Part I: Hyperbolic Systems of First-Order Equations

2. Linear Theory

Spaces of Functions

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pages 253–309

Space H^{-1} and H_0^1

$$H_0^1(\Omega) = \overline{C_0^\infty(\Omega)}^{H^1}$$

$$\| $$

$$\left\{ u \;\middle|\; \begin{array}{l} \exists\, u_k \in C_0^\infty(\Omega) \text{ s.t.} \\ \|(u_k - u,\ Du_k - Du)\|_{L^2} \xrightarrow{k\to\infty} 0 \end{array} \right\}$$

$$H_0^1(\mathbb{R}^n) = H^1(\mathbb{R}^n)$$

$$H^{-1}(\Omega) = \left(H_0^1(\Omega)\right)^* \quad \text{Dual space}$$

$$\begin{cases} f \in H^{-1}(\Omega) \iff |\langle f, u\rangle| < \infty \quad \forall\, u \in H_0^1(\Omega). \\ \|f\|_{H^{-1}(\Omega)} := \sup\left\{ \langle f, u\rangle \;\middle|\; u \in H_0^1(\Omega),\ \|u\|_{H_0^1(\Omega)} \le 1 \right\} \end{cases}$$

Thm (Characterization of H^{-1})

$f \in H^{-1}(\Omega)$

$\Rightarrow$ (i) $\exists\, f^0, f^1, \cdots, f^n \in L^2(\Omega)$ s.t.

$$(*) \quad \langle f, v \rangle = \int_\Omega \left(f^0 v + \sum_{i=1}^n f^i v_{x_i} \right) dx$$

$$\forall\, v \in H_0^1(\Omega).$$

(ii) $$\|f\|_{H^{-1}(\Omega)} = \inf \left\{ \left(\int_\Omega \sum_{i=0}^n |f^i|^2 dx \right)^{1/2} \,\middle|\, f \text{ satisfies } (*) \text{ for } f^0, \dots, f^n \in L^2(\Omega) \right\}$$

Notation We write

$$f = f^0 - \underbrace{\sum_{i=1}^n \frac{\partial}{\partial x_i} f^i}$$

whenever $(*)$ holds.

Weak derivatives

Spaces Involving Time

X — real Banach space
with norm $\|\cdot\|_X$

Space $L^p(0,T;X)$, $1 \le p \le \infty$

$u : [0,T] \to X$ measurable functions

$$\begin{cases} \|u\|_{L^p(0,T;X)} := \left(\int_0^T \|u(t)\|_X^p \, dt \right)^{\frac{1}{p}} < \infty & 1 \le p < \infty \\ \|u\|_{L^\infty(0,T;X)} := \operatorname*{ess\,sup}_{0 \le t \le T} \|u(t)\|_X < \infty \end{cases}$$

Space $C([0,T];X)$:

$u : [0,T] \to X$ continuous function

$$\|u\|_{C([0,T];X)} := \max_{0 \le t \le T} \|u(t)\|_X < \infty$$

Sobolev Space $W^{1,p}(0,T;X)$

$$\| $$

$$\left\{ u \in L^p(0,T;X) \,\middle|\, \|u\|_{W^{1,p}(0,T;X)} < \infty \right\}$$

$$\|u\|_{W^{1,p}(0,T;X)} := \begin{cases} \left(\int_0^T \left(\|u(t)\|_X^p + \|u'(t)\|_X^p \right) dt \right)^{\frac{1}{p}} & 1 \le p < \infty \\ \operatorname*{ess\,sup}_{0 \le t \le T} \left(\|u(t)\|_X + \|u'(t)\|_X \right) & p = \infty \end{cases}$$

$v = u' \in L^1(0,T;X)$ is the weak derivative of $u \in L^1(0,T;X)$ provided

$$\int_0^T \phi'(t)\, u(t)\, dt = -\int_0^T \phi(t)\, v(t)\, dt$$

$$\forall \phi \in C_c^\infty(0,T)$$

Thm $u \in W^{1,p}(0,T;X)$ for some $1 \le p \le \infty$

$\Longrightarrow$

(i) $u \in C([0,T];X)$

(after possibly being redefined on a set of measure zero)

(ii) $u(t) = u(s) + \int_s^t u'(\tau)\,d\tau$

for all $0 \le s \le t \le T$.

(iii) $\max_{0 \le t \le T} \|u(t)\|_X \le \underbrace{C}_{\text{only } T.} \|u\|_{W^{1,p}(0,T;X)}$

Thm $\begin{cases} u \in L^2(0,T; H_0^1(\Omega)) \\ u' \in L^2(0,T; H^{-1}(\Omega)) \end{cases}$

$\Rightarrow$

(i) $u \in C([0,T]; L^2(\Omega))$

(after possibly being modified on a set of measure zero)

(ii) $\begin{cases} \text{The mapping } t \longmapsto \|u(t)\|^2_{L^2(\Omega)} \text{ absolutely continuous} \\ \dfrac{d}{dt}\|u(t)\|^2_{L^2(\Omega)} = 2\langle u'(t), u(t)\rangle \quad \text{a.e. } 0 \le t \le T \end{cases}$

(iii) $\max_{0\le t\le T} \|u(t)\|_{L^2(\Omega)}$

$$\le \underbrace{C}_{\text{only } T}\left(\|u\|_{L^2(0,T;H_0^1(\Omega))} + \|u'\|_{L^2(0,T;H^{-1}(\Omega))}\right)$$

Thm (Mapping into Better Spaces)

$$\begin{cases} \Omega \text{ open, bdd; } \partial\Omega \text{ smooth} \\ m \text{ — nonnegative integer} \\ u \in L^2(0,T; H^{m+2}(\Omega)) \\ u' \in L^2(0,T; H^m(\Omega)) \end{cases}$$

$\Rightarrow$

(i) $u \in C([0,T]; H^{m+1}(\Omega))$
(after possibly being redefined on a set of measure zero)

(ii) $$\max_{0\le t\le T} \|u(t)\|_{H^{m+1}(\Omega)} \le C\left(\|u\|_{L^2(0,T;H^{m+2}(\Omega))} + \|u'\|_{L^2(0,T;H^m(\Omega))}\right)$$

Exercise

Hyperbolic Systems of First-Order Equations

$$(*)\begin{cases} U_t + \sum_{j=1}^{n} B_j U_{x_j} = f, & \mathbb{R}^n \times (0,\infty) \\ U|_{t=0} = g \end{cases}$$

- $U = (u^1, \cdots, u^m)^T \in \mathbb{R}^m$
- $B_j = B_j(x,t)$ —— $m \times m$ Matrices.
- $f : \mathbb{R}^n \times [0,\infty) \longrightarrow \mathbb{R}^m$ Given
- $g : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ Given.

