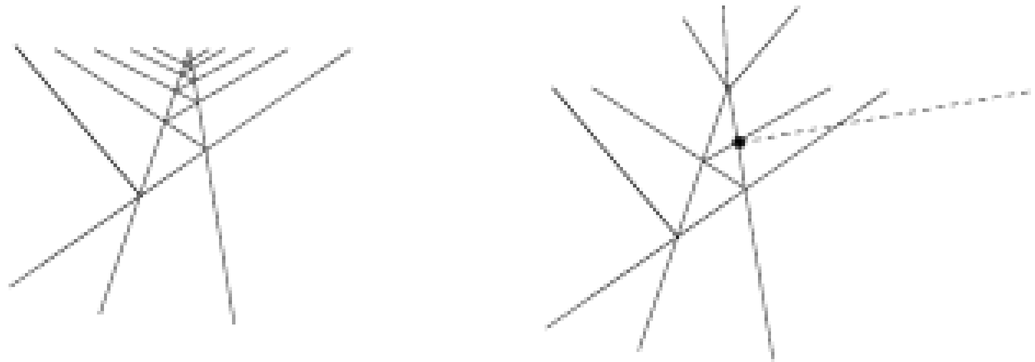
is nonincreasing, as long as V remains small

Total variation initially small $\implies$ global BV bounds

$$\text{Tot.Var.}\{u(t,\cdot)\} \leq V(t) \leq V(0) + C_0 Q(0)$$

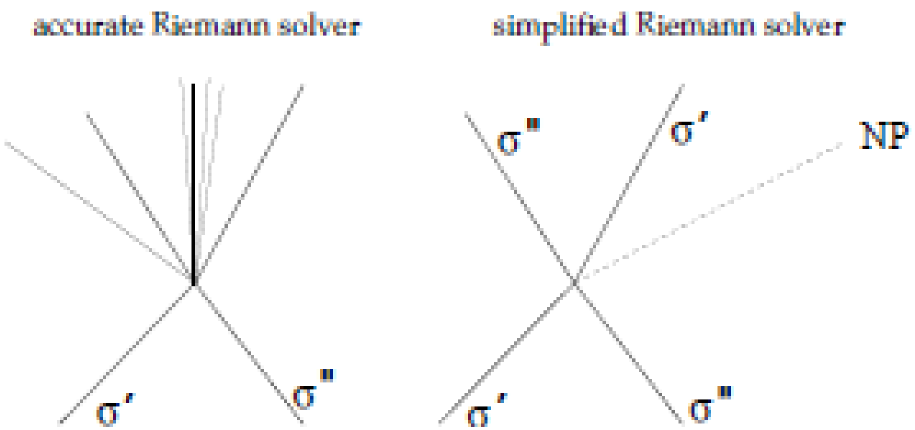
Front tracking approximations can be constructed for all $t \geq 0$

Keeping finite the number of wave fronts



At each interaction point, the **Accurate Riemann Solver** yields a solution, possibly introducing several new fronts

The total number of fronts can become infinite in finite time



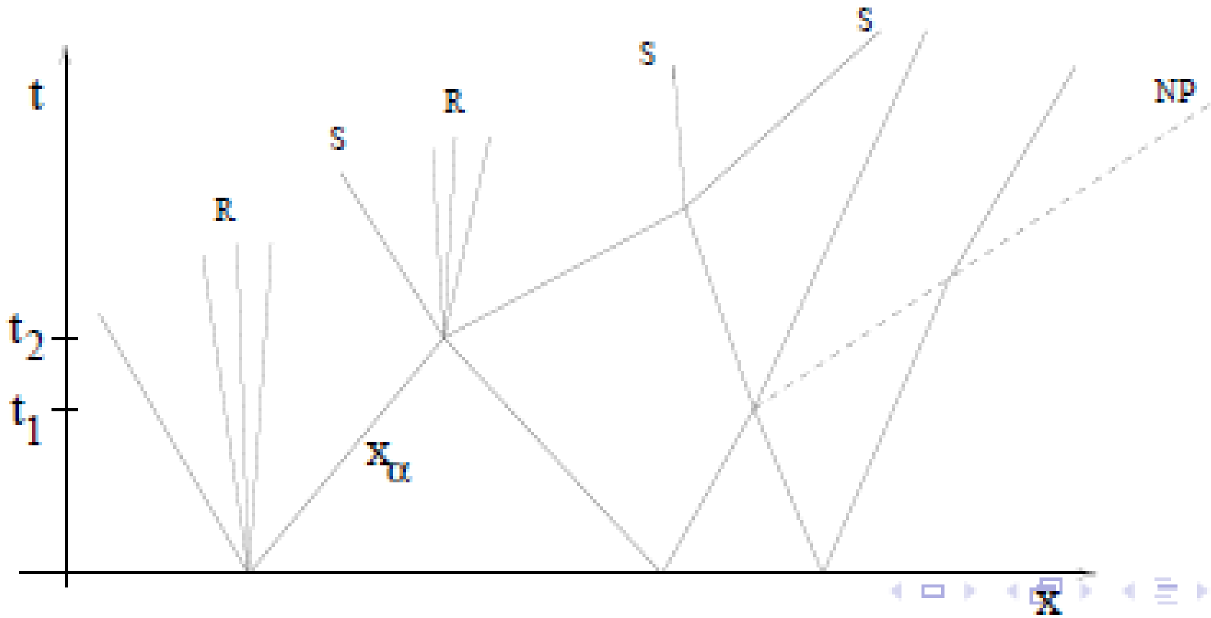
Need: a **Simplified Riemann Solver**, producing only one *"non-physical"* front

