PDE-CDT Core Course

Analysis of Partial Differential Equations-Part III

EPSRC Centre for Doctoral Training in
Partial Differential Equations
Trinity Term
1 May – 19 June 2019 (16 hours; Wednesdays)
Final Exam: 26 June 2019 (Wednesday)
Course format: Teaching Course (TT)

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1. Introduction

Overview: Theory of Hyperbolic PDEs

is a large subject, which has close connections with the other areas of mathematics including

Analysis, Differential Geometry, Topology, Mechanics, Relativity, Mathematical Physics, …

Besides its mathematical importance, it has a wide range of applications in

Engineering, Physics, Biology, Economics, …

- Backbone of the year: Introduction to most facets of the theory of hyperbolic PDEs and related PDEs: Features, methods, approaches, connections, ….
- Knowledge with PDE Foundation module and Analysis of PDEs (Parts I-II) is desirable

Synopsis - I:

1. Introduction

Part I: Hyperbolic Systems of First-Order Equations

2. Linear Theory:

Spaces involving time;
Hyperbolic systems of first-order equations, examples;
Weak solutions, well-posedness;
Vanishing viscosity method, energy methods,
Fourier transform method.

3. **Nonlinear Theory I - Multi-D Scalar Conservation Laws**:

L^1 - well-posedness theory, test function methods,
vanishing viscosity method;
*Other methods (numerical methods, kinetic method, relaxation method, the layering method, …);
*Further results (compactness, regularity, decay, trace, structure).

Synopsis - II:

4. **Nonlinear Theory II – One-Dimensional Systems of Conservation Laws:**
   Riemann problem, Cauchy problem;
   Elementary waves: shock waves, rarefaction waves,
   contact discontinuities;
   Lax entropy conditions;
   Glimm scheme, front-tracking, BV solutions;
   Compensated compactness, entropy analysis, L^p solutions,
   vanishing viscosity methods;
   *Uniqueness and continuous dependence; ...

5. ***Nonlinear Theory III - Multidimensional Systems of Conservation Laws:**
   Basic features/phenomena (re-visit);
   Local existence and stability; formation of singularities;
   Discontinuities and free boundary problems;
   Stability of shock waves, rarefaction waves, vortex sheets,
   entropy waves.

Synopsis - III:

Part II: Second-Order Wave Equations

6. Energy estimates and local existence, Galerkin method;
   Global existence of semi-linear wave equations with small data
   (Quasilinear case could be similarly treated);
   Lower regularity results for large data;
   *Littlewood-Paley theory and Strichartz estimates.

* Optional

References:

1. **P. D. Lax:** **Hyperbolic Systems of Conservation Laws and the Mathematical Theory of Shock Waves**. CBMS, SIAM, 1973.

2. **C. M. Dafermos:** **Hyperbolic Conservation Laws in Continuum Physics,** 4th Edition. Springer-Verlag: Berlin, 2016.

3. **L. C. Evans:** **Partial Differential Equations,** 2nd Edition. AMS: Providence, RI, 2010.

4. **L. Hormander:** **Lectures on Nonlinear Hyperbolic Differential Equations,** Springer-Verlag: Berlin-Heidelberg, 1997

5. **D. Serre:** **Systems of Conservation Laws**, **Vols. I, II**, Cambridge University Press: Cambridge, 1999, 2000.

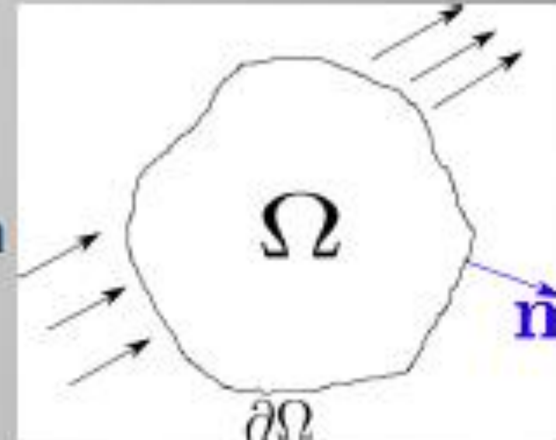
6. **C. D. Sogge:** **Lectures on Nonlinear Wave Equations,** 2nd edition. International Press, Boston, MA, 2008.

