Young Measures

$K \subset \mathbb{R}^m$, $\Omega \subset \mathbb{R}^n$ bounded open

$u^k : \Omega \longrightarrow \mathbb{R}^m$ measurable

$u^k(y) \in K$, a.e.

$\Longrightarrow$ $\exists \ \{\nu_y \in \mathrm{Prob.}(\mathbb{R}^m)\}_{y \in \Omega}$ s.t.

- $\boxed{\mathrm{Supp}\,\nu_y \subset \overline{K}}$ $\quad \forall y \in \Omega$
- $\forall f \in C(\mathbb{R}^m; \mathbb{R})$, $\exists \ \{u^{k_j}\}_{j=1}^{\infty} \subset \{u^k\}$.

$$\boxed{\begin{aligned} w^*\text{-}\lim f(u^{k_j}) &= \langle \nu_y(\lambda), f(\lambda) \rangle \\ &= \int f(\lambda)\, d\nu_y(\lambda) \end{aligned}}$$

- $u^{k_j} \longrightarrow u$ a.e. $\Longleftrightarrow$ $\nu_y(\lambda) = \delta_{u(y)}(\lambda)$

Dirac mass

* This theorem can be extended to more general cases.

Remarks

1. The deviation between the weak and strong convergence is measured by the spreading of the support of ν_y.

$$\boxed{\begin{aligned}&\| f(w^*\text{-}\lim u^k) - w^*\text{-}\lim f(u^k)\|_{L^\infty} \\ &\leq C \sup_y (\operatorname{diam}(\operatorname{supp} \nu_y))\end{aligned}}$$

for $f \in Lip(\mathbb{R}^m; \mathbb{R})$

$$\begin{aligned}&\| f(w^*\text{-}\lim u^k) - w^*\text{-}\lim f(u^k)\|_{L^\infty} \\ &= \| f(\langle \nu_y, \lambda \rangle) - \langle \nu_y, f(\lambda)\rangle \|_{L^\infty} \\ &= \| \langle \nu_y, f(\lambda) - f(\langle \nu_y, \lambda\rangle)\rangle \|_{L^\infty} \\ &\leq C \| \langle \nu_y, |\lambda - \langle \nu_y, \lambda \rangle| \rangle \|_{L^\infty} \\ &\leq C \sup_y (\operatorname{diam}(\operatorname{supp} \nu_y)).\end{aligned}$$

Remarks

2. The Young measure family $\{\nu_y\}_{y\in\Omega}$ can be thought of as the limiting probability distribution of the values of $\{u^k(y)\}$ near the point y as $k\to\infty$.

$\Omega\subset\mathbb{R}^n$, $y\in\Omega$.

$\exists\ \delta_0>0$ s.t. $B(y,\delta)\subset\Omega$, $0<\delta\le\delta_0$.

Define

$$\langle \nu_{y,\delta}^k, \phi\rangle = \frac{1}{|B(y,\delta)|}\phi(u^k(x))dx$$

$$\forall \phi\in C_c(\mathbb{R}^m;\mathbb{R})$$

$$\nu_{y,\delta}^k(\lambda) \triangleq \frac{1}{|B(y,\delta)|}\delta_{u^k(x)}dx$$

$$\boxed{\nu_y(\lambda) = \lim_{\delta\to0}\lim_{k\to\infty}\nu_{y,\delta}^k}$$

Weak Continuity of 2x2 Determinants

$\Omega \subset \mathbb{R}_+ \times \mathbb{R}$ bounded open

$U^k : \Omega \to \mathbb{R}^4$ measurable

$$\begin{cases} \text{w-}\lim_{k\to\infty} U^k = U & \text{in } L^2(\Omega; \mathbb{R}^4) \\ \begin{cases} \dfrac{\partial u_1^k}{\partial t} + \dfrac{\partial u_2^k}{\partial x} \\ \dfrac{\partial u_3^k}{\partial t} + \dfrac{\partial u_4^k}{\partial x} \end{cases} & \text{compact in } H^{-1}_{loc}(\Omega) \end{cases}$$

$$\Longrightarrow \quad \begin{vmatrix} u_1^k & u_2^k \\ u_3^k & u_4^k \end{vmatrix} \longrightarrow \begin{vmatrix} u_1 & u_2 \\ u_3 & u_4 \end{vmatrix} \quad \mathcal{D}'$$