$$\boxed{U_t + \sum_{j=1}^{n} B_j(x,t)\, U_{x_j} = f}$$

<u>Hyperbolicity</u>: For each $y \in \mathbb{R}^n$

the $m \times m$ Matrix

$$\boxed{B(x,t;y) = \sum_{j=1}^{n} y_j B_j(x,t)}$$

is $\boxed{\text{Diagonalizable}}$ for each $x \in \mathbb{R}^n$, $t \geq 0$

$\Rightarrow$ For each x, y, t,

$B(x,t;y)$ has m real eigenvalues

$$\lambda_1(x,t;y) \leq \lambda_2(x,t;y) \leq \cdots \leq \lambda_m(y)$$

and corresponding m linearly independent eigenvectors $\{r_j(x,t;y)\}_{j=1}^{m}$

* No hypothesis concerning $\{r_j(x,t;y)\}_{j=1}^{m}$

$$\boxed{u_t + \sum_{j=1}^{n} B_j(x,t)\, u_{x_j} = f}$$

Symmetric Hyperbolic Systems:

For each $x \in \mathbb{R}^n$, $t \geq 0$,

$B_j(x,t)$, $j=1,\cdots,m$, are all symmetric $m\times m$ matrices

Strict Hyperbolicity:

For each $x, y \in \mathbb{R}^n$, $y \neq 0$, $t \geq 0$.

$$\boxed{B(x,t;y) = \sum_{j=1}^{n} y_j B_j(x,t)}$$

has m distinct real eigenvalues:

$$\lambda_1(x,t;y) < \lambda_2(x,t;y) < \cdots < \lambda_m(x,t;y)$$

Motivation (B_j constant, $f \equiv 0$)

Look for a Plane Wave Solution with the form

$$\boxed{u(x,t) = w(y \cdot x - \sigma t)\, r(y), \quad x \in \mathbb{R}^n, t \geq 0}$$

for some direction $y \in \mathbb{R}^n$, velocity $\frac{\sigma}{|y|}$ ($\sigma \in \mathbb{R}$)

profile $w : \mathbb{R} \to \mathbb{R}$, where $r(y) \in \mathbb{R}^m$

$$\Rightarrow \boxed{\underbrace{w'(y \cdot x - \sigma t)}_{\neq 0} \Big(-\sigma I + \underbrace{\sum_{j=1}^{n} y_j B_j}_{= B(y)}\Big) r(y) = 0}$$

$\Rightarrow$ $r(y)$ is an eigenvector of $B(y)$ corresponding to the eigenvalue σ.

$\hookrightarrow$ The $\boxed{\text{Hyperbolicity Condition}}$ requires that

$\exists$ m distinct plane wave solutions for each direction y

with form

$$\Big\{ (y \cdot x - \lambda_k(y) t)\, r_k(y) \Big\}_{k=1}^{m}.$$

* The eigenvalues for $|y| = 1$ are called the wave speeds

Systems of First-Order Symmetric Hyperbolic PDE with Variable Coefficients

$$(*)\begin{cases} u_t + \sum_{j=1}^{n} B_j u_{x_j} = f & \mathbb{R}^n \times (0,T) \\ u|_{t=0} = g \end{cases}$$

- The matrices $B_j(x,t)$ are symmetric.

  $j = 1, 2, \cdots, n,\quad x \in \mathbb{R}^n,\quad 0 \le t \le T.$

- $B_j \in C^2$ with

  $$|B_j| + |D_{x,t} B_j| + |D^2_{x,t} B_j| \le C,$$

  $j = 1, 2, \cdots, n$

- $\begin{cases} g \in H^1(\mathbb{R}^n; \mathbb{R}^m) \\ f \in H^1(\mathbb{R}^n \times (0,T); \mathbb{R}^m). \end{cases}$

Weak Solutions

Bilinear Form: $u, v \in H^1(\mathbb{R}^n;\mathbb{R}^m)$

$$\boxed{B[u,v;t] = \int_{\mathbb{R}^n} \sum_{j=1}^{n} (B_j u_{x_j}) \cdot v \, dx}$$

$0 \le t \le T.$

Definition. We say $u \in L^2(0,T; H^1(\mathbb{R}^n;\mathbb{R}^m))$

with $u' \in L^2(0,T; L^2(\mathbb{R}^n;\mathbb{R}^m))$

is a weak solution of $(*)$, provided

(i) $\boxed{(u', v) + B[u,v;t] = (f, v)}$

Inner product in $L^2(\mathbb{R}^n;\mathbb{R}^m)$

$\forall\, v \in H^1(\mathbb{R}^n;\mathbb{R}^m)$

a.e. $0 \le t \le T.$

(ii) $u(0) = g.$

Thm in Spaces involving time

$\hookrightarrow$ $u \in C([0,T]; L^2(\mathbb{R}^n;\mathbb{R}^m))$

$\Rightarrow$ The initial condition (ii) makes sense

Vanishing Viscosity Method

Construct approximate solutions

$$u^\varepsilon = u^\varepsilon(x,t)$$

by the parabolic system

$$(\ast)\begin{cases} u^\varepsilon_t - \varepsilon \Delta u^\varepsilon + \sum_{j=1}^n B_j u^\varepsilon_{x_j} = f & \mathbb{R}^n \times (0,T) \\ u^\varepsilon|_{t=0} = g^\varepsilon \equiv \eta_\varepsilon * g \end{cases}$$

Ideas.

1. $\forall \varepsilon > 0$, $(\ast)$ has a unique smooth solution $u^\varepsilon = u^\varepsilon(x,t)$ s.t.

$$|u^\varepsilon| \longrightarrow 0 \quad \text{as } |x| \to \infty.$$

2. ? $u^\varepsilon \longrightarrow u \quad \varepsilon \to 0$?

$\llcorner$ Weak solution

? Which sense

Heat Equation

$$\begin{cases} w_t - \Delta w = f \\ w\big|_{t=0} = 0 \end{cases}$$

$$\Rightarrow \begin{cases} w(x,t) = \int_0^t \int_{\mathbb{R}^n} \frac{1}{(4\pi(t-s))^{\frac{n}{2}}} e^{-\frac{|x-y|^2}{4(t-s)}} f(y,s)\,dy \\ \qquad\qquad x \in \mathbb{R}^n, \quad t > 0 \\ \max_{0 \le t \le T} \|w(t)\|_{H^1(\mathbb{R}^n;\mathbb{R}^m)} \\ \qquad \le C \|f\|_{L^2(0,T;L^2(\mathbb{R}^n;\mathbb{R}^m))}. \\ w \in L^2(0,T;H^2(\mathbb{R}^n;\mathbb{R}^m)) \\ w' \in L^2(0,T;L^2(\mathbb{R}^n;\mathbb{R}^m)) \end{cases}$$