7. **G.-Q. Chen, M. Feldman: The Mathematics of Shock Reflection-Diffraction and Von Neumann's Conjectures**, Princeton University Press, 2019.

8. **P. D. Lax:** **Hyperbolic Differential Equations,** AMS: Providence, 2000.

9. **Bressan, G.-Q. Chen, M. Lewicka, D. Wang:** **Nonlinear Conservation Laws and Applications,** IMA Volume 153, Springer: New York, 2011.

Conservation Laws:



Rate of change of the total amount of certain quantity contained in a fixed region Ω

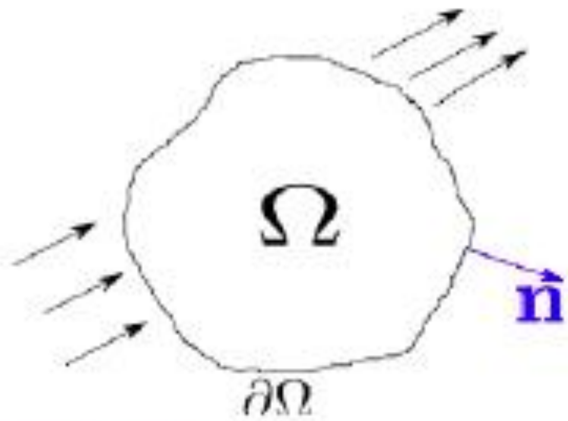
= Flux of this quantity across the boundary $\partial\Omega$ of the region

- The amount of such a quantity in any region can be measured by accounting for how much of it is currently present and how much of it enters or leaves the region in any fixed period of time.

Examples: Three Fundamental Laws of Nature

- Conservation Laws of Mass and Energy: Mass and Energy can be neither created nor destroyed.
- Conservation Law of Momentum: The total momentum of a closed system of objects remains constant through time.

Conservation Laws



Rate of Change of the Total Amount of
Certain Quantity in a Fixed Region Ω
= Flux of the Quantity across the Boundary $\partial\Omega$.

Conservation Law via Calculus

$$\Longleftrightarrow \qquad \frac{d}{dt}\int_{\Omega} u\,d\mathbf{x} = -\int_{\partial\Omega} \mathbf{f}\cdot\mathbf{n}\,dS$$

u – Density of the Quantity; $\mathbf{f}$ – Flux of the Quantity
$\mathbf{n}$ – Outward Normal to Ω; dS – Surface Element on $\partial\Omega$

Calculus Manipulations $\Longrightarrow$

$$\partial_t u + \nabla_{\mathbf{x}}\cdot\mathbf{f}(u) = 0$$

Physical Systems with $m \geq 2$ Quantities – Density Functions

$\Longrightarrow$ **Systems of Conservation Laws**:

$$\partial_t\,\mathbf{u} + \nabla\cdot\mathbf{f}(\mathbf{u}) = 0$$

$$\mathbf{u} = (u_1, \cdots, u_m)^\top \qquad \mathbf{f}(\mathbf{u}) = (\mathbf{f}_1(\mathbf{u}), \cdots, \mathbf{f}_d(\mathbf{u}))$$

Euler Equations for Compressible Fluids

$$
\begin{cases}
\partial_t \rho + \nabla_{\mathbf{x}} \cdot (\rho \mathbf{v}) = 0 & \text{(conservation of mass)} \\
\partial_t(\rho \mathbf{v}) + \nabla_{\mathbf{x}} \cdot (\rho \mathbf{v} \otimes \mathbf{v}) + \nabla_{\mathbf{x}} p = 0 & \text{(conservation of momentum)} \\
\partial_t(\frac{1}{2}\rho|\mathbf{v}|^2 + \rho e) + \nabla_{\mathbf{x}} \cdot ((\frac{1}{2}\rho|\mathbf{v}|^2 + \rho e + p)\mathbf{v}) = 0 & \text{(conservation of energy)}
\end{cases}
$$