Subsequentially

Another Form

$V^k = (V_1^k, V_2^k) \in L^2(\Omega; \mathbb{R}^2)$

$W^k = (W_1^k, W_2^k) \in L^2(\Omega; \mathbb{R}^2)$

$$\begin{cases} \text{w-}\lim_{k\to\infty} (V^k, W^k) = (V, W), & L^2(\Omega) \\ \begin{cases} \operatorname{div} V^k \\ \operatorname{curl} W^k \end{cases} & \text{compact in } H^{-1}_{loc}(\Omega) \end{cases}$$

$$\Longrightarrow \quad V^k \cdot W^k \longrightarrow V \cdot W \quad \mathcal{D}'.$$

Div-Curl Lemma

$\Omega \subset \mathbb{R}^n$ open, bounded

$p, q > 1, \quad \frac{1}{p} + \frac{1}{q} = 1$

$v^k \in L^p(\Omega; \mathbb{R}^n)$

$w^k \in L^q(\Omega; \mathbb{R}^n)$

$$\begin{cases} v^k \longrightarrow v & \text{weakly in } L^p(\Omega; \mathbb{R}^n) \\ w^k \longrightarrow w & \text{weakly in } L^q(\Omega; \mathbb{R}^n). \end{cases}$$

$$\begin{cases} \operatorname{div} v^k & \text{compact in } W^{-1,p}_{loc}(\Omega; \mathbb{R}) \\ \operatorname{curl} w^k & \text{compact in } W^{-1,q}_{loc}(\Omega; \mathbb{R}). \end{cases}$$

$\Longrightarrow$

$$v^k \cdot w^k \longrightarrow v \cdot w \quad \mathcal{D}'$$

Compensated Compact Embedding Lemma

$\Omega \subset \mathbb{R}^n$ bounded open

$\hookrightarrow$

(Compact set of $W^{-1,q}_{loc}(\Omega)$)

$\cap$ (Bounded set of $W^{-1,r}_{loc}(\Omega)$)

$\subset$ (Compact set of $W^{-1,p}_{loc}(\Omega)$)

for any $1 < q \leq p < r < \infty$

2x2 Hyperbolic Systems of Conservation Laws

$$\begin{cases} u_t + f(u)_x = 0 & u \in \mathbb{R}^2 \\ u|_{t=0} = u_0(x) \end{cases}$$

Assume

- $\exists$ a strictly convex entropy $\eta_*(u)$.

  $\nabla^2 \eta_*(u) > 0$

- $\exists$ globally defined Riemann Invariants

  $W = (W_1, W_2) : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ s.t.

  $\nabla W_j(u) \,//\, \ell_j(u)$

If $u \in C^1$:

$$\boxed{\partial_t W_j + \lambda_j(u(W))\, \partial_x W_j = 0}$$

Entropy Equation

Entropy $\eta(u)$, **Entropy Flux** $q(u)$

$$\nabla q(u) = \nabla \eta(u) \nabla f(u)$$

$$(\lambda_j \nabla \eta - \nabla q) \cdot r_j = 0$$

$$\boxed{\begin{aligned} & q_{w_j} = \lambda_j \eta_{w_j} \\ & \eta_{w_1 w_2} - \frac{\lambda_1 w_2}{\lambda_2 - \lambda_1} \eta_{w_1} + \frac{\lambda_2 w_1}{\lambda_2 - \lambda_1} \eta_{w_2} = 0 \end{aligned}}$$

Genuine Nonlinearity

$$\nabla \lambda_j(u) \cdot r_j(u) \neq 0, \qquad j = 1, 2.$$

$$\Longleftrightarrow \quad \frac{\partial \lambda_j}{\partial w_j} \neq 0. \qquad j = 1, 2,$$

Method of Vanishing Viscosity

$$\begin{cases} u_t + f(u)_x = 0 & u \in \mathbb{R}^2 \\ u|_{t=0} = u_0(x) \in L^\infty \cap L^1(\mathbb{R}) \end{cases}$$