Formal: $\begin{cases} u \text{ is smooth} \\ u \to 0 \text{ as } |x|\to\infty \text{ sufficiently rapidly.} \end{cases}$

$$\int_{\mathbb{R}^n} f^2\,dx = \int_{\mathbb{R}^n} (u_t - \Delta u)^2\,dx$$

$$= \int_{\mathbb{R}^n} \left(u_t^2 - 2\Delta u\, u_t + (\Delta u)^2\right) dx$$

$$= \int_{\mathbb{R}^n} \Big(u_t^2 + \underbrace{2Du\cdot Du_t}_{= \frac{d}{dt}(|Du|^2)} + (\Delta u)^2 \Big)\, dx.$$

$$\int_0^t \int_{\mathbb{R}^n} 2Du\cdot Du_t\, dx\,ds = \int_{\mathbb{R}^n} |Du|^2 dx \Big|_{s=0}^{s=t}$$

$$\int_{\mathbb{R}^n} (\Delta u)^2 dx = \sum_{i,j=1}^n \int_{\mathbb{R}^n} u_{x_i x_i} u_{x_j x_j}\, dx$$

$$= -\sum_{i,j=1}^n \int_{\mathbb{R}^n} u_{x_i x_i x_j} u_{x_j}\, dx$$

$$= \sum_{i,j=1}^n \int_{\mathbb{R}^n} u_{x_i x_j} u_{x_i x_j} = \int_{\mathbb{R}^n} |D^2 u|^2 dx$$

$\Longrightarrow$

$$\sup_{0\le t\le T} \int_{\mathbb{R}^n} |Du|^2 dx + \int_0^T \int_{\mathbb{R}^n} \left(u_t^2 + |D^2 u|^2\right) dx\,dt$$

$$\le C \int_0^T \int_{\mathbb{R}^n} f^2\, dx\,dt.$$

Thm (Existence of Approximate Solutions)

For each $\varepsilon > 0$, $\exists\, 1$ u^ε of ($\ast$) with

$$(\mathrm{V}) \begin{cases} u^\varepsilon \in L^2(0,T;\ H^3(\mathbb{R}^n;\mathbb{R}^m)) \\ u^{\varepsilon\prime} \in L^2(0,T;\ H^1(\mathbb{R}^n;\mathbb{R}^m)) \end{cases}$$

Proof

1. Set $X = L^\infty(0,T;\ H^1(\mathbb{R}^n;\mathbb{R}^m))$.

For each $v \in X$, consider the Cauchy problem

$$\begin{cases} \underline{u_t - \varepsilon \Delta u} = \underline{f - \sum_{j=1}^n B_j v_{x_j}} \in L^2(0,T;\ L^2(\mathbb{R}^n;\mathbb{R}^m)) \\ u|_{t=0} = g^\varepsilon \end{cases}$$

$$\Longrightarrow \exists\, 1 \text{ Solution } \begin{cases} u \in L^2(0,T;\ H^2(\mathbb{R}^n;\mathbb{R}^m)) \\ u' \in L^2(0,T;\ L^2(\mathbb{R}^n;\mathbb{R}^m)) \end{cases}$$

Similarly, let $\tilde{v} \in X$ and let $\tilde{u}$ solve

$$\begin{cases} \tilde{u}_t - \varepsilon \Delta \tilde{u} = f - \sum_{j=1}^n B_j \tilde{v}_{x_j} & \mathbb{R}^n \times (0,T) \\ \tilde{u}|_{t=0} = g^\varepsilon \end{cases}$$

2. Set $\hat{u} = u - \tilde{u}$, $\hat{v} = v - \tilde{v}$

$$\Rightarrow \begin{cases} \hat{u}_t - \varepsilon \Delta \hat{u} = -\sum_{j=1}^{n} B_j \hat{v}_{x_j} & \mathbb{R}^n \times (0,T) \\ \hat{u}|_{t=0} = 0 \end{cases}$$

Linear Theory

$$\hookrightarrow \max_{0 \le t \le T} \| \hat{u}(t) \|_{H^1(\mathbb{R}^n; \mathbb{R}^m)}$$

$$\le C(\varepsilon) \left\| \sum_{j=1}^{n} B_j \hat{v}_{x_j} \right\|_{L^2(0,T; L^2(\mathbb{R}^n; \mathbb{R}^m))}$$

$$\le C(\varepsilon) \| \hat{v} \|_{L^2(0,T; H^1(\mathbb{R}^n; \mathbb{R}^m))}$$

$$\le C(\varepsilon) T^{1/2} \max_{0 \le t \le T} \| \hat{v}(t) \|_{H^1(\mathbb{R}^n; \mathbb{R}^m)}.$$

$$\Rightarrow \quad \| \hat{u} \|_X \le C(\varepsilon) T^{1/2} \| \hat{v} \|_X$$

3. If T is so small that

$$C(\varepsilon) T^{1/2} \le \frac{1}{2}.$$

$\overset{2.}{\Longrightarrow}$ $$\|u - \tilde{u}\|_X \le \frac{1}{2} \|v - \tilde{v}\|_X.$$

Banach Fixed Point Thm

$\hookrightarrow$ The mapping $v \longmapsto u$

has a unique fixed pt.

$\Longrightarrow$ $u = u^\varepsilon$ solves $(*)$ if $C(\varepsilon) T^{1/2} = \frac{1}{2}$.

If $T > \frac{1}{2} C$. then we choose $T_1 C(\varepsilon)^2 = \frac{1}{4}$.

and repeat the above argument on the intervals

$[0, T_1]$, $[T_1, 2T_1]$, ...

the assertion (v) follows from the Regularity theory of the nonhomogeneous heat equation.

Energy Estimates Uniform in $\varepsilon > 0$

$\underline{\varepsilon \to 0}$? We need some uniform estimates.

<u>Thm (Energy Estimates).</u> $\exists\, C \sim n, B_j$ s.t.

$$\max_{0\le t\le T} \|u^\varepsilon(t)\|_{H^1(\mathbb{R}^n;\mathbb{R}^m)} + \|u^{\varepsilon\prime}\|_{L^2(0,T;\,L^2(\mathbb{R}^n;\mathbb{R}^m))}$$

$$\le C\left(\|g\|_{H^1(\mathbb{R}^n;\mathbb{R}^m)} + \|f\|_{L^2(0,T;\,H^1(\mathbb{R}^n;\mathbb{R}^m))} + \|f'\|_{L^2(0,T;\,L^2(\mathbb{R}^n;\mathbb{R}^m))} \right).$$

for each $\varepsilon > 0$.

