# Lecture 2

## Analysis of PDEs - 3.

# $L^1$ - Theory for Scalar Conservation Laws

$$(*)\quad \begin{cases} \partial_t u + \nabla_x \cdot f(u) = 0 \\ u|_{t=0} = u_0(x) \end{cases}$$

$f: \mathbb{R} \longrightarrow \mathbb{R}^d$ Given Smooth function on $\mathbb{R}$

$f(u) = (f_1(u), \cdots, f_d(u))$

## Admissible Solutions

$\boxed{u \in L^\infty(\mathbb{R}_+ \times \mathbb{R}^d)}$, $\mathbb{R}_+ = [0, \infty)$

$$(**)\quad \boxed{\int_0^\infty \int_{\mathbb{R}^d} \Big( \eta(u) \partial_t \varphi + \sum_{j=1}^d q_j(u) \partial_{x_j} \varphi \Big) dx\, dt + \int_{\mathbb{R}^d} \varphi(0,x)\, \eta(u_0(x))\, dx \geq 0.}$$

for any entropy $\eta(u)$, $\underline{\eta''(u) \geq 0}$, and Corresponding entropy flux $q(u) = \int^u \eta'(v) f'(v)\, dv$. and any test function $\varphi \in C_0^\infty(\mathbb{R}_+ \times \mathbb{R}^d)$, $\underline{\varphi \geq 0}$

(**) $\int_0^\infty \int_{\mathbb{R}^d} \left( \eta(u)\partial_t \varphi + q(u)\cdot\nabla_x \varphi \right) dx\,dt + \int_{\mathbb{R}^d} \varphi(0,x)\,\eta(u_0(x))\,dx \geq 0$

1. $\partial_t \eta(u(t,x)) + \nabla_x \cdot q(u(t,x)) \leq 0 \qquad \mathcal{D}'$

$\hookrightarrow$ $\boxed{\partial_t \eta(u) + \nabla_x q(u) =: \mu \quad \text{Radon Measure}}$

2. $(\eta(u), q(u)) = (\pm u, \pm f(u)) \Rightarrow$ $u(t,x)$ Weak Solution

3. Lipschitz Convex functions can be approximated by
$\left\{ c_0 u + \sum_{j=1}^{m} c_j (u-u_j)^+ \right\}$

$\boxed{(**)} \Leftarrow$ Sufficient for $(\eta, q) = \begin{cases} (\pm u, \pm f(u)) \\ ((u-\bar{u})^+, \operatorname{sign}(u-\bar{u})^+(f(u)-f(\bar{u}))) \end{cases}$ $\forall\, \bar{u} \in \mathbb{R}$.

$\Updownarrow$

$\boxed{\left( |u-\bar{u}|, \operatorname{sign}(u-\bar{u})(f(u)-f(\bar{u})) \right)}$

Kružkov's family of Entropy-Entropy Flux Pairs

# $L^1$- Theory

**Theorem 1** (Existence & Uniqueness) $\quad u_0 \in L^\infty(\mathbb{R}^d)$

$\Longrightarrow \exists\, 1$ admissible solution $u$ of $(*)$ and

$$u(t,\cdot) \in C^0([0,\infty); L^1_{loc}(\mathbb{R}^d)).$$

**Theorem 2** (Stability in $L^1$ & Monotonicity in $L^\infty$).

$$\begin{cases} u_0 \in L^\infty(\mathbb{R}^d) \longrightarrow u(t,\cdot) \in C^0([0,\infty); L^1_{loc}(\mathbb{R}^d)) \\ v_0 \in L^\infty(\mathbb{R}^d) \longrightarrow v(t,\cdot) \in C^0([0,\infty); L^1_{loc}(\mathbb{R}^d)) \end{cases}$$

$\Longrightarrow \exists\; s = s(\|u_0\|_{L^\infty}, \|v_0\|_{L^\infty}) > 0$ such that, $\forall\, t>0,\ r>0$.

$$\begin{cases} \int_{|x|<r} [u(t,x)-v(t,x)]^+ dx \le \int_{|x|<r+st} [u_0(x)-v_0(x)]^+ dx \\ \|u(t,\cdot)-v(t,\cdot)\|_{L^1(B_r)} \le \|u_0(\cdot)-v_0(\cdot)\|_{L^1(B_{r+st})} \end{cases}$$

$$\hookrightarrow \begin{cases} \text{If } u_0(x) \le v_0(x),\ \text{a-e} \Longrightarrow u(t,x) \le v(t,x)\ .\ \text{a-e on } \mathbb{R}_+\times\mathbb{R}^d \\ \inf u_0(x) \le u(t,x) \le \sup u_0(x); \quad \inf v_0(x) \le v(t,x) \le \sup v_0(x) \end{cases}$$

# Ideas of Proof : Theorem 2

1. $\left(\eta(u;v),\ q(u;v)\right) = \left((u-v)^+,\ \mathrm{sign}(u-v)^+(f(u)-f(v))\right)$

   Entropy-Entropy Flux Pair in the Variable $u$, for fixed $v$

   – – Variable $v$, for fixed $u$

   $\phi(t,x;\tau,y) \geq 0$ Lipschitz, $\mathrm{supp}\,\phi \subset\subset \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{R}^d$