Viscosity Approximation

$$\begin{cases} u_t + f(u)_x = \varepsilon u_{xx} \\ u|_{t=0} = u_0^\varepsilon(x) \longrightarrow u_0(x) \quad \text{a.e.} \end{cases}$$

$\hookrightarrow$ $u^\varepsilon = u^\varepsilon(t,x)$

- Invariant Regions or L^p Estimates

$$\|u^\varepsilon\|_{L^\infty} \le C \qquad \text{or} \qquad \|u^\varepsilon\|_{L^p} \le C$$

- Dissipation Estimate

$$\|\sqrt{\varepsilon}\, u_x^\varepsilon\|_{L^2} \le C \quad \not\sim \ \varepsilon .$$

Dissipation Estimate

$$\nabla\bar{\eta}_*(u^\varepsilon)\times\left[\ u_t^\varepsilon + f(u^\varepsilon)_x = \varepsilon u_{xx}^\varepsilon\right]$$

$\hookrightarrow$

$$\varepsilon(u_x^\varepsilon)^T\nabla^2\bar{\eta}_*(u^\varepsilon)u_x^\varepsilon \geq c_0\varepsilon|u_x^\varepsilon|^2$$

$$= -\bar{\eta}_*(u^\varepsilon)_t - \bar{q}_*(u^\varepsilon)_x + \varepsilon\bar{\eta}_*(u^\varepsilon)_{xx}$$

$\hookrightarrow$

$$c_0\int_0^T\!\!\int \varepsilon|u_x^\varepsilon|^2\,dx\,dt$$

$$\leq \int\bar{\eta}_*(u_0^\varepsilon)\,dx - \int\bar{\eta}_*(u^\varepsilon(T,x))\,dx.$$

$$\leq \int\bar{\eta}_*(u_0^\varepsilon)\,dx \leq C \ \cancel{\ \ } \ \varepsilon.$$

- $\bar{\eta}_*(u) = \eta_*(u) - \eta_*(0) - \nabla\eta(0)\,u \gtrsim c_0 > 0$

$\bar{q}_*(u) = q_*(u) - q_*(0) - \nabla\eta(0)(f(u) - f(0)).$

H^{-1}- Compactness

$$\eta(u^\varepsilon)_t + q(u^\varepsilon)_x$$
$$= \varepsilon(\nabla\eta(u^\varepsilon)\, u_x^\varepsilon)_x - \varepsilon\,(u_x^\varepsilon)^T \nabla^2\eta(u^\varepsilon)\, u_x^\varepsilon$$
$$= I_1^\varepsilon + I_2^\varepsilon$$

- $\|I_1^\varepsilon\|_{H^{-1}(\Omega)} \le \sqrt{\varepsilon}\, \|\sqrt{\varepsilon}\, u_x^\varepsilon\|_{L^2}\, \|\nabla\eta(u^\varepsilon)\|_{L^\infty} \le \sqrt{\varepsilon}\, C \to 0$
- $\|I_2^\varepsilon\|_{L^1(\Omega)} \le \|\nabla^2\eta(u^\varepsilon)\|_{L^\infty}\, \|\sqrt{\varepsilon}\, u_x^\varepsilon\|_{L^2}^2 \le C$

$\hookrightarrow$ I_2^ε compact in $W^{-1,q}(\Omega)$, $1<q<2$

$\hookrightarrow$ $I_1^\varepsilon + I_2^\varepsilon$ compact in $W^{-1,q}(\Omega)$, $1<q<2$

But $\eta(u^\varepsilon)_t + q(u^\varepsilon)_x$ bounded in $W^{-1,\infty}(\Omega)$

$\hookrightarrow$

<u>Lemma</u>

$$\boxed{\eta(u^\varepsilon)_t + q(u^\varepsilon)_x \text{ is compact in } H^{-1}_{loc}}$$

$\forall\, (\eta, q) \in C^2$

Commutation Identity for Young Measure. $\{\nu_{t,x}\}_{(t,x)\in\mathbb{R}_+^2}$

$$\Updownarrow$$

$\{u^\varepsilon\}_{\varepsilon>0}$.