<u>Proof</u>

1. $$\frac{d}{dt}\left(\frac{1}{2}\|u^\varepsilon\|^2_{L^2(\mathbb{R}^n;\mathbb{R}^m)}\right) = (u^\varepsilon, u^\varepsilon_t)$$

$$= \Big(u^\varepsilon,\ f - \sum_{j=1}^n B_j u^\varepsilon_{x_j} + \varepsilon \Delta u^\varepsilon\Big).$$

$$= (u^\varepsilon, f) - \underbrace{\Big(u^\varepsilon, \sum_{j=1}^n B_j u^\varepsilon_{x_j}\Big)} + \underbrace{\varepsilon(u^\varepsilon, \Delta u^\varepsilon)}_{= -\varepsilon \|Du^\varepsilon\|^2_{L^2(\mathbb{R}^n;\mathbb{R}^m)} \le 0}$$

$$|(u^\varepsilon, f)| \le \|u^\varepsilon\|^2_{L^2(\mathbb{R}^n;\mathbb{R}^m)} + \|f\|^2_{L^2(\mathbb{R}^n;\mathbb{R}^m)}$$

2. If $v \in C_c^\infty(\mathbb{R}^n;\mathbb{R}^m)$.

$$\left(v, \sum_{j=1}^n B_j v_{x_j}\right) = \int_{\mathbb{R}^n} (B_j v_{x_j})\cdot v\,dx$$

$$\parallel$$

$$\int_{\mathbb{R}^n} \left((B_j v)_{x_j}\cdot v - (B_{j,x_j} v)\cdot v \right) dx$$

$$\parallel$$

$$\int_{\mathbb{R}^n} ((B_j v)\cdot v)_{x_j}\,dx - \int_{\mathbb{R}^n} (B_{j,x_j} v)\cdot v\,dx$$

$$\underline{- \int_{\mathbb{R}^n} (B_j v)\cdot v_{x_j}\,dx}$$

$$\parallel \; B_j \text{ symmetric}$$

$$\boxed{- \int_{\mathbb{R}^n} (B_j v_{x_j})\cdot v\,dx} = 0$$

$$\Rightarrow \left(v, \sum_{j=1}^n B_j v_{x_j}\right) = \boxed{\frac{1}{2}\int_{\mathbb{R}^n} ((B_j v)\cdot v)_{x_j}\,dx} - \frac{1}{2}\int_{\mathbb{R}^n} (B_{j,x_j} v)\cdot v\,dx$$

$$\Rightarrow \left|\left(v, \sum_{j=1}^n B_j v_{x_j}\right)\right| \le \frac{1}{2}\int_{\mathbb{R}^n} |(B_{j,x_j} v)\cdot v|\,dx$$

$$\le C\|v\|^2_{L^2(\mathbb{R}^n;\mathbb{R}^m)}$$

By Approximation

$$\left|\left(u^\varepsilon, \sum_{j=1}^n B_j u^\varepsilon_{x_j}\right)\right| \le C\|u^\varepsilon\|^2_{L^2(\mathbb{R}^n;\mathbb{R}^m)}$$

3. ① + ②

$$\hookrightarrow \frac{d}{dt}\left(\|u^\varepsilon\|^2_{L^2(\mathbb{R}^n;\mathbb{R}^m)}\right)$$

$$\leq C\left(\|u^\varepsilon\|^2_{L^2(\mathbb{R}^n;\mathbb{R}^m)} + \|f\|^2_{L^2(0,T;L^2(\mathbb{R}^n;\mathbb{R}^m))}\right)$$

$$\xrightarrow[\text{Ineq.}]{\text{Gronwall}} \max_{0\leq t\leq T}\|u^\varepsilon(t)\|^2_{L^2(\mathbb{R}^n;\mathbb{R}^m)}$$

$$\leq C\left(\underbrace{\|g^\varepsilon\|^2_{L^2(\mathbb{R}^n;\mathbb{R}^m)}}_{\substack{\wedge\!\|\\ \|g\|^2_{L^2(\mathbb{R}^n;\mathbb{R}^m)}}} + \|f\|^2_{L^2(0,T;L^2(\mathbb{R}^n;\mathbb{R}^m))}\right)$$

$$\leq C\left(\|g\|^2_{L^2(\mathbb{R}^n;\mathbb{R}^m)} + \|f\|^2_{L^2(0,T;L^2(\mathbb{R}^n;\mathbb{R}^m))}\right).$$

4. Fix $k \in \{1, 2, \cdots, n\}$.

Write $v^k = u^\varepsilon_{x_k}$

$$\Rightarrow \begin{cases} (v^k)_t - \varepsilon \Delta v^k + \sum_{j=1}^n B_j v^k_{x_j} \\ \qquad = f_{x_k} - \sum_{j=1}^n B_{j,x_k} u^\varepsilon_{x_j}, \quad \mathbb{R}^n \times (0,T). \\ v^k|_{t=0} = g^\varepsilon_{x_k} \end{cases}$$

Reasoning as above, we find

$$(\blacksquare)_k \quad \frac{d}{dt}\left(\|v^k\|^2_{L^2(\mathbb{R}^n;\mathbb{R}^m)}\right)$$
$$\le C\left(\|Du^\varepsilon\|^2_{L^2(\mathbb{R}^n;\mathbb{R}^m)} + \|Df\|^2_{L^2(\mathbb{R}^n; M^{m\times n})}\right).$$

$$\xrightarrow{\sum_{k=1}^n (\blacksquare)_k} \frac{d}{dt}\left(\|Du^\varepsilon\|^2_{L^2(\mathbb{R}^n;\mathbb{R}^m)}\right)$$
$$\le C\left(\|Du^\varepsilon\|^2_{L^2(\mathbb{R}^n;\mathbb{R}^m)} + \|Df\|^2_{L^2(0,T;L^2(\mathbb{R}^n;M^{m\times n}))}\right)$$

$$\xrightarrow[\text{Ineq.}]{\text{Gronwall}} \max_{0\le t\le T} \|Du^\varepsilon(t)\|^2_{L^2(\mathbb{R}^n;M^{m\times n})}$$
$$\le C\left(\|Dg\|^2_{L^2(\mathbb{R}^n;M^{m\times n})} + \|Df\|^2_{L^2(0,T;L^2(\mathbb{R}^n;M^{m\times n}))}\right)$$

5. Next, Set $v = u^{\varepsilon'}$

$$\begin{cases} v_t - \varepsilon \Delta v + \sum_{j=1}^{n} B_j v_{x_j} = f_t - \sum_{j=1}^{n} B_{j,t} u^{\varepsilon}_{x_j}, & \mathbb{R}^n \times (0,T), \\ v|_{t=0} = f - \sum_{j=1}^{n} B_j g^{\varepsilon}_{x_j} + \varepsilon \Delta g^{\varepsilon}, \end{cases}$$

$\Longrightarrow$

$$\max_{0 \le t \le T} \| u^{\varepsilon'}(t) \|^2_{L^2(\mathbb{R}^n; \mathbb{R}^m)}$$

$$\le C \left(\| Dg \|^2_{L^2(\mathbb{R}^n; \mathbb{R}^m)} + \varepsilon^2 \| D^2 g^{\varepsilon} \|^2_{L^2(\mathbb{R}^n; \mathbb{R}^m)} + \underbrace{\| f(0) \|^2_{L^2(\mathbb{R}^n; \mathbb{R}^m)}}_{\wedge\!\backslash} + \| f' \|^2_{L^2(0,T; L^2(\mathbb{R}^n; \mathbb{R}^m))} \right)$$

$$C \left(\| f \|^2_{L^2(0,T; L^2(\mathbb{R}^n; \mathbb{R}^m))} + \| f' \|^2_{L^2(0,T; L^2(\mathbb{R}^n; \mathbb{R}^m))} \right).$$

$$\| D^2 g^{\varepsilon} \|^2_{L^2(\mathbb{R}^n; \mathbb{R}^m)} \le \frac{C}{\varepsilon^2} \| Dg \|^2_{L^2(\mathbb{R}^n; M^{m \times n})}.$$

$$\uparrow$$

$$g^{\varepsilon} = \eta_{\varepsilon} * g$$

$\square$

Existence of Weak Solutions

Thm. $\exists$ a weak solution of $(*)$.