A sequence of approximate solutions

$$u_t + f(u)_x = 0 \qquad u(0,x) = \bar{u}(x)$$

$(u_\nu)_{\nu \geq 1}$ sequence of approximate front tracking solutions

- initial data satisfy $\|u_\nu(0,\cdot) - \bar{u}\|_{\mathbf{L}^1} \leq \varepsilon_\nu \to 0$
- all shock fronts in u_ν are entropy-admissible
- each rarefaction front in u_ν has strength $\leq \varepsilon_\nu$
- at each time $t \geq 0$, the total strength of all non-physical fronts in $u_\nu(t,\cdot)$ is $\leq \varepsilon_\nu$



Existence of a convergent subsequence

$$\text{Tot.Var.}\{u_\nu(t,\cdot)\} \leq C$$

$$\begin{aligned} \big\|u_\nu(t) - u_\nu(s)\big\|_{\mathbf{L}^1} &\leq (t-s)\cdot[\text{total strength of all wave fronts}]\cdot[\text{maximum speed}] \\ &\leq L\cdot(t-s) \end{aligned}$$

Helly's compactness theorem $\Longrightarrow$ a subsequence converges

$$u_\nu \to u \qquad\qquad \text{in } \ \mathbf{L}^1_{loc}$$

Claim: $u = \lim_{\nu \to \infty} u_\nu$ is a weak solution

$$\iint \left\{ \phi_t u + \phi_x f(u) \right\} dxdt = 0 \qquad \phi \in \mathcal{C}_c^1\Big( ]0, \infty[\times \mathbb{R} \Big)$$

Need to show:

$$\lim_{\nu \to \infty} \iint \left\{ \phi_t u_\nu + \phi_x f(u_\nu) \right\} dxdt = 0$$

$$\int_0^\infty \int_{-\infty}^{\infty} \left\{ \phi_t(t,x) u_\nu(t,x) + \phi_x(t,x) f\left(u_\nu(t,x)\right) \right\} dxdt$$
$$= \sum_j \int_{\partial \Gamma_j} \Phi_\nu \cdot \mathbf{n} \, d\sigma$$

$$\limsup_{\nu \to \infty} \left| \sum_j \int_{\partial \Gamma_j} \Phi_\nu \cdot \mathbf{n} \, d\sigma \right|$$

$$\leq \limsup_{\nu \to \infty} \left| \sum_{\alpha \in \mathcal{S} \cup \mathcal{R} \cup \mathcal{NP}} \left[\dot{x}_\alpha(t) \cdot \Delta u_\nu(t, x_\alpha) - \Delta f\left(u_\nu(t, x_\alpha)\right) \right] \phi(t, x_\alpha(t)) \right|$$

$$\leq \left(\max_{t,x} \left| \phi(t,x) \right| \right) \cdot \limsup_{\nu \to \infty} \left\{ \mathcal{O}(1) \cdot \sum_{\alpha \in \mathcal{R}} \varepsilon_\nu |\sigma_\alpha| + \mathcal{O}(1) \cdot \sum_{\alpha \in \mathcal{NP}} |\sigma_\alpha| \right\}$$