2. Fix $(\tau, y)$, in $(**)$, choose

$$\begin{cases} (\eta, q) = \left(\eta(u; v(\tau,y)),\ q(u; v(\tau,y))\right) & \text{w.r.t } (t,x) \\ \psi(t,x) := \phi(t,x;\tau,y) \end{cases}$$

$$\Longrightarrow \boxed{\begin{aligned} &\int_0^\infty \int_{\mathbb{R}^d} \left\{ \partial_t \phi(t,x;\tau,y)\, \eta\big(u(t,x); v(\tau,y)\big) + \nabla_x \phi(t,x;\tau,y)\, q\big(u(t,x); v(\tau,y)\big) \right\} dx\,dt \\ &+ \int_{\mathbb{R}^d} \phi(0,x;\tau,y)\, \eta\big(u_0(x); v(\tau,y)\big)\, dx \geq 0 \end{aligned}} \quad \text{(A)}$$

Fix $(t,x)$, in $(**)$, choose

$$\begin{cases} (\eta, q) = (\eta(u(t,x); v), \; q(u(t,x); v)) & \text{w.r.t. } (\tau, y) \\ \psi(\tau, y) := \phi(t, x; \tau, y) \end{cases}$$

$$\Rightarrow \boxed{\begin{aligned} &\int_0^\infty \int_{\mathbb{R}^d} \Big\{ \partial_\tau \phi(t,x;\tau,y)\, \eta(u(t,x); v(\tau,y)) + \nabla_y \phi(t,x;\tau,y)\, q(u(t,x); v(\tau,y)) \Big\} \, dy \, d\tau \\ &\quad + \int_{\mathbb{R}^d} \phi(t,x;0,y)\, \eta(u(t,x); v_0(y)) \, dy \geq 0 \end{aligned}} \quad (B)$$

$$\boxed{\int_0^\infty \int_{\mathbb{R}^d} (A)\, d\tau\, dy + \int_0^\infty \int_{\mathbb{R}^d} (B)\, dt\, dx}$$

$$\Rightarrow \int_0^\infty \int_{\mathbb{R}^d} \int_0^\infty \int_{\mathbb{R}^d} \Big\{ \underline{(\partial_t + \partial_\tau)\phi(t,x;\tau,y)}\, \eta(u(t,x);v(\tau,y)) + \sum_{j=1}^d \underline{(\partial_{x_j} + \partial_{y_j})\phi(t,x;\tau,y)}\, q_j(u(t,x);v(\tau,y)) \Big\}\, dx\, dt\, dy\, d\tau$$

$$+ \int_0^\infty \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \phi(0,x;\tau,y)\, \eta(u_0(x);v(\tau,y))\, dx\, dy\, d\tau$$

$$+ \int_0^\infty \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \phi(t,x;0,y)\, \eta(u(t,x);v_0(y))\, dx\, dy\, d\tau \geq 0$$

$$\forall \begin{cases} \phi(t,x;\tau,y) \geq 0 \quad \text{Lipschitz} \\ \operatorname{supp}\phi \subset\subset \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{R}^d \end{cases}$$

3. Choose

$$\boxed{\phi(t,x;\tau,y) = \frac{1}{\varepsilon^{d+1}}\psi\left(\frac{x+y}{2},\frac{t+\tau}{2}\right)\rho\left(\frac{t-\tau}{2\varepsilon}\right)\prod_{j=1}^{d}\rho\left(\frac{x_j-y_j}{2\varepsilon}\right), \quad \varepsilon>0}$$

$$\rho\in C_0^\infty(\mathbb{R}),\quad \rho\geq 0 \quad \int_{-\infty}^{\infty}\rho(z)\,dz = 1$$

$$\psi\in Lip(\mathbb{R}_+\times\mathbb{R}^d),\quad \psi\geq 0,\quad \operatorname{supp}\psi\subset\subset \mathbb{R}_+\times\mathbb{R}^d$$

$$\left\{\begin{aligned}
(\partial_t+\partial_\tau)\phi(t,x;\tau,y) &= \frac{1}{\varepsilon^{d+1}}\partial_t\psi\left(\frac{x+y}{2},\frac{t+\tau}{2}\right)\rho\left(\frac{t-\tau}{\varepsilon}\right)\prod_{j=1}^{d}\rho\left(\frac{x_j-y_j}{\varepsilon}\right),\\
(\partial_{x_j}+\partial_{y_j})\phi(t,x;\tau,y) &= \frac{1}{\varepsilon^{d+1}}\partial_{x_j}\psi\left(\frac{x+y}{2},\frac{t+\tau}{2}\right)\rho\left(\frac{t-\tau}{\varepsilon}\right)\prod_{j=1}^{d}\rho\left(\frac{x_j-y_j}{\varepsilon}\right),
\end{aligned}\right.$$

$$\left\{\begin{aligned}
|\eta(u(t,x);v_0(y))-\eta(u_0(x);v_0(y))| &\leq |u(t,x)-u_0(x)|,\\
|\eta(u_0(x);v(t,x))-\eta(u_0(x);v_0(y))| &\leq |v(\tau,y)-v_0(y)|.
\end{aligned}\right.$$

$$\frac{1}{\varepsilon^{d+1}}\rho\left(\frac{t-\tau}{\varepsilon}\right)\prod_{j=1}^{d}\rho\left(\frac{x_j-y_j}{\varepsilon}\right) \longrightarrow \delta_{t=\tau}\prod_{j=1}^{d}\delta_{x_j=y_j} \qquad (M)$$

4. $\begin{cases} \varepsilon \longrightarrow 0 \\ \text{Convergence Theorem} \end{cases}$