Leonhard Euler

Constitutive Relations: $p = p(\rho, e)$

- ρ – density, $\mathbf{v} = (v_1, v_2, v_3)^\top$ – fluid velocity
- p – pressure, e – internal energy

***Govern the Flows when Convective Motions Dominate Diffusion/Dispersion, …**

e.g. , shock waves in Gases, Elastic Fluids, Shallow Water, ……

Poisson, Challis, Stokes, Kelvin, Rayleigh, Airy, Earnshaw, Riemann, Rankine, Christoffel, Mach, Clausius, Kirchhoff, Gibbs, Hugoniot, Duhem, Hadamard, Jouguet, Zamplen, Weber, Taylor, Becker, Bethe, Weyl, von Neumann, Courant, Friedrichs, ……

George Stokes

Euler Equations for Potential Flow

$$\begin{cases} \partial_t \rho + \nabla_{\mathbf{x}} \cdot (\rho \nabla_{\mathbf{x}} \Phi) = 0, & \text{(Conservation of mass)} \\ \partial_t \Phi + \frac{1}{2}|\nabla_{\mathbf{x}} \Phi|^2 + \frac{\rho^{\gamma-1}}{\gamma-1} = \frac{B_\infty}{\gamma-1} := \frac{\rho_\infty^{\gamma-1} + \frac{\gamma-1}{2} u_\infty^2}{\gamma-1}, & \text{(Bernoulli's law)} \end{cases}$$

for $\gamma > 1$ or, equivalently,

$$\partial_t \rho(\partial_t \Phi, \nabla_{\mathbf{x}} \Phi, B_\infty) + \nabla_{\mathbf{x}} \cdot \big(\rho(\partial_t \Phi, \nabla_{\mathbf{x}} \Phi, B_\infty) \nabla_{\mathbf{x}} \Phi\big) = 0,$$

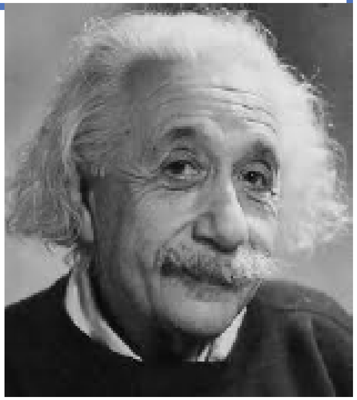
with

$$\rho(\partial_t \Phi, \nabla_{\mathbf{x}} \Phi, B_\infty) = \big(B_\infty - (\gamma - 1)(\partial_t \Phi + \tfrac{1}{2}|\nabla_{\mathbf{x}} \Phi|^2)\big)^{\frac{1}{\gamma-1}}.$$

- Aerodynamics/Gas Dynamics: Fundamental PDE
- The potential flow equations and the full Euler equations coincide or are close each other in many important physical situations

  J. Hadamard: Leçons sur la Propagation des Ondes,
  Hermann: Paris 1903

Conservation Laws and Einstein Equations



$$\mathbf{G}_{\mu\nu} = 8\pi \mathbf{T}_{\mu\nu}$$

$T_{\mu\nu}$ – Stress-energy tensor (Energy-momentum tensor)
$G_{\mu\nu}$ – Einstein tensor (Function of the metric)

These equations, with the geodesic equation, form the core of the mathematical formulation of General Relativity

Structure of the Einstein Equations

$\Longrightarrow$ **Conservation Laws of Energy and Momentum:**

$$\nabla_b \mathbf{T}^{ab} = \mathbf{T}^{ab}{}_{;b} = 0$$

CALCULUS OF VARIATIONS

A Field of Mathematics that deals with extremizing functionals, as opposed to ordinary calculus which deals with functions:

$$\boxed{I[w] = \int_\Omega L(\nabla_x w(x), w(x), x)dx}$$

- Energy or Action Functionals in Physics/Engineering/industry....
- Distance/Metric Functional in Optics (light),
  Geometry (geodesics, minimal surfaces, ...), ...
- Cost Functionals in Optimization (controls, games, image processing, design,
  finance, transportation, ...), ...