$$= 0$$

The Glimm scheme

$$u_t + f(u)_x = 0 \qquad u(0,x) = \bar{u}(x)$$

Assume: all characteristic speeds satisfy $\lambda_i(u) \in [0,1]$

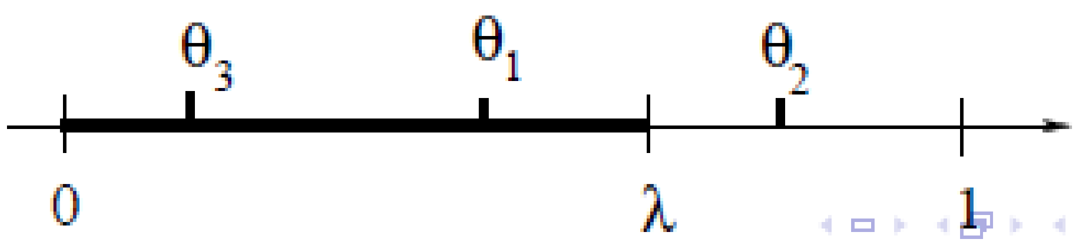
This is not restrictive. If $\lambda_i(u) \in [-M, M]$, simply change coordinates:

$$y = x + Mt, \qquad \tau = 2Mt$$

Choose:

- a grid in the t-x plane with step size $\Delta t = \Delta x$
- a sequence of numbers $\theta_1, \theta_2, \theta_3, \ldots$ *uniformly distributed* over $[0,1]$

$$\lim_{N\to\infty} \frac{\#\{j\ ;\ \ 1 \leq j \leq N,\ \ \theta_j \in [0,\lambda]\ \}}{N} = \lambda \qquad \text{for each } \lambda \in [0,1].$$