Proof.

1. **Energy Estimates**

$\hookrightarrow$ $\exists$ $\begin{cases} \varepsilon_k \to 0 \\ u \in L^2(0,T; H^1(\mathbb{R}^n;\mathbb{R}^m)) \\ \quad \text{with } u' \in L^2(0,T; L^2(\mathbb{R}^n;\mathbb{R}^m)) \end{cases}$

s.t.

$$\begin{cases} u^{\varepsilon_k} \rightharpoonup u & \text{weakly in } L^2(0,T; H^1(\mathbb{R}^n;\mathbb{R}^m)) \\ u^{\varepsilon_k\prime} \rightharpoonup u' & \text{---} \quad L^2(0,T; L^2(\mathbb{R}^n;\mathbb{R}^m)) \end{cases}$$

2. Choose $v \in C^1([0,T]; H^1(\mathbb{R}^n;\mathbb{R}^m))$.

$\hookrightarrow$

$$\int_0^T (u^{\varepsilon\prime}, v)\,dt + \int_0^T B[u^\varepsilon, v; t]\,dt = \int_0^T (f, v)\,dt + \varepsilon\int_0^T (\Delta u^\varepsilon, v)\,dt \tag{V-1}$$

$\underline{\varepsilon = \varepsilon_k \to 0}$

$$\hookrightarrow \int_0^T (u', v)\,dt + \int_0^T B[u, v; t]\,dt = \int_0^T (f, v)\,dt$$

Valid for all $v \in C([0,T]; H^1(\mathbb{R}^n; \mathbb{R}^m))$. (V-2).

$$\Longrightarrow (u', v) + B[u, v; t] = (f, v)$$

a.e. t for each $v \in H^1(\mathbb{R}^n; \mathbb{R}^m)$.

3. Assume now $v(T) = 0$

$$(V\text{-}1) \Longrightarrow -\int_0^T (u^\varepsilon, v')\,dt + \int_0^T B[u^\varepsilon, v; t]\,dt = \int_0^T (f, v)\,dt + \varepsilon \int_0^T (\Delta u^\varepsilon, v)\,dt + (g^\varepsilon, v(0))$$

$\underline{\varepsilon = \varepsilon_k \to 0}$

$$\hookrightarrow -\int_0^T (u, v')\,dt + \int_0^T B[u, v; t]\,dt = \int_0^T (f, v)\,dt + (g, v(0))$$

(V-2)

$\hookrightarrow$
$$-\int_0^T (u, v')\,dt + \int_0^T B[u, v; t]\,dt = \int_0^T (f, v)\,dt + (u(0), v(0))$$

$$\Longrightarrow (g, v(0)) = (u(0), v(0)) \quad \forall\, v(0) \in H^1(\mathbb{R}^n; \mathbb{R}^n)$$

$$\Longrightarrow \quad u(0) = g.$$

Uniqueness of Weak Solutions

Thm A weak solution is unique.

Proof. It suffices to show that the only weak solution with $f = g = 0$ is $u \equiv 0$.

Definition of Weak solutions

$$(u', v) + B[u, v; t] = 0 \quad \forall v \in H^1(\mathbb{R}^n; \mathbb{R}^m)$$

for a.e. $0 \le t \le T$.

Choose $v = u$

$\hookrightarrow$

$$\underbrace{(u', u)}_{\frac{d}{dt}\left(\|u(t)\|^2_{L^2(\mathbb{R}^n; \mathbb{R}^m)}\right)} + \underbrace{B[u, u; t]}_{\ge C\|u(t)\|^2_{L^2(\mathbb{R}^n; \mathbb{R}^m)}} = 0$$

$$\Rightarrow \frac{d}{dt}\left(\|u(t)\|^2_{L^2(\mathbb{R}^n; \mathbb{R}^m)}\right) \le C\|u(t)\|^2_{L^2(\mathbb{R}^n; \mathbb{R}^m)}.$$

$$\xrightarrow[\text{Ineq.}]{\text{Gronwall}} \quad \|u(t)\|^2_{L^2(\mathbb{R}^n; \mathbb{R}^m)} = 0 \qquad 0 \le t \le T.$$

Remarks

1. The methods developed above also apply more general systems with the form:

$$
(**)\begin{cases} B_0 U_t + \sum_{j=1}^{n} B_j U_{x_j} = f \\ B_j(x,t) \text{ are symmetric}, \quad j = 0, 1, \cdots, n. \end{cases}
$$

2. Symmetric hyperbolic systems of form (**) generalize the second-order hyperbolic PDE.

$$
V_{tt} - \sum_{i,j=1}^{n} a_{ij} V_{x_i x_j} = 0, \qquad a_{ij} = a_{ji}
$$

$$
U = (u^1, \cdots, u^{n+1}) = (V_{x_1}, V_{x_2}, \cdots, V_{x_n}, V_t)
$$

$$
B_0 = \begin{pmatrix} a_{11} & \cdots & a_{1n} & 0 \\ & \ddots & & \vdots \\ a_{1n} & \cdots & a_{nn} & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix}_{(n+1)\times(n+1)}
$$

$$
B_j = \begin{pmatrix} 0 & \cdots & 0 & -a_{1j} \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & -a_{nj} \\ -a_{1j} & \cdots & -a_{nj} & 0 \end{pmatrix}_{(n+1)\times(n+1)} \qquad j = 1, \cdots, n.
$$

Hyperbolic Systems With Constant Coefficients

$$\boxed{U_t + \sum_{j=1}^{n} B_j U_{x_j} = 0 \qquad \mathbb{R}^n \times [0,\infty)}$$

- $U = (U^1, \cdots, U^m)^T \in \mathbb{R}^m$
- B_j — $m\times m$ Constant Matrices

Hyperbolicity: For each $y \in \mathbb{R}^n$ the $m\times m$ Matrix $\boxed{B(y) := \sum_{j=1}^{n} y_j B_j}$ is Diagonalizable

$\Rightarrow$ $B(y)$ has m real eigenvalues

$$\lambda_1(y) \le \lambda_2(y) \le \cdots \le \lambda_m(y)$$

and corresponding m linearly independent eigenvectors $\{r_j(y)\}_{j=1}^m$

---

* No hypothesis concerning $\{r_j(y)\}_{j=1}^m$
* No assumption of Symmetry for $\{B_j\}_{j=1}^m$

Fourier Transform Methods

Fourier Transform on L^1: $u \in L^1(\mathbb{R}^n)$

$$\hat{u}(y) := \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-ix\cdot y} u(x)\, dx, \quad y \in \mathbb{R}^n$$

The Inverse Fourier Transform

$$\check{u}(y) := \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{ix\cdot y} u(x)\, dx, \quad y \in \mathbb{R}^n$$

? on L^2 ?