POINT: Seek a Minimizer or Critical Point u of $I[\cdot]$:

$$\boxed{I'[u]=0}$$

Great Progress has been made in the recent four decades...

Conservation Laws and Calculus of Variations

- Systems of Euler-Lagrange Equations

- Noether's Theorem:

Any Differentiable Symmetry of the Action of a Physical System Has a Corresponding Conservation Law.

Any Invariance of the Variational Integral I[w] leads to a corresponding Conservation Law for the critical point of I[·]

Hyperbolicity

$$\partial_t \mathbf{u} + \nabla \cdot \mathbf{f}(\mathbf{u}) = 0, \quad \mathbf{u} \in \mathbb{R}^m, \ \mathbf{x} \in \mathbb{R}^d$$

Plane Wave Solutions: $\mathbf{u}(t, \mathbf{x}) = \mathbf{w}(t, \omega \cdot \mathbf{x})$

$\mathbf{w}(t, \xi)$ is determined by: $\partial_t \mathbf{w} + (\nabla_{\mathbf{w}} \mathbf{f}(\mathbf{w}) \cdot \omega)\, \partial_\xi \mathbf{w} = 0$

?? **Existence** of stable plane wave solutions ??

Hyperbolicity in D: For any $\omega \in S^{d-1}$, $\mathbf{u} \in D$,

$$(\nabla_{\mathbf{u}} \mathbf{f}(\mathbf{u}) \cdot \omega)_{m \times m}\, \mathbf{r}_j(\mathbf{u}, \omega) = \lambda_j(\mathbf{u}, \omega)\, \mathbf{r}_j(\mathbf{u}, \omega), \ 1 \le j \le m$$

$\lambda_j(\mathbf{u}, \omega)$ **are real**

Main Features:

Finiteness of Propagation Speeds;

Discontinuities of Solutions, $\cdots\cdots$

Well-Posedness: Existence, Uniqueness, Stability, $\cdots$

Hadamard's Example: $\mathbf{u} \in \mathbb{R}^2$, $x \in \mathbb{R}$

Cauchy Problem: $$\begin{cases} \partial_t \mathbf{u} + \partial_x \mathbf{f}(\mathbf{u}) = 0 \\ \mathbf{u}|_{t=0} = \begin{pmatrix} \frac{\sin(nx)}{n} \\ 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{as } n \to \infty. \end{cases}$$

$$\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad \mathbf{f}(\mathbf{u}) = \begin{pmatrix} u_2 \\ -u_1 \end{pmatrix}, \quad \nabla \mathbf{f}(\mathbf{u}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \lambda_j = (-1)^j \sqrt{-1}$$

Well-Posedness: Expect that **the Limit Solution $\mathbf{u}^0(t,x) \equiv 0$**

Fixed $n > 0$, **the Unique Solution:**

$$\mathbf{u}^n(t,x) = \left(\frac{\sin(nx)(e^{nt} + e^{-nt})}{2n}, \frac{\cos(nx)(e^{nt} - e^{-nt})}{2n}\right)^\top$$

$$\overline{\lim_{n\to\infty}} |\mathbf{u}^n(t,x) - \mathbf{u}^0(t,x)| = \infty \qquad \text{for } t > 0 \text{ for any reasonable topology}$$

***Small Changes in the Data $\Longrightarrow$ Large Changes in the Solutions**

***Mapping: "Data Space" $\to$ "Solutions Space" Is Not Continuous**

UNSTABLE!!

The Simplest Example

Scalar Conservation Laws

$$\partial_t u + \nabla \cdot \mathbf{f}(u) = 0, \qquad u \in \mathbb{R},\ \mathbf{x} \in \mathbb{R}^d$$

$$\mathbf{f} : \mathbb{R} \to \mathbb{R}^d$$

Then

$$\lambda(u,\omega) = \mathbf{f}'(u) \cdot \omega, \qquad r(u,\omega) \equiv 1$$

$\Longrightarrow$ **Scalar conservation laws are always hyperbolic**

*** This is not the case for Systems, or High-Order Equations**