$\Longrightarrow$

$$\text{(c)} \quad \boxed{\begin{aligned} &\int_0^\infty \int_{\mathbb{R}^d} \Big\{ \partial_t \psi(t,x)\, \eta\big(u(t,x); v(t,x)\big) \\ &\qquad + \sum_{j=1}^{d} \partial_{x_j} \psi(t,x)\, q_j\big(u(t,x); v(t,x)\big) \Big\}\, dx\,dt \\ &+ \int_{\mathbb{R}^d} \psi(0,x)\, \eta\big(u_0(x); v_0(x)\big)\, dx \geq 0 \end{aligned}}$$

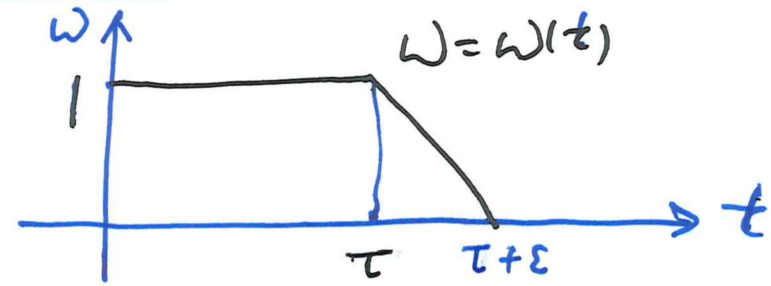
$\forall\, \psi \in Lip(\mathbb{R}_+ \times \mathbb{R}^d),\ \psi \geq 0,\ \ \mathrm{Supp}\,\psi \subset\subset \mathbb{R}_+ \times \mathbb{R}^d$

5. $|g(u;v)| \leq s\,\eta(u;v)$ $\quad \forall u \in [\inf u_0(x), \sup u_0(x)]$

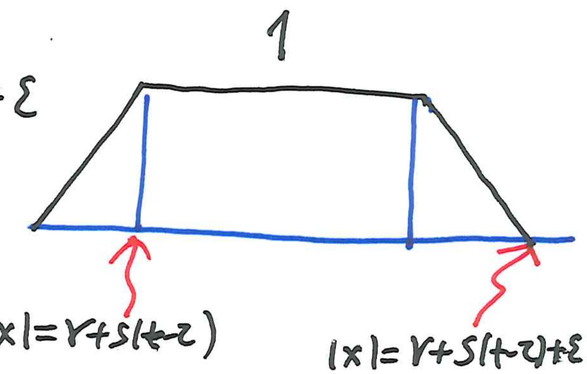
$\forall v \in [\inf v_0(x), \sup v_0(x)]$

For $\underline{r>0,\ t\geq 0,\ \varepsilon>0}$ $\quad \boxed{\psi(t,x) = \omega(t)\chi(\tau, x)}$

$$\omega(t) = \begin{cases} 1 & 0<t<\tau \\ \frac{\tau-t}{\varepsilon}+1 & \tau \leq t < \tau+\varepsilon \\ 0 & \tau+\varepsilon \leq t < \infty \end{cases}$$




$$\chi(t,x) = \begin{cases} 1 & |x| < r+s(t-\tau) \\ \frac{r+s(t-\tau)-|x|}{\varepsilon}+1 & r+s(t-\tau) \leq |x| < r+s(t-\tau)+\varepsilon \\ 0 & |x| \geq r+s(t-\tau)+\varepsilon \end{cases}$$




$$(C) \Rightarrow \frac{1}{\varepsilon}\int_t^{t+\varepsilon}\int_{|x|<r}[u(\tau,x)-v(\tau,x)]^+\,dx\,d\tau$$

$$\leq \int_{|x|<r+st}[u_0(x)-v_0(x)]^+\,dx - \frac{1}{\varepsilon}\int_0^t\int_{r+s(t-\tau)\leq|x|<r+s(t-\tau)+\varepsilon}\left[s\eta(u;v)+\frac{g(u;v)\cdot x}{|x|}\right]dx\,d\tau + O(\varepsilon).$$

$$\xRightarrow{\varepsilon\to 0+} \int_{|x|<r}[u(t,x)-v(t,x)]^+\,dx \leq \int_{|x|\leq r+st}[u_0(x)-v_0(x)]^+\,dx$$

6. Interchanging the roles of $u$ and $v$

$$\Longrightarrow \int_{|x|<r} [v(t,x)-u(t,x)]^+ \, dx \le \int_{|x|\le r+st} [v_0(x)-u_0(x)]^+ \, dx$$

7. <u>Steps 5–6</u>

$$\Longrightarrow \| u(t,\cdot)-v(t,\cdot)\|_{L^1(B_r)} \le \| u_0(\cdot)-v_0(\cdot)\|_{L^1(B_{r+st})}.$$

8. $u_0(x) \le v_0(x)$ a.e $\Longrightarrow$ $u(t,x) \le v(t,x)$ a.e.

$v_0(x) = \sup_x u_0(x) \Longrightarrow u(t,x) \le \sup_x u_0(x)$ a.e

$v_0(x) = \inf_x u_0(x) \Longrightarrow u(t,x) \ge \inf_x u_0(x)$ a.e.