Thm (Plancherel's Thm): $u \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$

$$\Rightarrow \begin{cases} \hat{u},\ \check{u} \in L^2(\mathbb{R}^n) \\ \|\hat{u}\|_{L^2(\mathbb{R}^n)} = \|\check{u}\|_{L^2(\mathbb{R}^n)} = \|u\|_{L^2(\mathbb{R}^n)} \end{cases}$$

Proof.

1. Basic Facts

(i) $v, w \in L^1(\mathbb{R}^n) \Rightarrow \hat{v}, \hat{w} \in L^\infty(\mathbb{R}^n)$

(ii) $\int_{\mathbb{R}^n} v(x)\,\hat{w}(x)\,dx = \int_{\mathbb{R}^n} \hat{v}(y)\,w(y)\,dy$

(iii) $\int_{\mathbb{R}^n} e^{ix\cdot y - t|x|^2}\,dx = \left(\frac{\pi}{t}\right)^{\frac{n}{2}} e^{-\frac{|y|^2}{4t}}$, $t>0$

$\hookrightarrow$ For $v_\varepsilon(x) = e^{-\varepsilon|x|^2}$, $\varepsilon > 0$

$$\Rightarrow \quad \hat{v}_\varepsilon(x) = \frac{e^{-\frac{|y|^2}{4\varepsilon}}}{(2\varepsilon)^{\frac{n}{2}}}$$

2. For $u \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. Set $v(x) := \bar{u}(-x)$.

Define $w := u * v \in L^1(\mathbb{R}^n) \cap C(\mathbb{R}^n)$.

$$\hat{w} = (2\pi)^{\frac{n}{2}}\,\hat{u}\,\hat{v} \in L^\infty(\mathbb{R}^n). \quad \text{(check!)}$$

Also

$$\hat{v}(y) = \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n} e^{-ix\cdot y}\,\bar{u}(-x)\,dx = \overline{\hat{u}(y)}$$

$$\Rightarrow \quad \hat{w} = (2\pi)^{\frac{n}{2}}\,|\hat{u}|^2$$

3. W is continuous

$$\Rightarrow \lim_{\varepsilon\to 0}\int_{\mathbb{R}^n} \underbrace{\Phi_\varepsilon(x)}_{\| } W(x)\,dx = (2\pi)^{\frac{n}{2}} W(0)$$

$$\Phi_\varepsilon(x) = \frac{1}{(2\varepsilon)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4\varepsilon}} = \hat{v}_\varepsilon(x)$$

$$\int_{\mathbb{R}^n} \underbrace{\Phi_\varepsilon(x)}\,dx = (2\pi)^{\frac{n}{2}}$$

$$\Phi_\varepsilon(x) \to (2\pi)^{\frac{n}{2}}\delta_0(x)$$

(ii) $\Rightarrow$

$$\int_{\mathbb{R}^n} \hat{W}(y)\underbrace{e^{-\varepsilon|y|^2}}_{= v_\varepsilon(y)}\,dy = \int_{\mathbb{R}^n} W(y)\,\hat{v}_\varepsilon(y)\,dy$$

$$\downarrow \varepsilon\to 0 \qquad\qquad \downarrow \varepsilon\to 0$$

$$\int_{\mathbb{R}^n} \hat{W}(y)\,dy = (2\pi)^{\frac{n}{2}} W(0)$$

$\|$ $\hat{W}$ is summable

$$(2\pi)^{\frac{n}{2}} W(0) = (2\pi)^{\frac{n}{2}}\int_{\mathbb{R}^n} u(x)\,\bar{u}(-x)\,dx = (2\pi)^{\frac{n}{2}}\int_{\mathbb{R}^n}|u|^2\,dx$$

$$\int_{\mathbb{R}^n} \hat{W}(y)\,dy = (2\pi)^{\frac{n}{2}}\int_{\mathbb{R}^n}|\hat{u}|^2\,dy$$

$$(2\pi)^{\frac{n}{2}}\int_{\mathbb{R}^n}|\hat{u}|^2\,dy = (2\pi)^{\frac{n}{2}}\int_{\mathbb{R}^n}|u|^2\,dx$$

* The proof for $\check{u}$ is similar.

Plancherel's Thm

↳ Definition of Fourier Transform on L^2

For $u \in L^2(\mathbb{R}^n)$. Choose $\{u_k\}_{k=1}^{\infty} \subset L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$

s.t. $u_k \longrightarrow u$ in $L^2(\mathbb{R}^n)$

$\Longrightarrow$

$$\|\hat{u}_k - \hat{u}_j\|_{L^2(\mathbb{R}^n)} = \|\widehat{u_k - u_j}\|_{L^2(\mathbb{R}^n)}$$

$$= \|u_k - u_j\|_{L^2(\mathbb{R}^n)} \xrightarrow{k, j \to \infty} 0$$

$\Longrightarrow \exists\ \hat{u} \in L^2$ s.t.

$$\hat{u}_k \longrightarrow \hat{u} \quad \text{in } L^2(\mathbb{R}^n)$$

Define

$$\boxed{\hat{u} = \lim_{k \to \infty} \hat{u}_k \quad \text{in } L^2(\mathbb{R}^n)}.$$

$\not\sim$ choice of $\{u_k\}_{k=1}^{\infty}$

$\Downarrow$

$\boxed{\text{* Similarly for } \check{u}}$ $\boxed{\text{Well-defined}}$

Useful Properties of Fourier Transform

$u, v \in L^2(\mathbb{R}^n)$

$\Rightarrow$ (i) $\int_{\mathbb{R}^n} u \bar{v} \, dx = \int_{\mathbb{R}^n} \hat{u} \bar{\hat{v}} \, dy$

(ii) $\widehat{D^\alpha u} = (iy)^\alpha \hat{u}$

for each multiindex α such that $D^\alpha u \in L^2(\mathbb{R}^n)$

(iii) If $u, v \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$

$\Rightarrow (u * v)^\wedge = (2\pi)^{\frac{n}{2}} \hat{u} \hat{v}$

(iv) $(\hat{u})^\vee = u$

Exercise

Space $H^k(\mathbb{R}^n)$, k - integer

$$H^k(\mathbb{R}^n) = \left\{ u \in L^2(\mathbb{R}^n) \,\middle|\, \underline{(1+|y|^k)\,\hat{u}(y) \in L^2(\mathbb{R}^n)} \right\}$$

$$\Updownarrow$$

$$\boxed{\begin{array}{c} D^\alpha u \in L^2(\mathbb{R}^n) \\ \forall\, |\alpha| \le k. \end{array}}$$

Extension to

Space $H^s(\mathbb{R}^n)$: $0 < s < \infty$

$$H^s(\mathbb{R}^n) = \left\{ u \in L^2(\mathbb{R}^n) \,\middle|\, (1+|y|^s)\,\hat{u}(y) \in L^2(\mathbb{R}^n) \right\}.$$

Norm:

$$\|u\|_{H^s(\mathbb{R}^n)} := \|(1+|y|^s)\,\hat{u}\|_{L^2(\mathbb{R}^n)}$$

$\hookrightarrow$ $H^s(\mathbb{R}^n)$, $0<s<\infty$, are Hilbert spaces

Cauchy Problem

$$\begin{cases} U_t + \sum_{j=1}^{n} B_j U_{x_j} = 0 & \mathbb{R}^n \times (0,\infty) \\ U|_{t=0} = g : \mathbb{R}^n \to \mathbb{R}^m & \text{given} \end{cases}$$

Thm $g \in H^k(\mathbb{R}^n; \mathbb{R}^m)$ for $k > \frac{n}{2} + m$

$\Rightarrow$ The Cauchy problem has a unique solution

$U \in C^1([0,\infty); \mathbb{R}^m)$.