- $\mathrm{Supp}\,\nu_{t,x} \subset\subset \mathbb{R}^2$

- For any entropy pairs (η, q),

$$(*)\quad \left\langle \nu_{t,x}, \begin{vmatrix} \eta_1 & q_1 \\ \eta_2 & q_2 \end{vmatrix} \right\rangle = \begin{vmatrix} \langle \nu_{t,x}, \eta_1\rangle & \langle \nu_{t,x}, q_1\rangle \\ \langle \nu_{t,x}, \eta_2\rangle & \langle \nu_{t,x}, q_2\rangle \end{vmatrix} \quad \text{a.e. } (t,x)$$

$$\overset{?}{\Longrightarrow} \quad \nu_{t,x} = \delta_{u(t,x)}(\lambda) \quad ??$$

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* If $f(u) = Au$ (linear)

$\hookrightarrow$ $(*)$ is trivial.

The imbalance of $(*)$ is enforced by the nonlinearity of $f(u)$.

Proof of (*)

$\forall\ (\eta_i, q_i) \in C$, $i = 1, 2.$

$U^\varepsilon = (\eta_1(u^\varepsilon), q_1(u^\varepsilon), \eta_2(u^\varepsilon), q_2(u^\varepsilon))$ uniformly bdd

$\hookrightarrow\ \exists\ \{\varepsilon_k\}_{k=1}^\infty$, s.t $\varepsilon_k \to 0$ as $k \to \infty$.

- $U^{\varepsilon_k} \xrightarrow{*} \underbrace{(\langle \nu_{t,x}, \eta_1(\lambda)\rangle, \langle \nu_{t,x}, q_1(\lambda)\rangle, \langle \nu_{t,x}, \eta_2(\lambda)\rangle, \langle \nu_{t,x}, q_2(\lambda)\rangle)}_{= U(t,x)}$

- $$\begin{vmatrix} U_1^{\varepsilon_k} & U_2^{\varepsilon_k} \\ U_3^{\varepsilon_k} & U_4^{\varepsilon_k} \end{vmatrix} \xrightarrow{*} \boxed{\left\langle \nu_{t,x}, \begin{vmatrix} \eta_1(\lambda) & q_1(\lambda) \\ \eta_2(\lambda) & q_2(\lambda) \end{vmatrix} \right\rangle}$$

Div-Curl

$$\begin{vmatrix} U_1 & U_2 \\ U_3 & U_4 \end{vmatrix} = \boxed{\begin{vmatrix} \langle \nu_{t,x}, \eta_1(\lambda)\rangle & \langle \nu_{t,x}, q_1(\lambda)\rangle \\ \langle \nu_{t,x}, \eta_2(\lambda)\rangle & \langle \nu_{t,x}, q_2(\lambda)\rangle \end{vmatrix}}.$$

The two boxed expressions are equal (//).

Reduction of the Young Measure:

Scalar Conservation Laws: $u^\varepsilon \xrightarrow{*} u$, a.e.

$\hookrightarrow u(t,x) = \langle \nu_{t,x}, \lambda \rangle$

$$\boxed{\left\langle \nu_{t,x}, \begin{vmatrix} \eta_1(\lambda) & q_1(\lambda) \\ \eta_2(\lambda), & q_2(\lambda) \end{vmatrix} \right\rangle = \begin{vmatrix} \langle \nu_{t,x}, \eta_1(\lambda) \rangle & \langle \nu_{t,x}, q_1(\lambda) \rangle \\ \langle \nu_{t,x}, \eta_2(\lambda) \rangle & \langle \nu_{t,x}\, q_2(\lambda) \rangle \end{vmatrix}}$$

Choose: $(\eta_1(\lambda), q_1(\lambda)) = (\lambda - u(t,x),\ f(\lambda) - f(u(t,x)))$