Similarly. We have

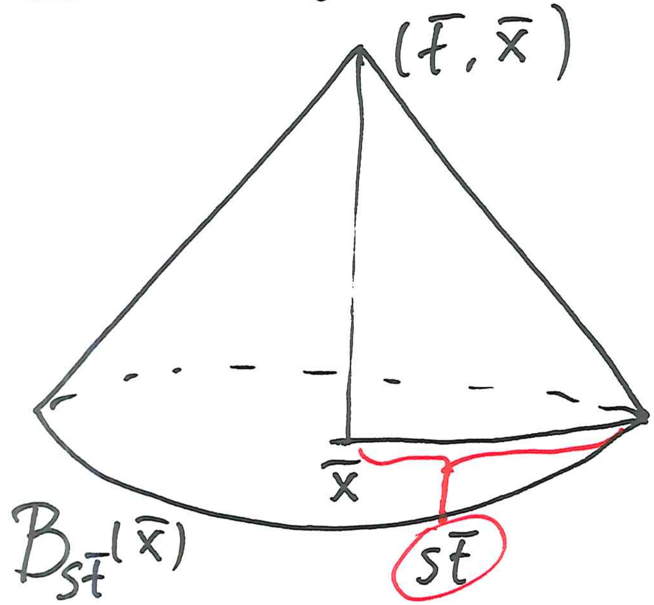
$$\inf_x v_0(x) \le v(t,x) \le \sup_x v_0(x) \quad \text{a.e}$$

# Direct Applications

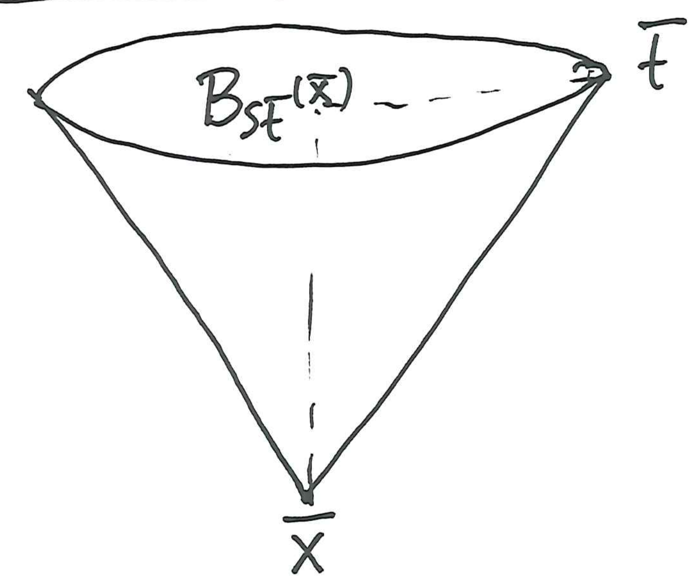
1. $\exists$ at most one admissible weak solution of $(*)$
2. The value $u(\bar{t},\bar{x})$ depends solely on the restriction of the initial data to the ball $B_{s\bar{t}}(\bar{x})$

$\hookrightarrow$ $\boxed{\text{Finite Propagation Speed}}$

## Domain of Dependence




## Domain of Influence



## Direct Applications

3. $$u_0 \in BV_{loc}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \Longrightarrow u(t,x) \in BV_{loc}(\mathbb{R}_+ \times \mathbb{R}^d).$$

$\Uparrow$

For fixed $t>0$ $\begin{cases} u(t,\cdot) \in BV_{loc}(\mathbb{R}^d) \\ TV_{B_r}(u(t,\cdot)) \le TV_{B_{r+st}}(u_0(\cdot)), \quad \forall r>0 \end{cases}$

$s \sim \|u_0\|_{L^\infty}$

Proof ① $\{e_j\}_{j=1}^d$ — standard orthonormal basis of $\mathbb{R}^d$

For $j=1,\cdots,d$, $\quad v(t,x) := u(t, x+he_i) \quad h>0$

[admissible solution of (*). with initial data $v_0(x) = u_0(x+he_i)$

$\xRightarrow{\text{Theorem 2}}$ $$\int_{|x|<r} |u(t, x+he_i) - u(t,x)|\,dx \le \int_{|x|<r+st} |u_0(x+he_i) - u_0(x)|\,dx \underset{u_0 \in BV}{\le} C|h|.$$

$\Longrightarrow$ $u(t,\cdot) \in BV_{loc}(\mathbb{R}^d)$

<u>Proof</u> (Conti.)

(2) (1) $\Longrightarrow$ $\partial_{x_j} u(t,\cdot) \in \mathcal{M}_{loc}(\mathbb{R}_+ \times \mathbb{R}^d)$

$\hookrightarrow$ $\nabla_x \cdot f(u(t,x)) \in \mathcal{M}_{loc}(\mathbb{R}_+ \times \mathbb{R}^d)$

$\hookrightarrow$ $u_t = -\nabla_x \cdot f(u(t,x)) \in \mathcal{M}_{loc}(\mathbb{R}_+ \times \mathbb{R}^d)$

$\hookrightarrow$ $u \in BV_{loc}(\mathbb{R}_+ \times \mathbb{R}^d)$

## Direct Applications

4. $u_0 \in L^\infty(\mathbb{R}^d)$

$$\Rightarrow \exists\ S = S(\|u_0\|_{L^\infty(\mathbb{R}^d)}) \text{ s.t. } \forall\ p \in [1,\infty),\ t>0,\ r>0$$

$$\boxed{\|u(t,\cdot)\|_{L^p(B_r)} \le \|u_0(\cdot)\|_{L^p(B_{r+St})}}$$

Similar Proof

# Existence Proof Via the Method of Vanishing Viscosity.

**Consider**

$$(\ast)\quad \begin{cases} \partial_t u + \nabla_x \cdot f(u) = \varepsilon \Delta u \\ u|_{t=0} = u_0(x) \in L^\infty \cap L^1(\mathbb{R}^d). \end{cases} \qquad \varepsilon > 0$$

$$\Delta = \sum_{j=1}^{d} \partial_{x_j}^2 \qquad \text{Laplace's operator}$$

**The Parabolic Theory**

$\hookrightarrow$ $\exists$ 1 Global Smooth Solution $u^\varepsilon(t,x)$

**Question** $\qquad \underbrace{u^\varepsilon(t,x) \longrightarrow u(t,x) \quad \text{a.e.}}_{\text{Admissible Solution}}$ ?? ??