Proof

1. First assume u is a smooth solution

Apply the Fourier Transform in x for fixed t

$$\hat{u} = (\hat{u}^1, \cdots \hat{u}^m)^T$$

$$\Longrightarrow \begin{cases} \hat{u}_t + iB(y)\hat{u} = 0, & \mathbb{R}^n \times (0,\infty) \\ \hat{u}|_{t=0} = \hat{g} \end{cases}$$

$$\Longrightarrow \hat{u}(y,t) = e^{-itB(y)}\hat{g}(y)$$

$y \in \mathbb{R}^n$, $t \geq 0$.

$$\Longrightarrow \quad (*) \quad \boxed{u(x,t) = \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n} e^{ix\cdot y} e^{-itB(y)}\hat{g}(y)\,dy \qquad x \in \mathbb{R}^n,\ t \geq 0}$$

* Necessary Condition $\xrightarrow{\text{If Verified}}$ Uniqueness.

2. We have derived formula $(*)$ under the assumption that u is a smooth solution.

We now verify that, in fact, the function u defined by $(*)$ is a solution when $g \in H^k(\mathbb{R}^n; \mathbb{R}^m)$

$$\Downarrow$$

$$\boxed{\begin{array}{l} \exists\, f \in L^2(\mathbb{R}^n; \mathbb{R}^m) \quad \text{s.t.} \\ |\hat{g}(y)| \le C\,(1+|y|^k)^{-1}\,|f(y)|, \quad y \in \mathbb{R}^n \end{array}}$$

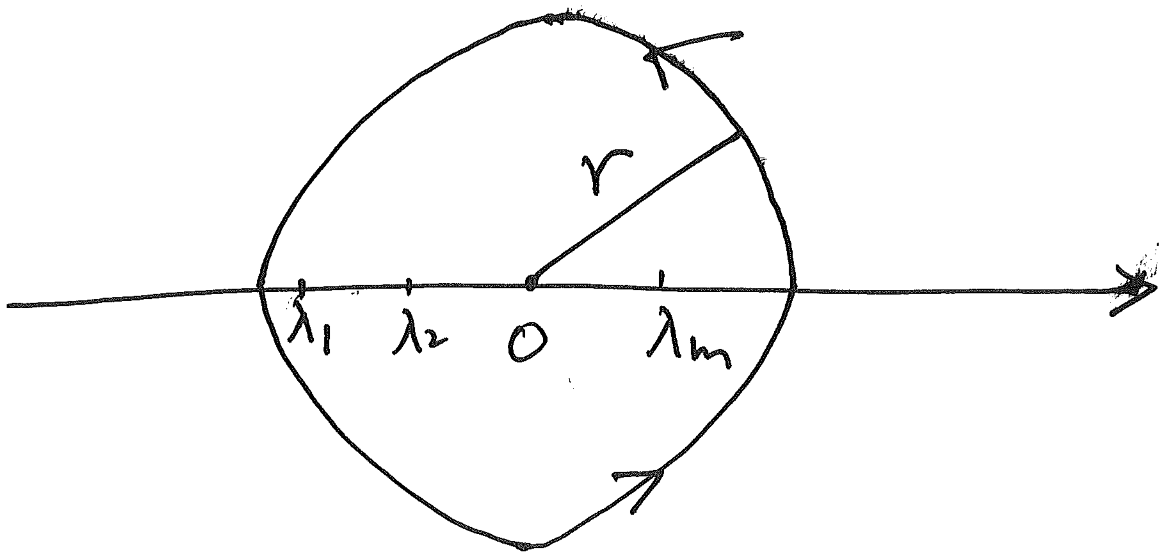
Main Point:

Estimate $\|e^{-itB(y)}\|$

$\hookrightarrow$ Convergence of Integral in $(*)$.

3. Fix $y \in \mathbb{R}^n$

Claim: $e^{-itB(y)} = \underbrace{\frac{1}{2\pi i}\int_{\Gamma} e^{-itz}(zI - B(y))^{-1}dz}_{\displaystyle = A(t,y)}$



$\Gamma = \partial B(0,r)$, $r >> 1$.

?
$$\begin{cases} \left(\frac{d}{dt} + iB(y)\right)A(t,y) = 0 \\ A(0,y) = I \end{cases}$$

$\forall\, x \in \mathbb{R}^m$

$$B(y)A(t,y)x = \frac{1}{2\pi i}\int_\Gamma e^{-it\zeta}\underbrace{B(y)}_{(B(y)-\zeta I)+\zeta I}(\zeta I - B(y))^{-1}x\,d\zeta$$

$$= \frac{1}{2\pi i}\int_\Gamma e^{-it\zeta}\left(\zeta(\zeta I - B(y))^{-1}x - x\right)d\zeta$$

$$\boxed{\int_\Gamma e^{-it\zeta}\,d\zeta = 0}$$

$$= \frac{1}{2\pi i}\int_\Gamma e^{-it\zeta}\,\zeta\,(\zeta I - B(y))^{-1}x\,d\zeta$$

$$= -\frac{1}{i}\frac{d}{dt}A(t,y)x. \qquad \forall\, x \in \mathbb{R}^m$$

$$\Rightarrow \left(\frac{d}{dt} + iB(y)\right)A(t,y) = 0$$

$\forall\, x \in \mathbb{R}^m$

$$A(0,y)x = \frac{1}{2\pi i}\int_\Gamma \underbrace{(\zeta I - B(y))^{-1}}_{\frac{1}{\zeta}I + \frac{B(y)(\zeta I - B(y))^{-1}}{\zeta}} x \, d\zeta$$

$$= x + \frac{1}{2\pi i}\int_\Gamma \frac{B(y)(\zeta I - B(y))^{-1}x}{\zeta} d\zeta$$

Set $w := (\zeta I - B(y))^{-1}x$

$$\Rightarrow \quad |w|\Big|_\Gamma \le \frac{c}{|\zeta|}, \quad c \sim x, y$$

(Exercise).