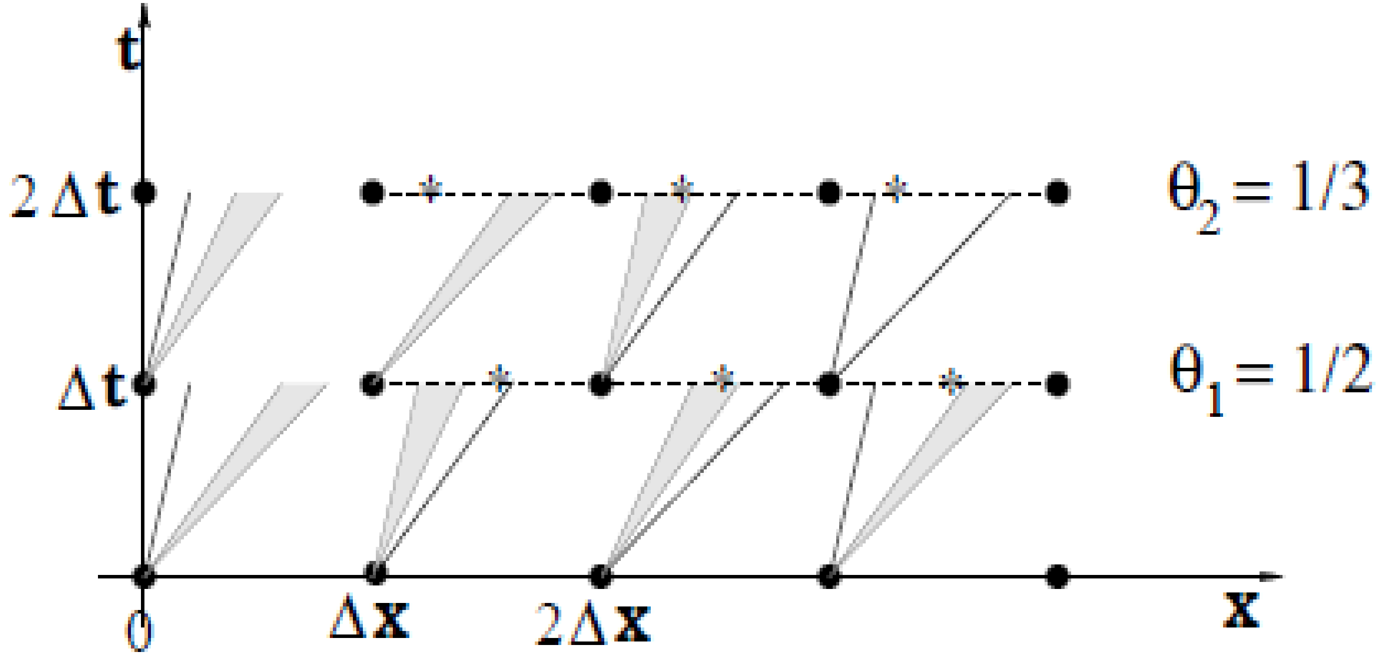


Glimm approximations

Grid points : $x_j = j \cdot \Delta x, \qquad t_k = k \cdot \Delta t$

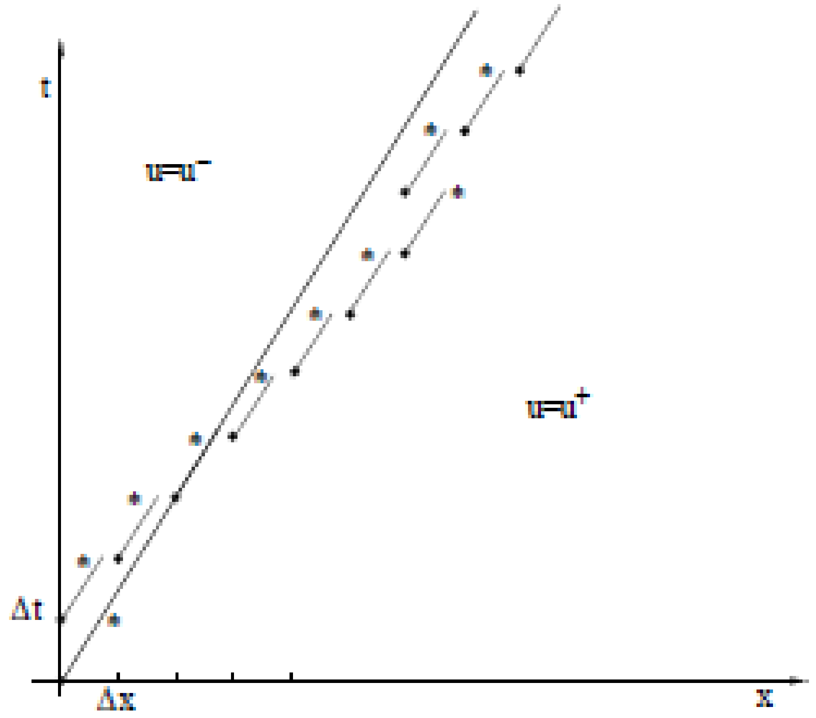
• for each $k \geq 0$, $u(t_k, \cdot)$ is piecewise constant, with jumps at the points x_j. The Riemann problems are solved exactly, for $t_k \leq t < t_{k+1}$

• at time t_{k+1} the solution is again approximated by a piecewise constant function, by a sampling technique



Example: Glimm's scheme applied to a solution containing a single shock

$$U(t,x) = \begin{cases} u^+ & \text{if} \quad x > \lambda t \\ u^- & \text{if} \quad x < \lambda t \end{cases}$$



Fix $T > 0$, take $\Delta x = \Delta t = T/N$

$$x(T) = \#\{j\ ;\ \ 1 \leq j \leq N,\ \ \theta_j \in [0, \lambda]\ \} \cdot \Delta t$$

$$= \frac{\#\{j\ ;\ \ 1 \leq j \leq N,\ \ \theta_j \in [0, \lambda]\ \}}{N} \cdot T \ \to\ \lambda T \qquad \text{as}\ \ N \to \infty$$

Rate of convergence for Glimm approximations

Random sampling at points determined by the equidistributed sequence $(\theta_k)_{k\geq 1}$

$$\lim_{N\to\infty} \frac{\#\{j\ ;\ \ 1 \leq j \leq N,\ \ \theta_j \in [0,\lambda]\ \}}{N} = \lambda \qquad \text{for each } \lambda \in [0,1].$$

Need fast convergence to uniform distribution. Achieved by choosing:

$$\theta_1 = 0.1\,, \quad \ldots \quad , \quad \theta_{759} = 0.957\,, \quad \ldots \quad , \quad \theta_{39022} = 0.22093\,, \quad \ldots$$

Convergence rate: $$\lim_{\Delta x\to 0} \frac{\left\| u^{\mathrm{Glimm}}(T,\cdot) - u^{\mathrm{exact}}(T,\cdot)\right\|_{\mathbf{L}^1}}{\sqrt{\Delta x}\cdot |\ln \Delta x|} = 0$$

(A.Bressan & A.Marson, 1998)

Bressan, A.: **Hyperbolic Systems of Conservation Laws. The One-Dimensional Cauchy Problem**. Oxford University Press: Oxford, 2000.

Dafermos, C: **Hyperbolic Conservation Laws in Continuum Physics,** 4rd Edition, Springer-Verlag: Berlin, 2016.

Functional Analytic Approaches for the Existence Theory:

- Compensated Compactness
- Weak Convergence Methods
- Geometric Measure Arguments
- ……

1. **C. M. Dafermos:** **Hyperbolic Conservation Laws in Continuum Physics,** Third edition. Springer-Verlag: Berlin, 2010.

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