<u>Fact I.</u>

$$\begin{cases} u_0 \in L^\infty \cap L^1 \longrightarrow u^\varepsilon(t,x) \\ v_0 \in L^\infty \cap L^1 \longrightarrow v^\varepsilon(t,x) \end{cases}$$

$\Longrightarrow$ $\forall\, t > 0$

- $$\begin{cases} \int_{\mathbb{R}^d} \left[u^\varepsilon(t,x) - v^\varepsilon(t,x)\right]^+ dx \le \int_{\mathbb{R}^d} \left[u_0(x) - v_0(x)\right]^+ dx \\ \int_{\mathbb{R}^d} \left|u^\varepsilon(t,x) - v^\varepsilon(x,t)\right| dx \le \int_{\mathbb{R}^d} \left|u_0(x) - v_0(x)\right| dx. \end{cases}$$

- If $u_0(x) \le v_0(x)$ a.e $\Longrightarrow$ $u^\varepsilon(t,x) \le v^\varepsilon(t,x)$ a.e.

- $\inf u_0(\cdot) \le u^\varepsilon(t,x) \le \sup u_0(\cdot)$

  $\inf v_0(\cdot) \le v^\varepsilon(t,x) \le \sup v_0(\cdot)$

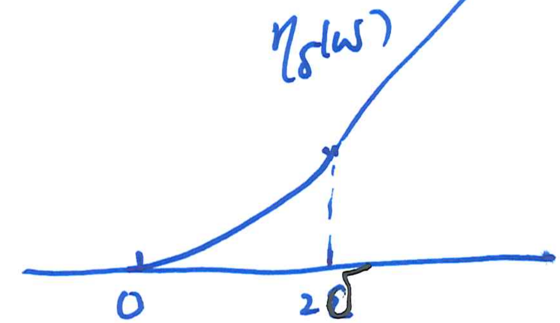
**Proof of Fact I** To simplify the notation, we drop the subscript $\varepsilon$, denote $u^\varepsilon, v^\varepsilon$ by $u, v$

① **The Parabolic Theory**: $u_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$

$\hookrightarrow$ $u(t,x)$, $D_x^\alpha u(t,x) \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, $t>0$

② For $\delta > 0$

$$\eta_\delta(w) = \begin{cases} 0 & -\infty < w \le 0 \\ \frac{w^2}{4\delta} & 0 < w \le 2\delta \\ w-\delta & 2\delta < w < \infty \end{cases}$$




$\hookrightarrow$

$$(V)\begin{cases} \partial_t \eta_\delta(u-v) + \nabla\cdot\left(\eta_\delta'(u-v)\left(f(u)-f(v)\right)\right) \\ \qquad - \eta_\delta''(u-v)\left(f(u)-f(v)\right)\cdot\nabla(u-v) \\ = \varepsilon\Delta\eta_\delta(u-v) - \varepsilon\underbrace{\eta_\delta''(u-v)}_{\geq 0}|\nabla_x(u-v)|^2 \end{cases}$$

③ Fix $0<s<t<\infty$.

$$\int_s^t \int_{\mathbb{R}^d} (\text{V})\,dx\,d\tau$$

$$\hookrightarrow \int_{\mathbb{R}^d} \eta_\delta\big(u(t,x)-v(t,x)\big)\,dx - \int_{\mathbb{R}^d} \eta_\delta\big(u(s,x)-v(s,x)\big)\,dx$$

$$\leq \int_s^t \int_{\mathbb{R}^d} \underbrace{\eta_\delta''(u-v)\big(-f(u)-f(v)\big)\cdot \nabla_x(u-v)}_{\text{Uniformly bdd for fixed } \varepsilon>0}\,dx\,d\tau$$

$\delta \to 0$ pointwise $0$

$$\xrightarrow{\text{Dominated Convergence Theorem}} 0$$

3 (Conti).

$$(\hat{A}) \quad \int_{\mathbb{R}^d} [u(t,x)-v(t,x)]^+ dx \le \int_{\mathbb{R}^d} [u(s,x)-v(s,x)]^+ dx$$

$$\Big\downarrow \; s \to 0$$

$$\int_{\mathbb{R}^d} [u_0(x)-v_0(x)]^+ dx$$

4 <u>Interchange the role of $u$ and $v$</u>

$$(\hat{B}) \quad \int_{\mathbb{R}^d} [v(t,x)-u(t,x)]^+ dx \le \int_{\mathbb{R}^d} [v_0(x)-u_0(x)]^+ dx$$

$$\xRightarrow{(\hat{A})+(\hat{B})} \int_{\mathbb{R}^d} |u(t,x)-v(t,x)|\, dx \le \int_{\mathbb{R}^d} |u_0(x)-v_0(x)|\, dx$$

5 $\quad u_0(x) \le v_0(x) \xRightarrow{(\hat{A})} u(t,x) \le v(t,x)$ a.e.

<u>choose</u> $v(t,x) = \sup u_0(x) \xRightarrow{(\hat{A})} u(t,x) \le \sup u_0(x)$. a.e.

choose $v(t,x) = \inf u_0(x) \xRightarrow{(\hat{B})} u(t,x) \ge \inf u_0(x)$. a.e.