$$\Rightarrow \frac{1}{2\pi i}\int_\Gamma \frac{B(y)(\zeta I - B(y))^{-1}x}{\zeta} d\zeta \xrightarrow[r\to\infty]{} 0$$

$$\Rightarrow A(0,y)x = x$$

$$\Rightarrow A(0,y) = I.$$

$\hookrightarrow$ <u>claim</u>

4. Define a new path Δ in the complex plane:

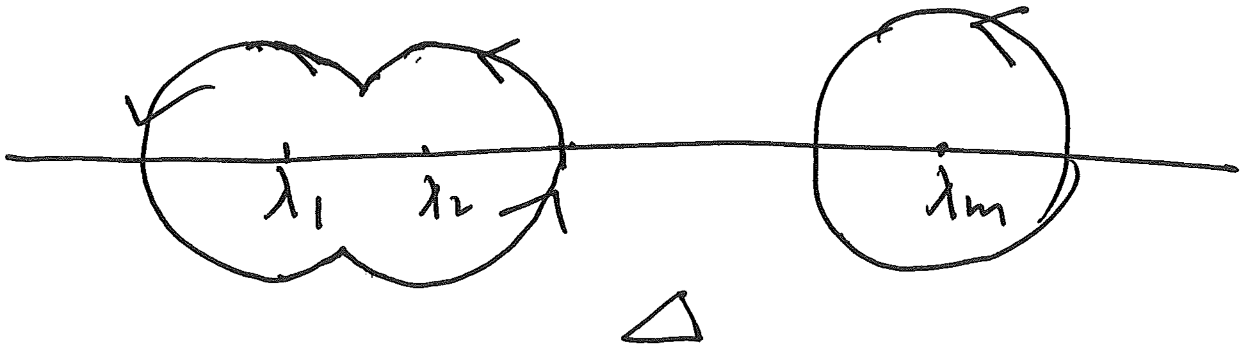
<u>Fix y,</u> draw circles

$$B_i = B_i(\lambda_i(y), 1) \text{ of radius } 1$$

centered at $\lambda_i(y)$, $i = 1, 2, \cdots, m$

Take Δ to be the bdry of $\bigcup_{k=1}^{m} B_k$

<u>traversed counterclockwise</u>



<u>Deform the path Γ into Δ to obtain</u>

$$\boxed{e^{-itB(y)} = \frac{1}{2\pi i} \int_{\Delta} e^{-it\zeta} (\zeta I - B(y))^{-1} \, d\zeta}$$

$$(\zeta I - B(y))^{-1} = \frac{\mathrm{Cof}(\zeta I - B(y))^T}{\det(\zeta I - B(y))}$$

The cofactor matrix

$$|\det(\zeta I - B(y))| = \left|\prod_{k=1}^{m} (\zeta - \lambda_k(y))\right| \geq 1$$

if $\zeta \in \Delta$

$$\|\mathrm{Cof}(\zeta I - B(y))^T\| \leq C\left(1 + |\zeta|^{m-1} + \|B(y)\|^{m-1}\right)$$

$$\leq C\,(1 + |y|^{m-1})$$

$|\lambda_k(y)| \leq C|y|$, $k = 1, 2 \cdots m$.

$$|e^{-it\zeta}| \leq e^t, \quad \forall\, \zeta \in \Delta$$

$$\Rightarrow \|e^{-itB(y)}\| \leq \frac{1}{2\pi}\left\|\int_\Delta e^{-it\zeta}(\zeta I - B(y))^{-1}\, d\zeta\right\|$$

$$\leq C e^t (1 + |y|^{m-1}), \quad \forall\, y \in \mathbb{R}^n$$

5. $\left|\int_{\mathbb{R}^n} e^{ix\cdot y} e^{-itB(y)} \hat{g}(y)\,dy\right|$

$\leq C \int_{\mathbb{R}^n} \|e^{-itB(y)}\| \, (1+|y|^k)^{-1} |f(y)|\,dy$

$\leq C e^t \int_{\mathbb{R}^n} |f(y)| \, (1+|y|^{m-1}) \, (1+|y|^k)^{-1} dy$

$$\leq C \|f\|_{L^2} \underbrace{\int_{\mathbb{R}^n} \frac{dy}{1+|y|^{2(k-m+1)}}}_{<\infty}$$

if $2(k-m+1) > n$

$\Updownarrow$

$\boxed{k > \frac{n}{2} + m - 1}$

Similarly

$|\nabla_x u(x,t)| < \infty \Longleftarrow \boxed{k > \frac{n}{2} + m}$

$\Longrightarrow \quad u \in C^1([0,\infty); \mathbb{R}^n)$

Remarks

① Uniqueness: $g \equiv 0 \implies u \equiv 0$

② Continuous Dependence.

$$g_n \longrightarrow g \quad \text{in } H^k(\mathbb{R}^n; \mathbb{R}^m)$$

$$k > \frac{n}{2} + m$$

$$\implies u_n(x,t) \longrightarrow u(x,t) \quad \text{in } C^1([0,\infty); \mathbb{R}^m)$$

③ We don't need

* Symmetry of the matrices $\{B_j\}_{j=1}^m$
* strict Hyperbolicity

$$\lambda_1(y) < \lambda_2(y) < \cdots < \lambda_m(y) \quad \forall\, y \in \mathbb{R}^n.$$

Exercise Read Lax: Chapters 2-3

pages 5-36

Higher-Order Hyperbolic Equations With Constant Coefficients

$$(*)\begin{cases} P(D,\tau)u = 0 \\ \tau^k u = g_k(x) \end{cases}$$

$$D=(D_1,\cdots,D_n)=\left(\frac{\partial}{\partial x_1},\cdots,\frac{\partial}{\partial x_n}\right),\quad \tau=\frac{\partial}{\partial t}$$

$$P(D,\tau)=\tau^m+P_1(D)\tau^{m-1}+\cdots+P_m(D).$$

$P_k(D)$ is a polynomial of degree $\le k$ in $D_1,\cdots,D_n$.

Gårding's Hyperbolicity Condition.

$$\boxed{\begin{array}{l}\exists\ c\in\mathbb{R}\ \text{s.t} \\ P(i\xi,i\lambda)\neq 0 \text{ for all } \xi\in\mathbb{R}^n \\ \qquad\qquad \text{and all } \lambda \text{ with } \operatorname{Im}\lambda\le -c.\end{array}}$$

$$\Updownarrow$$

$$\boxed{\begin{array}{l}\text{All of the } m \text{ roots } \lambda \text{ of } P(i\xi,i\lambda)=0 \\ \text{lies in one and the same half plane } \operatorname{Im}\lambda>-c \\ \text{of the complex number plane for all real } \xi\in\mathbb{R}^n\end{array}}$$

See

Fritz John: PDE, P143-157.
Springer-Verlag: 1982

Solvability of (*)
Via Fourier Transform Methods.

* Higher-Order Hyperbolic Equations
$\Longrightarrow$ First-Order Hyperbolic Systems