$(\eta_2(\lambda), q_2(\lambda)) = \left(f(\lambda) - f(u(t,x)),\ \int_{u(t,x)}^{\lambda} (f'(s))^2 ds\right)$

$$\left\langle \nu_{t,x} \begin{vmatrix} \lambda - u & f(\lambda) - f(u) \\ f(\lambda) - f(u) & \int_u^\lambda (f'(s))^2 ds \end{vmatrix} \right\rangle$$

$$= \begin{vmatrix} \underbrace{\boxed{\langle \nu_{t,x}\ \lambda - u \rangle}}_{=0} & \langle \nu_{t,x}\ f(\lambda) - f(u) \rangle \\ \langle \nu_{t,x}, f(\lambda) - f(u) \rangle & \langle \nu_{t,x}, \int_u^\lambda (f'(s))^2 ds \rangle \end{vmatrix}$$

$$\boxed{\left\langle \nu_{t,x}, (\lambda - u)\int_u^\lambda (f'(s))^2 ds - (f(\lambda) - f(u))^2 \right\rangle + \langle \nu_{t,x}, f(\lambda) - f(u) \rangle^2 = 0}$$

$$(\lambda-u)\int_u^\lambda (f'(s))^2\,ds - (f(\lambda)-f(u))^2$$

$$= (\lambda-u)\int_u^\lambda \left(f'(s) - \frac{1}{\lambda-u}\int_u^\lambda f'(\tau)\,d\tau\right)^2 ds \geq 0.$$

$\Rightarrow$

$$\begin{cases} \langle \nu_{t,x},\ f(\lambda)-f(u) \rangle = 0 \\ \langle \nu_{t,x},\ \underbrace{(\lambda-u)\int_u^\lambda \left(f'(s) - \frac{1}{\lambda-u}\int_u^\lambda f'(\tau)\,d\tau\right)^2 ds}_{\|} \rangle = 0 \end{cases}$$

$$(\lambda-u)\int_u^\lambda \left(\int_u^\lambda f''(\lambda+\theta(s-\tau))(s-\tau)\,d\tau\right)^2 ds$$

$\Rightarrow$

- $\langle \nu_{t,x}, f(\lambda) \rangle = f(u(t,x))$
- If $f''(u) > 0$

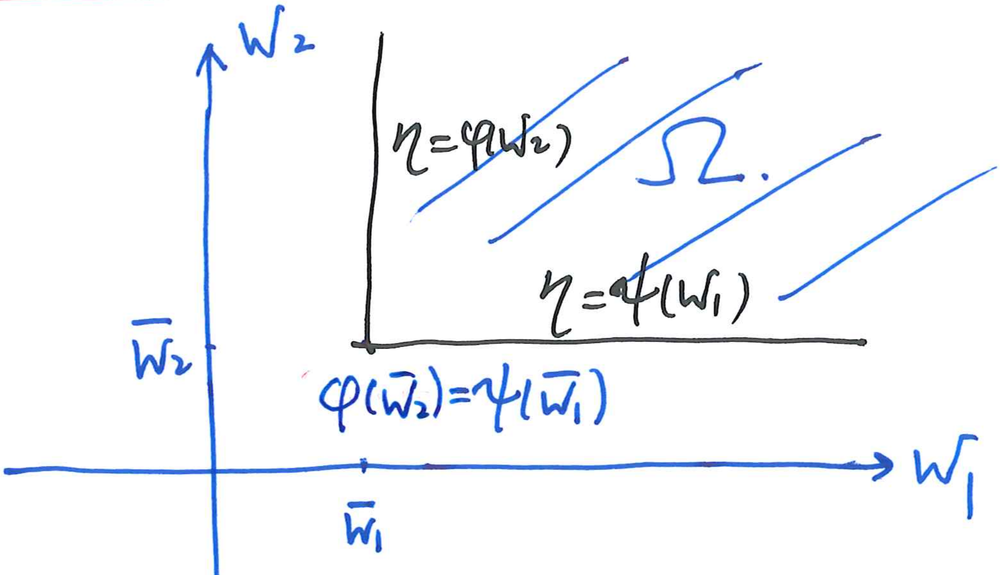
$\hookrightarrow$ $\boxed{\nu_{t,x} = \delta_{u(t,x)}}$

The Goursat Entropy Pairs for 2x2 Hyperbolic Systems of Conservation Laws

$$\eta_{w_1 w_2} - \frac{\lambda_{1w_2}}{\lambda_2-\lambda_1}\eta_{w_1} + \frac{\lambda_{2w_1}}{\lambda_2-\lambda_1}\eta_{w_2} = 0 \qquad (*)$$

$$q_{w_j} = \lambda_j \eta_{w_j} \qquad (**)$$

Goursat Problem for $(*)$



Well-posed !