<u>Similarly</u>

$$\inf v_0(x) \le v(t,x) \le \sup v_0(x)$$

$\square$

**Fact II**

$$\begin{cases} u_0 \in L^\infty \cap L^1(\mathbb{R}^d) \\ \int_{\mathbb{R}^d} |u_0(x+y) - u_0(x)|\, dx \le \omega(|y|), \quad \forall\, y \in \mathbb{R}^d \end{cases}$$

$$\omega(r) \nearrow_{r \nearrow \infty}, \qquad \omega(r) \downarrow 0 \text{ as } r \downarrow 0$$

$$\Longrightarrow \quad \exists\; C = C(\|u_0\|_{L^\infty}) \quad \text{s.t.} \quad \forall\, t > 0$$

$$\begin{cases} \int_{\mathbb{R}^d} |u^\varepsilon(t, x+y) - u^\varepsilon(t,x)|\, dx \le \omega(|y|), \quad y \in \mathbb{R}^d \\ \int_{\mathbb{R}^d} |u^\varepsilon(t+h, x) - u^\varepsilon(t,x)|\, dx \le C\left(h^{\frac{2}{3}} + \varepsilon h^{\frac{1}{3}}\right) \|u_0\|_{L^1(\mathbb{R}^d)} + 2\omega(h^{\frac{1}{3}}) \qquad h > 0 \end{cases}$$

## Ideas of the Proof

1. Fix $t>0$. $\forall\, y\in\mathbb{R}^d$

$w^{\varepsilon}(t,x) = u^{\varepsilon}(t,x+y)$ is a solution with initial data $w^{\varepsilon}\big|_{t=0} = u_0(x+y)$

$$\Longrightarrow \int_{\mathbb{R}^d} |u^{\varepsilon}(t,x+y)-u^{\varepsilon}(t,x)|\,dx \le \int_{\mathbb{R}^d} |u_0(x+y)-u_0(x)|\,dx$$

2. Fix $h>0$. W.O.L.G. $f(0)=0$.

$\forall\, \phi\in C^{\infty}(\mathbb{R}^d)$, Multiply the equation by $\phi$ and integrate the resulting equation over $(t,t+h)\times\mathbb{R}^d$.

$$\int_{\mathbb{R}^d}\phi(x)\,\big[\underbrace{u^{\varepsilon}(t+h,x)-u^{\varepsilon}(t,x)}_{=v(x)}\big]\,dx$$
$$=\int_t^{t+h}\int_{\mathbb{R}^d}\Big(\nabla\phi(x)\cdot f(u^{\varepsilon}(\tau,x)) + \varepsilon\,\Delta\phi(x)\,u^{\varepsilon}(\tau,x)\Big)\,dx\,d\tau$$

Formally. $\phi(x) = \operatorname{sign} v(x)$ ✓

3. For $\rho \in C_0^\infty(\mathbb{R})$, $\rho \geq 0$, $\operatorname{supp} \rho \subset \left[-\frac{1}{\sqrt{d}}, \frac{1}{\sqrt{d}}\right]$, $\int \rho(x)\,dx = 1$.

Define $$\phi(x) = \int_{\mathbb{R}^d} \frac{1}{h^{\frac{d}{3}}} \prod_{j=1}^{d} \rho\left(\frac{x_j - \xi_j}{h^{1/3}}\right) \operatorname{sign} v(\xi)\, d\xi$$

$\Longrightarrow$ $|\nabla_x \phi| \leq C h^{-\frac{1}{3}}$ $\quad |\Delta_x \phi| \leq C h^{-\frac{2}{3}}$

<u>Note</u> $\|u(\tau,\cdot)\|_{L^1(\mathbb{R}^d)} \leq \|u_0\|_{L^1(\mathbb{R}^d)}$

<u>Step 2</u>

$$\int_{\mathbb{R}^d} \phi(x) v(x)\, dx \leq \underbrace{C}_{\|u_0\|_{L^\infty(\mathbb{R}^d)}} \left(h^{\frac{2}{3}} + \varepsilon h^{\frac{1}{3}}\right) \|u_0\|_{L^1(\mathbb{R}^d)}$$

4. Note

$$|U(x)| - U(x)\,\mathrm{sign}\,U(z) = |U(x)| - |U(z)| + [U(z) - U(x)]\,\mathrm{sign}\,U(z)$$
$$\leq 2|U(x) - U(z)|$$

$$\hookrightarrow \quad |U(x)| - \phi(x)U(x) = \int_{\mathbb{R}^d} \frac{1}{h^{\frac{d}{3}}} \prod_{j=1}^{d} \rho\left(\frac{x_j - z_j}{h^{1/3}}\right)\left[|U(x)| - U(x)\,\mathrm{sign}\,U(z)\right] dz$$
$$\leq 2\int_{|\xi|\leq 1} \prod_{j=1}^{d} \rho(\xi_j)\,|U(x) - U(x - h^{1/3}\xi)|\,d\xi$$
$$\leq 2\,\omega(h^{\frac{1}{3}})$$

(+) Step 3

$$\hookrightarrow \quad \int_{\mathbb{R}^d} |U(x)|\,dx \leq \int_{\mathbb{R}^d} \left(|U(x)| - \phi(x)U(x)\right) dx + \int_{\mathbb{R}^d} \phi(x)U(x)\,dx$$
$$\leq C\left(h^{\frac{2}{3}} + \varepsilon h^{\frac{1}{3}}\right)\|u_0\|_{L^1(\mathbb{R}^d)} + 2\,\omega(h^{\frac{1}{3}}).$$

无 $\varepsilon$

## Proof of Theorem 1 (Existence)

1. $u_0 \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$

$\hookrightarrow \displaystyle\int_{\mathbb{R}^d} |u_0(x+y) - u_0(x)|\, dx \le \omega(|y|) \xrightarrow[|y|\to 0]{} 0$

$\hookrightarrow u^\varepsilon \in C^0(\mathbb{R}_+ ; L^1(\mathbb{R}^d))$.

