Foundation Module Course

Introduction to Partial Differential Equations

Overview:

This is an introductory course on PDEs that are central to the other CDT courses. The course emphasizes rigorous treatment and analysis of PDEs through examples, representation formulas, and properties that can be understood by using relatively elementary mathematical tools and techniques.
Topics will include: The transport equation, the heat equation, the wave equation, Laplace's equation, conservation laws, and Hamilton-Jacobi equations.
Methods introduced through these topics will include: Method of characteristics, fundamental solutions and Green's functions, separation of variables, spherical means, Hadamard’s method of descent, energy methods, maximum principles, Duhamel's principle,, transform methods, asymptotics, numerical methods, and many more.

Recommended prerequisites include undergraduate-level advanced calculus, linear algebra, and ODE, and some exposure to complex analysis. Though this is an introductory course, it will move quickly and require considerable mathematical maturity.

Learning Outcomes:

Students will learn basic rigorous treatment and analysis of partial differential equations with emphasis on prototypical linear/nonlinear PDEs, as well as various techniques to represent solutions of these PDEs.

Synopsis:

1. Introduction

Brief history, definitions, terminologies, notations. Mathematical setup, challenges, well-posedness. Examples, connections to other areas in Mathematics and other sciences.

2. Revision of Ordinary Differential Equations

Local well-posedness for the Cauchy problem. Solutions in the large. Generalized solutions.....

3. Transport Equation

Method of characteristics, Cauchy problem, nonhomogeneous problem, explicit formulas of global solutions.

4. Laplace's Equation and Related Topics

Examples of problems leading to Laplace's equation, or closely related equations (including probability and fluid dynamics). Mean-value property, maximum principle, Hanack inequality, regularity of solutions. Fundamental solution, separation of variables, Green's function, Neumann function. Energy methods, uniqueness, variational principles. Some non-constant-coefficient and nonlinear problems that can be done using similar techniques.

5. Heat Equation and Related Topics

Cauchy problem, initial-boundary value problem. Uniqueness via energy argument and via maximum principle. Fundamental solution, separation of variables. Solution formulas for the Cauchy problem, half-space problems, and bounded domains. Mean-value formula, properties of solutions, infinite propagation speeds, regularity. Some nonconstant-coefficient and nonlinear problems that can be done using similar techniques.

6. Linear Wave Equation and Related Topics

Derivation of the wave equation from Newton's second law. Energy methods, uniqueness, domain of dependence. D'Alembert's formula for 1D. Spherical means, Euler-Poisson-Darboux equation. Formula of the solution for odd dimension by the method of spherical means, Kirchhoff's formula for 3D, Huygens's principle. Formula of the solution for even dimension by Hadamard's method of descent, Poisson's formula for 2D. Nonhomogeneous problem, Duhamel's principle. Some non-constant-coefficient and nonlinear problems that can be done using similar techniques.

7. Introduction to Nonlinear First-Order PDE

Examples, quasilinear equations, method of characteristics, local well-posedness, Non-characteristics surfaces, *Cauchy-Kowalewsky theorem.

8. Introduction to Conservation Laws and Related Topics

Examples of problems leading to scalar conservation laws and systems of conservation laws (eg. traffic flow, shallow water flow, gas flow). Method of characteristics, shock formation, shocks, entropy condition, entropy solutions, Lax-Oleinik formula and uniqueness of entropy solutions, Riemann¡¯s problem, large time behaviour.

9. *Other Topics: Hamilton-Jacobi equations; the method of characteristics; transform methods; asymptotics,...

Examples of problems leading to Hamilton-Jacobi equations (optimal control, interface motion laws). Link to scalar conservation laws in 1D setting. Hopf-Lax solution formula. Brief discussion of viscosity solutions. Solution by the method of characteristics. Basic numerical schemes. Connections with material discussed earlier.

*Optional

Preparatory Reading:

1. W. Strauss, Partial Differential Equations: An Introduction, John Wiley and Sons, 1992.

The best undergraduate-level text I know. Many of the topics we will discuss are present in Strauss, at least in some measure, with an exposition that may be more accessible, especially if your PDE background is weak. I strongly recommend reading the relevant sections of this book alongside the more sophisticated texts below. (Amazon's price: about $70.)

2. H. Weinberger, A First Course in Partial Differential Equations, with Complex Variables and Transform Methods, Dover, 1965.

The first half is a lot like Strauss: A discussion of PDE making heavy use of separation of variables, but also emphasizing that there's much more to the theory than that. The second half is a good, application-oriented introduction to complex variables. Feels a little dated by now, but this material hasn't changed since 1965 and you can't beat the price. (Amazon's price: about $15.)

Core Reading:

1. L.-C. Evans, Partial Differential Equations, American Mathematical Society, 2nd edition, 2010. Chapters 2 (Four important linear pde), 3 (Nonlinear first-order pde), and 4 (Other ways to represent solutions) are largely at the level of this class. Then the book continues with more advanced material (at the level of PDE II). Evans is especially good for mathematical aspects of scalar conservation laws, for the link between optimal control and Hamilton-Jacobi equations, and for material on viscosity solutions. (Amazon's price: about $65.)

2. F. John, Partial Differential Equations, 4th edition, Springer-Verlag 1982. The exposition is a model of clarity. Downloadable for free from within the nyu.edu domain from the website http://link.springer.com/book/10.1007/978-1-4684-0059-5/page/1

(The same page will also offer you the opportunity to order an inexpensive soft-cover edition.)

Further Reading:

1. **Jürgen Jost: Partial Differential Equations**, Springer-Verlag: New York, 2002.

2. **R. Courant and D. Hilbert, Methods of Mathematical Physics I-II**, John Wiley & Sons, 1953, 1989.

3. **Michael Taylor, Partial Differential Equations, Vols. 1-3,** Springer-Verlag: New York, 1996.

4. **M. Renardy and R. Rogers, An Introduction to Partial Differential Equations,** 2nd edition, Springer-Verlag, 2004.

5. **G. Folland, Introduction to Partial Differential Equations**, 2nd edition, Princeton University Press, 1995.

6. **J. Ockendon, S. Howison, A. Lacey, and A. Movchan, Applied Partial Differential Equations**, Oxford University Press, Revised Edition 2003.

7. **J. Kevorkian, Partial Differential Equations: Analytical Solution Techniques**, 2nd edition, Springer-Verlag, 1999.

8. **R. Guenther and J. Lee, Partial Differential Equations of Mathematical Physics and Integral Equations**, Dover, 1996.

9. **Q. Han, A Basic Course in Partial Differential Equations**, American Mathematical Society, 2011.

10. **P. Garabedian, Partial Differential Equations, 2nd revised edn**, AMS Chelsea Publishing, 1998.

11. **R. McOwen**, **Partial Differential Equations, 2nd edn,** Prentice Hall, 2003

12. **E. DiBenedetto, Partial Differential Equations, 2nd edn**, Birkhäuser, 2010