Goursat Entropy Pairs

$\exists$ two families of entropy pairs:

$$\begin{cases} \eta_a(w) = I_1(w)\,a(w_1) + \int_{\bar{w}_1}^{w_1} J_1(\xi; w)\,a(\xi)\,d\xi \\ q_a(w) = K_1(w)\,a(w_1) + \int_{\bar{w}_1}^{w_1} L_1(\xi; w)\,a(\xi)\,d\xi \end{cases}$$

$$\begin{cases} \eta_b(w) = I_2(w)\,b(w_2) + \int_{\bar{w}_2}^{w_2} J_2(\xi; w)\,b(\xi)\,d\xi \\ q_b(w) = K_2(w)\,b(w_2) + \int_{\bar{w}_2}^{w_2} L_2(\xi; w)\,b(\xi)\,d\xi \end{cases}$$

where (I_i, J_i, K_i, L_i), $i = 1, 2$, are unique smooth functions and independent of $\bar{w}_1$ and $\bar{w}_2$:

$$\begin{cases} I_i(w) > 0, \quad \begin{cases} I_1(w_1, \bar{w}_2) = 1 \\ J_1(\xi; w_1, \bar{w}_2) = 0 \end{cases} \quad \begin{cases} I_2(\bar{w}_1, w_2) = 1 \\ J_2(\xi; \bar{w}_1, w_2) = 0 \end{cases} \\ K_i = \lambda_i I_i \\ \frac{\partial K_i(w)}{\partial w_1} + L_i(w_i; w) = \lambda_i(w)\left(\frac{\partial I_i(w)}{\partial w_i} + J(w_i; w)\right) \\ \frac{\partial L_i(\xi; w)}{\partial w_i} = \lambda_i(w)\,\frac{\partial J_i(\xi; w)}{\partial w_i} \\ \frac{\partial K_i(w)}{\partial w_j} = \lambda_j(w)\,\frac{\partial I_i(w)}{\partial w_j}, \qquad i \neq j \\ \frac{\partial L_i(\xi; w)}{\partial w_j} = \lambda_j(w)\,\frac{\partial J_i(\xi; w)}{\partial w_j} \end{cases}$$

Reduction of the Young Measure

<u>Thm</u>. If $\frac{\partial \lambda_j}{\partial w_j} \neq 0$, $j=1,2$ (Genuinely Nonlinear)

$$\Longrightarrow \quad \nu_{t,x} = \delta_{u(t,x)}$$

<u>Proof</u>. If $\nu_{t,x} \neq \delta_{u(t,x)}$, we denote

$[w_1^-, w_1^+] \times [w_2^-, w_2^+]$ the smallest rectangle containing $\operatorname{supp} \nu_{t,x}$.

1. <u>Claim</u>: If $w_1^- < w_1^+$, then $\exists\ c_1(t,x)$ s.t.

$$\langle \nu, q_a \rangle = c_1 \langle \nu, \eta_a \rangle$$

$\forall\ a \in C$, $a(w_1) = 0$ when $\begin{cases} w_1 \geq \bar{w}_1 \\ \text{or} \\ w_1 \leq \bar{w}_1 \end{cases}$ for $\bar{w}_1 \in (w_1^-, w_1^+)$

• Choose $\begin{cases} a_0(w_1) = (w_1 - w_1^*)_+, & w_1^* \geq \bar{w}_1, \quad |w_1^+ - w_1^*| \ll 1 \\ a(w_1) = 0, & w_1 \geq \bar{w}_1 \end{cases}$

$\hookrightarrow \begin{cases} \eta_{a_0}(w) > 0 & \forall\ w \in \{\bar{w}_1 < w_1 < w_1^+\} \cap \operatorname{supp} \nu \\ \eta_{a_0} q_a - \eta_a q_{a_0} \equiv 0 \end{cases}$