$\xrightarrow{\text{Fact II}} \{u^\varepsilon\}_{\varepsilon>0} \subset L^1([0,T]\times\mathbb{R}^d)$ Uniformly Equicontinuous w.r.t. $\varepsilon > 0$ for any fixed $T>0$

$\xrightarrow[\text{Diagonal Process}]{\text{Compactness}} \exists \{\varepsilon_k\}_{k=1}^\infty$ s.t.

$u^{\varepsilon_k}(t,x) \longrightarrow u(t,x)$ a.e $\mathbb{R}_+\times\mathbb{R}^d$

2. $\forall \eta \in C^2, \quad \eta''(u) \geq 0$

$$\partial_t \eta(u^\varepsilon) + \nabla_x \cdot q(u^\varepsilon) = \varepsilon \Delta \eta(u^\varepsilon) - \varepsilon \eta''(u^\varepsilon) |\nabla u^\varepsilon|^2$$
$$\leq \varepsilon \Delta \eta(u^\varepsilon).$$

$\forall \psi \in C_0^\infty(\mathbb{R}_+ \times \mathbb{R}), \quad \psi \geq 0$

$$\int_0^\infty \int_{\mathbb{R}^d} \left[ \partial_t \psi \, \eta(u^\varepsilon) + \nabla_x \psi \cdot q(u^\varepsilon) \right] dx\,dt + \int_{\mathbb{R}^d} \psi(0,x) \eta(u_0(x))\,dx$$
$$\geq -\varepsilon \int_0^\infty \int_{\mathbb{R}^d} \Delta \psi \, \eta(u^\varepsilon)\,dx\,dt$$

$$\varepsilon = \varepsilon_k \to 0. \Longrightarrow \quad u(t,x) \text{ is an admissible solution}$$

3. The Limit is unique ( Theorem 2)

$\hookrightarrow \quad u^{\varepsilon}(t,x) \longrightarrow \underset{C^0([0,\infty);\, L^1(\mathbb{R}^d))}{\underset{\Cap}{u(t,x)}} \quad$ a.e. $\quad$ ( Theorem 2)

4. $\underline{u_0 \in L^\infty(\mathbb{R}^d)}$

For $r>0$, $\quad \chi_r = \chi|_{B_r(0)}$.

$\hookrightarrow \quad \exists\, 1$ admissible solution $u^r(t,x)$ with

$$u^r|_{t=0} = \chi_r u_0 \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d). \quad \xrightarrow[r\to\infty]{L^1_{loc}(\mathbb{R}^d)} u_0$$

$\xrightarrow{\text{Thm 2}} \quad \exists \quad u^{r_k}(t,x) \longrightarrow u(t,x) \quad L^1_{loc}$

Unique admissible solution

$\hookrightarrow \quad \underset{C^0([0,\infty);\, L^1(\mathbb{R}^d)).}{\underset{\Cap}{u^r(t,x)}} \longrightarrow u(t,x). \quad$ a.e. $\quad r\to\infty$

$\because \quad u \equiv u^r$ on any compact subset of $\mathbb{R}_+ \times \mathbb{R}^d \Rightarrow \quad u(t,x) \in C^0([0,\infty); L^1_{loc}(\mathbb{R}^d))$

# Admissible Solutions as Trajectories of a Contraction Semigroup.

For $t \in [0, \infty)$. Define the map

$$S(t): \begin{cases} L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d) \longrightarrow L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d) \\ u_0 \longmapsto S(t)u_0(\cdot) := u(t,x) \end{cases}$$

$\Longrightarrow$ $S(t)$ is a $L^1$-contraction semigroup on $L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$

$$\begin{cases} S(0) = I \quad \text{(the identity)} \\ S(t+\tau) = S(t)\, S(\tau) \qquad \forall\, t, \tau \in [0, \infty) \\ S(\cdot)\, u_0 \in C^0([0,\infty); L^1(\mathbb{R}^d)). \\ \|S(t)u_0 - S(t)\bar{u}_0\|_{L^1(\mathbb{R}^d)} \le \|u_0 - \bar{u}\|_{L^1(\mathbb{R}^d)} \qquad \forall\, t \in [0,\infty) \end{cases}$$

* A direct, indept. proof of the existence theorem via the theory of nonlinear Contraction Semigroup in Banach space.

$\boxed{\text{Dafermos §6.4}}$

## Other Methods

1. Numerical Schemes.

   Lax-Friedrichs Scheme, . . .

2. The Layering Method

3. Relaxation.

. . . .

See Dafermos. §6.5–§6.11

# A Kinetic Formulation

An alternative characterization of admissible weak solutions

$$\partial_t\{|u-v|-|v|\}+\operatorname{div}\big(\operatorname{sign}(u-v)(f(u)-f(v))-\operatorname{sign}(v)f(v)\big) = -2\,m(v;t,x) \quad \forall\, v\in(-\infty,\infty)$$

$\in \mathcal{M}^{+}_{t,x}(\mathbb{R}_+\times\mathbb{R})$

Differentiate the equation above w.r.t. $v$ in the sense of distributions

$$\partial_t \chi(v;u)+\sum_{j=1}^{d} f_j'(v)\,\partial_{x_j}\chi(v;u) = \partial_v m(v;t,x).$$

(*) where

$$\chi(v;u)=\begin{cases} 1 & 0<v<u \\ -1 & u<v<0 \\ 0 & \text{otherwise}\end{cases}$$

