Foundation Module: PDE Problems

October 2018

Instructions: Work out at least **5** of the following 6 problems. Please specify which are the 5 problems you want to be graded. Please work on these problems only by yourself.

1. Let $\Omega \subset R^n$ be open, bounded, smooth domain, and $f \in L^2(\Omega)$. Consider Poisson's problem:

$$(*) \qquad \begin{cases} -\Delta u = f & \text{for } x \in \Omega, \\ u|_{\partial\Omega} = g, \end{cases}$$

and the variational problem:

$$(**) \qquad I[u] = \min_{\omega \in \mathcal{A}} I[\omega],$$

with

$$I[\omega] \equiv \int_\Omega \Big(\frac{1}{2}|\nabla\omega|^2 - \omega f\Big)dx, \quad \mathcal{A} \equiv \{\omega \in C^2(\Omega) \cap C^1(\bar{\Omega}) \,:\, \omega|_{\partial\Omega} = g\}.$$

Assume $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$. Show that $u(x)$ solves Poisson's problem (*) if and only if $u(x)$ is a minimizer of the variational problem (**).

2. Let $\Omega \subset \mathbb{R}^n$ open and bounded. Consider the initial-boundary value problem:

$$\begin{cases} u_t - \Delta u = f & \text{for } x \in \Omega,\ t > 0, \\ u|_{\partial\Omega \times \{t>0\}} = g, \\ u|_{t=0} = u_0(x), \end{cases} \tag{1}$$

where f, g, and u_0 are given smooth functions depending only on $x \in \mathbb{R}^n$.

(a). Using the energy method to show the uniqueness of solutions in the class of functions in $C^{2,1}(\Omega \times (0,\infty)) \cap C^{1,0}(\overline{\Omega} \times [0,\infty))$.

(b). Consider the corresponding stationary problem:

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u|_{\partial\Omega} = g. \end{cases} \tag{2}$$

Using the energy method to show the uniqueness of solutions in the class of functions in $C^2(\Omega) \cap C^1(\overline{\Omega})$.

(c). For the unique solution $u(x,t)$ of problem (1) and the unique solution $u_*(x)$ of problem (2), show that there exists a constant $C > 0$ such that

$$\int_\Omega |u(x,t) - u_*(x)|^2 dx \le \int_\Omega |u_0(x) - u_*(x)|^2 dx\, e^{-Ct} \qquad \text{for } t > 0.$$

3. Let M be a first-order operator of the form:

$$M = \partial_t + a \cdot \nabla_x + b, \qquad x = (x_1, \cdots, x_n) \in \mathbb{R}^n,$$

where $a = (a_1, \cdots, a_n) \in R^n$ and $b \in R$ are constants. Consider the following mixed problem:

$$\begin{cases} u_{tt} - \Delta_x u = 0 & \text{for } x_1 > 0, t > 0, \\ u|_{t=0} = f(x), \quad u_t|_{t=0} = g(x) & \text{for } x_1 > 0, \\ Mu|_{x_1=0} = 0, & \text{for } t > 0, \end{cases}$$

where f, g vanish for all sufficiently small $x_1 > 0$ and are smooth. Prove that there exists a solution $u(x,t)$ provided $a_1 \leq 0$.

4. Let $a \in C(R)$. Show that the solution $u(x,t)$ of the Cauchy problem of the quasilinear partial differential equation

$$u_t + a(u)u_x = 0$$

with initial condition $u(x,0) = u_0(x)$ is given implicitly by

$$u(x,t) = u_0(x - a(u(x,t))t).$$

Show that the solution becomes singular for some positive t, unless $a(u_0(s))$ is a nondecreasing function of s.

5. Derive an explicit formula for the solution of the nonhomogeneous initial-boundary problem for the heat equation:

$$\begin{cases} u_t - u_{xx} = f(x,t) & \text{for } x > 0, \\ u|_{x=0} = g(t) & \text{for } t > 0, \\ u|_{t=0} = u_0(x) & \text{for } x \geq 0, \end{cases}$$

for smooth and compact supported functions f, g, and u_0. Hint: Divide the problem into three problems.

6. Consider Burgers' equation

$$u_t + (\frac{u^2}{2})_x = 0,$$

with initial data:

$$u|_{t=0} = \begin{cases} 1 & \text{for } x < -1, \\ 0 & \text{for } -1 < x < 0, \\ 2 & \text{for } 0 < x < 1, \\ 0 & \text{for } x > 1. \end{cases}$$

Construct the entropy solution for all $t > 0$.