$\hookrightarrow \quad \langle \nu, \eta_{a_0} \rangle \langle \nu, q_a \rangle = \langle \nu, \eta_a \rangle \langle \nu, q_{a_0} \rangle$

$\hookrightarrow$ $$\langle \nu, \xi_a \rangle = c_1(\bar{w}_1) \langle \nu, \eta_a \rangle$$

$$\forall a \in C, \quad a(w_1) = 0, \quad w_1 \geq \bar{w}_1.$$

$$\left[c_1(\bar{w}_1) = \frac{\langle \nu, \xi_{a_0} \rangle}{\langle \nu, \eta_{a_0} \rangle} \right]$$

Similarly, $\forall a \in C$, $a(w_1) = 0$ when $w_1 \leq \bar{w}_1$,

$$\langle \nu, \xi_a \rangle = c_1(\bar{w}_1) \langle \nu, \eta_a \rangle$$

- <u>claim</u> $c_1(\bar{w}_1) \not\to \bar{w}_1$.

For any $\tilde{w}_1 \in (w_1^-, w_1^+)$, choose

$$\begin{cases} a_1(w_1) = 0 & w_1 \leq \tilde{w}_1 \\ a_2(w_1) = 0 & w_1 \geq \bar{w}_1 \end{cases} \quad \text{for } \tilde{w}_1 < \bar{w}_1$$

$$\begin{cases} a_1(w_1) = 0 & w_1 \geq \tilde{w}_1 \\ a_2(w_1) = 0 & w_1 \leq \bar{w}_1 \end{cases} \quad \text{for } \tilde{w}_1 > \bar{w}_1$$

$\Rightarrow$ $$\eta_{a_1} \xi_{a_2} - \eta_{a_2} \xi_{a_1} \equiv 0$$

$\hookrightarrow$ $$c_1(\bar{w}_1) = c_1(\tilde{w}_1)$$

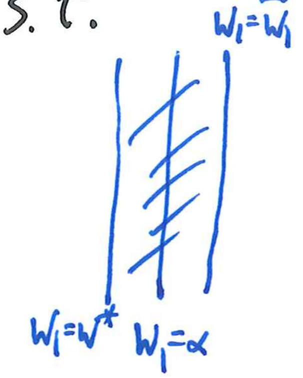
2. <u>claim</u> If $W_2^- < W_2^+$, then $\exists\ C_2(t,x)$ s.t.

$$\langle \nu, \xi_b \rangle = C_2 \langle \nu, \eta_b \rangle$$

$\forall b \in C$, $\quad b(W_2) = 0$ when $\begin{cases} W_2 \geq \overline{W_2} \\ \text{or} \\ W_2 \leq \overline{W_2} \end{cases}$ $\overline{W_2} \in (W_2^-, W_2^+)$.

3. $\forall\ \alpha \in (W_1^-, W_1^+)$, choose W_1^*, $\overline{W_1}$ s.t.

$$W_1^* < \alpha < \overline{W_1}, \qquad \overline{W_1} - W_1^* \ll 1$$



Choose (η_a, ξ_a): $\quad a(W_1) = (W_1 - W_1^*)_+$

Choose $(\eta_{\bar a}, \xi_{\bar a})$: $\quad \bar a(W_1) = (W_1 - \overline{W_1})_-$

$$\langle \nu, \eta_a \xi_{\bar a} - \eta_{\bar a} \xi_a \rangle = \langle \nu, \eta_a \rangle \langle \nu, \xi_{\bar a} \rangle - \langle \nu, \eta_{\bar a} \rangle \langle \nu, \xi_a \rangle$$
$$= 0$$

<u>We know</u> that, on $\{W_1 < W_1^*\} \cup \{W_1 > \overline{W_1}\}$,

$$\eta_a \xi_{\bar a} - \eta_{\bar a} \xi_a \equiv 0$$

On $\{ w_1^* \le w_1 \le \bar{w}_1 \}$.