## Remarks

1. For $|v| > \sup|u_0| = \sup|u|$,

$$2m(v;t,x) = |\partial_t u + \operatorname{div} f(u)| = 0.$$

2. The entropy production by any entropy-entropy flux pair $(\eta, q)$ is easily expressed in terms of $m(v;t,x)$

$$\xrightarrow{\int_{-\infty}^{\infty} \eta'(v)\,(*)\,dv}$$

$$\partial_t \Big( \underbrace{\int_{-\infty}^{\infty} \eta'(v)\chi(v;u)\,dv}_{=\,\eta(u)-\eta(0)} \Big) + \operatorname{div}\Big( \underbrace{\int_{-\infty}^{\infty} q'(v)\chi(v;u)\,dv}_{=\,q(u)-q(0)} \Big) = -\underbrace{\int_{-\infty}^{\infty} \eta''(v)\,dm(v;t,x)}_{\geq\, 0 \text{ if } \eta''(u)\geq 0}$$

For $\eta(u) = \frac{u^2}{2}$

$$\hookrightarrow \int_0^{\infty}\int_{\mathbb{R}^d}\int_{-\infty}^{\infty} dm(v;t,x) \leq \frac{1}{2}\int_{\mathbb{R}^d} u_0^2(x)\,dx$$

**Theorem** $u(t,x) \in L^\infty(\mathbb{R}_+ \times \mathbb{R}^d)$ is the admissible solution

$\Longleftrightarrow$ $\chi(v; u(t,x))$ satisfies $(*)$ on $\mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}^d$ for some $m \in \mathcal{M}^+(\mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}^d)$, together with the initial data:

$$\chi(v; u(0,x)) = \chi(v; u_0(x)) \qquad \begin{array}{l} v \in (-\infty, \infty) \\ x \in \mathbb{R}^d \end{array}$$

**Ideas of Proof.** "$\Longrightarrow$" Done

"$\Longleftarrow$" ??

1. Equation $(*)$ admits solutions $\chi(\cdot; u(t,\cdot)) \in C^0([0,\infty); L^1(\mathbb{R} \times \mathbb{R}^d))$

$\hookrightarrow$ The initial condition $u|_{t=0} = u_0(x)$ is attained strongly in $L^1(\mathbb{R}^d)$

It suffices to show that the entropy inequality holds for every entropy-entropy flux pair $(\eta, q)$ with $\eta''(u) \geq 0$.

$\because \|u\|_{L^\infty} \leq C < \infty \Longrightarrow$ It suffices to establish the entropy inequality for entropies with linear growth, i.e. $|\eta'(u)|$ bdd on $(-\infty, \infty)$

2. $\forall \eta$, with linear growth, $\eta''(v) \geq 0$.

$\forall k = 1, 2, \cdots$, set $\boxed{\eta_k(v) = \eta(v)\phi\left(\frac{v}{k}\right)}$

where
$$\begin{cases} \phi(-v) = \phi(v) & \\ \phi(v) = 1 & \text{for } |v| \leq 1 \\ \phi(v) = 0 & \text{for } |v| \geq 2 \\ \phi'(v) < 0 & v \in (1, 2) \\ \phi'(v) > 0 & v \in (-2, -1) \end{cases}$$

$$\partial_t \underbrace{\eta_k(u)}_{\overset{k\gg 1}{=}\ \eta(u)} + \nabla_x \cdot \underbrace{q_k(u)}_{\overset{k\gg 1}{=}\ q(u)} = -\int_{-\infty}^{\infty} \eta_k''(v)\, dm(v;t,x)$$

$$= -\int_{-\infty}^{\infty} \Big[ \underbrace{\eta''(v)\phi\!\left(\tfrac{v}{k}\right)}_{\geq 0} + \frac{2}{k} \underbrace{\eta'(v)\phi'\!\left(\tfrac{v}{k}\right)}_{= O(1)} + \frac{1}{k^2} \underbrace{\eta(v)\phi''\!\left(\tfrac{v}{k}\right)}_{= O(k)} \Big]\, dm(v;t,x)$$

$$\downarrow$$

$$-\int_{-\infty}^{\infty} \eta''(v)\, dm(v;t,x) \leq 0.$$

The kinetic formulation provides a powerful instrument for discovering properties of the admissible solutions

① An alternative, direct proof of the $L^1$-contraction property even under the more general assumption that $u_0 \in L^1(\mathbb{R}^d)$ (not necessarily in $L^\infty(\mathbb{R}^d)$).

② An observation of the compactness & smoothing effects by Nonlinearity

Theorem. $\exists\ r \in (0,1],\ C \geq 0$. s.t

$$(\mathrm{V}) \quad \boxed{\operatorname{meas}\{v : |v| \leq \|u_0\|_{L^\infty},\ |\tau + f'(v)\cdot\xi| < \delta\} \leq C\delta^r}$$

$$\forall \delta \in (0,1],\ \tau^2 + |\xi|^2 = 1$$

$$\Longrightarrow \quad u(t,\cdot) \in C^0((0,\infty);\ W^{s,1}_{loc}(\mathbb{R}^d)). \qquad s \in \left(0, \tfrac{r}{2r+1}\right)$$

* $f(u)$ is linear $\Rightarrow$ (V) fails; $\quad f_j''(u) > 0,\ j = 1, 2, \cdots, d \Longrightarrow r = 1$.

Refs.

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Oxford, Oxford University Press, 2002

2. C. Dafermos. §6.7

$$\partial_t u + \nabla \cdot f(u; t, x) = \nabla_x \cdot (A(u; t, x) \nabla_x u).$$


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