$$\begin{cases} \eta_a(w) = I_1(w)(w_1 - w_1^*) + \frac{1}{2} J_1(\alpha; w)(w_1 - w_1^*)^2 + O(|w_1 - w_1^*|^3) \\ \zeta_a(w) = K_1(w)(w_1 - w_1^*) + \frac{1}{2} L_1(\alpha; w)(w_1 - w_1^*)^2 + O(|w_1 - w_1^*|^3) \end{cases}$$

$$\begin{cases} \eta_{\bar{a}}(w) = I_1(w)(w_1 - \bar{w}_1) - \frac{1}{2} J_1(\alpha; w)(w_1 - \bar{w}_1)^2 + O(|w_1 - w_1^*|^3) \\ \zeta_{\bar{a}}(w) = K_1(w)(w_1 - \bar{w}_1) - \frac{1}{2} L_1(\alpha; w)(w_1 - \bar{w}_1)^2 + O(|w_1 - w_1^*|^3) \end{cases}$$

$$\begin{aligned} &(\eta_a \zeta_{\bar{a}} - \eta_{\bar{a}} \zeta_a)(w) \\ &= \frac{1}{2}(\bar{w}_1 - w_1^*)(\bar{w}_1 - w_1)(w_1 - w_1^*)\left(I^2 \frac{\partial \lambda_1}{\partial w_1}\right)(\alpha, w_2) \\ &\quad + O\left((\bar{w}_1 - w_1^*)^2 (\bar{w}_1 - w_1)(w_1 - w_1^*)\right) \end{aligned}$$

$$\Rightarrow \quad \left\langle \nu, \ (\bar{w}_1 - w_1)_+ (w_1 - w_1^*)_+ \left(\underset{\neq 0}{\underline{\left(I^2 \frac{\partial \lambda_1}{\partial w_1}\right)(\alpha, w_2)}} + O(|\bar{w}_1 - w_1^*|) \right) \right\rangle = 0$$

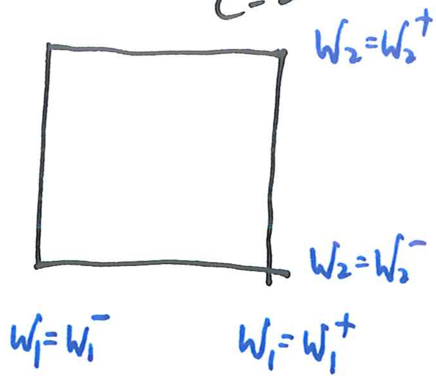
$$\Rightarrow \quad \operatorname{Supp} \nu \cap \{ w_1^* \le w_1 \le \bar{w}_1 \} = \phi \qquad \forall \ \begin{matrix} w_1^* < \bar{w}_1 \\ |\bar{w}_1 - w_1^*| \ll 1. \end{matrix}$$

$$\hookrightarrow \quad \operatorname{Supp} \nu \cap \{ w_1^- < w_1 < w_1^+ \} = \phi$$

Similarly.

$$\hookrightarrow \quad \operatorname{Supp} \nu \cap \{ w_2^- < w_2 < w_2^+ \} = \phi$$

4. If



$$\mathrm{Supp}\,\nu \cap \left(\{W_1 = W_1^{\pm}\} \cup \{W_2 = W_2^{\pm}\} \right) \neq \phi$$

for example

$$\mathrm{Supp}\,\nu \cap \{W_1 = W_1^-\} \neq \phi$$

$$\Rightarrow \quad \nu(\{W_1 = W_1^-\}) \neq 0$$

then we follow <u>Step 3</u> to choose

$$\alpha = W_1^-, \quad \bar{W}_1 = W_1^- + \varepsilon, \quad W_1^* = W_1^- - \varepsilon.$$

to conclude

$$\mathrm{Supp}\,\nu \cap \{W_1^- - \varepsilon \le W_1 \le W_1^- + \varepsilon\} = \phi$$

for sufficiently small $\varepsilon \geq 0$

$\Rightarrow$ Contradiction

5. <u>Conclusion</u>

$$\mathrm{Supp}\,\nu \cap \left([W_1^-, W_1^+] \times [W_2^-, W_2^+] \right) = \phi$$

$\Rightarrow$ Contradiction

$$\Longrightarrow \quad \nu_{t,x} = \delta_{u(t,x)} \quad \text{Single point support.}$$