

# Young Measures

$K \subset \mathbb{R}^m$ ,  $\Omega \subset \mathbb{R}^n$       bounded open

$u^k: \Omega \rightarrow \mathbb{R}^m$       measurable

$u^k(y) \in K$ , a.e.

$\Rightarrow \exists \{ \nu_y \in \text{Prob.}(\mathbb{R}^m) \}_{y \in \Omega}$  s.t.

•  $\boxed{\text{Supp } \nu_y \subset \bar{K}} \quad \forall y \in \Omega$

•  $\forall f \in C(\mathbb{R}^m; \mathbb{R}), \exists \{ u^{k_j} \}_{j=1}^{\infty} \subset \{ u^k \}$ .

$$\boxed{\begin{aligned} W^* \text{-lim } f(u^{k_j}) &= \langle \nu_y(\lambda), f(\lambda) \rangle \\ &= \int f(\lambda) d\nu_y(\lambda) \end{aligned}}$$

•  $u^{k_j} \rightarrow u$  a.e.  $\iff \nu_y(\lambda) = \delta_{u(y)}$

Dirac mass

\* This theorem can be extended to more general cases.

## Remarks

1. The deviation between the Weak and Strong convergence is measured by the spreading of the support of  $\nu_y$ .

$$\|f(w^*\text{-lim } u^k) - w^*\text{-lim } f(u^k)\|_{L^\infty} \leq C \sup_y (\text{diam}(\text{supp } \nu_y))$$

↑ for  $f \in \text{Lip}(\mathbb{R}^m; \mathbb{R})$

$$\begin{aligned} & \|f(w^*\text{-lim } u^k) - w^*\text{-lim } f(u^k)\|_{L^\infty} \\ &= \|f(\langle \nu_y, \lambda \rangle) - \langle \nu_y, f(\lambda) \rangle\|_{L^\infty} \\ &= \|\langle \nu_y, f(\lambda) - f(\langle \nu_y, \lambda \rangle) \rangle\|_{L^\infty} \\ &\leq C \|\langle \nu_y, |\lambda - \langle \nu_y, \lambda \rangle| \rangle\|_{L^\infty} \\ &\leq C \sup_y (\text{diam}(\text{supp } \nu_y)). \end{aligned}$$

## Remarks

2. The Young measure family  $\{\nu_y\}_{y \in \Omega}$  can be thought of as the limiting probability distribution of the values of  $\{u^k(y)\}$  near the point  $y$  as  $k \rightarrow \infty$ .

$\uparrow$   $\Omega \subset \mathbb{R}^n, y \in \Omega.$

$\hookrightarrow \exists \delta_0 > 0$  s.t.  $B(y, \delta) \subset \Omega, 0 < \delta \leq \delta_0$

Define

$$\langle \nu_{y, \delta}^k, \phi \rangle = \frac{1}{|B(y, \delta)|} \int \phi(u^k(x)) dx$$

$$\forall \phi \in C_c(\mathbb{R}^m; \mathbb{R})$$

$\Downarrow$

$$\nu_{y, \delta}^k(\lambda) \triangleq \frac{1}{|B(y, \delta)|} \int \delta_{u^k(x)} dx$$

$\hookrightarrow$

$$\nu_y(\lambda) = \lim_{\delta \rightarrow 0} \lim_{k \rightarrow \infty} \nu_{y, \delta}^k$$

# Weak Continuity of

## 2x2 Determinants

$\Omega \subset \mathbb{R}_+ \times \mathbb{R}$  bounded open

$U^k: \Omega \rightarrow \mathbb{R}^4$  measurable

$$\left\{ \begin{array}{l} \text{W-}\lim_{k \rightarrow \infty} U^k = U \text{ in } L^2(\Omega; \mathbb{R}^4) \\ \left\{ \begin{array}{l} \frac{\partial U_1^k}{\partial t} + \frac{\partial U_2^k}{\partial x} \\ \frac{\partial U_3^k}{\partial t} + \frac{\partial U_4^k}{\partial x} \end{array} \right. \text{ compact in } H_{loc}^1(\Omega) \end{array} \right.$$

$$\Rightarrow \begin{vmatrix} U_1^k & U_2^k \\ U_3^k & U_4^k \end{vmatrix} \longrightarrow \begin{vmatrix} U_1 & U_2 \\ U_3 & U_4 \end{vmatrix} \quad \mathcal{D}'!$$

Subsequentially

Another Form

$$U^k = (U_1^k, U_2^k) \in L^2(\Omega; \mathbb{R}^2)$$

$$W^k = (W_1^k, W_2^k) \in L^2(\Omega; \mathbb{R}^2)$$

$$\left\{ \begin{array}{l} \text{W-}\lim_{k \rightarrow \infty} (U^k, W^k) = (U, W), \quad L^2(\Omega) \\ \left\{ \begin{array}{l} \text{div } U^k \\ \text{curl } W^k \end{array} \right. \text{ compact in } H_{loc}^1(\Omega) \end{array} \right.$$

$$\Rightarrow U^k \cdot W^k \longrightarrow U \cdot W \quad \mathcal{D}'!$$

# Div-Curl Lemma

$\Omega \subset \mathbb{R}^n$  open, bounded

$$p, q > 1, \quad \frac{1}{p} + \frac{1}{q} = 1$$

$$v^k \in L^p(\Omega; \mathbb{R}^n)$$

$$w^k \in L^q(\Omega; \mathbb{R}^n)$$

$$\left\{ \begin{array}{l} v^k \longrightarrow v \text{ weakly in } L^p(\Omega; \mathbb{R}^n) \\ w^k \longrightarrow w \text{ weakly in } L^q(\Omega; \mathbb{R}^n). \end{array} \right.$$

$$\left\{ \begin{array}{l} \operatorname{div} v^k \text{ compact in } W_{loc}^{-1,p}(\Omega; \mathbb{R}) \\ \operatorname{curl} w^k \text{ compact in } W_{loc}^{-1,q}(\Omega; \mathbb{R}). \end{array} \right.$$

$$\Rightarrow v^k \cdot w^k \longrightarrow v \cdot w \quad \mathcal{D}'$$

# Compensated Compact

## Embedding Lemma

$\Omega \subset \mathbb{R}^n$  bounded open



(Compact set of  $W_{loc}^{-1,q}(\Omega)$ )

$\cap$  (Bounded set of  $W_{loc}^{-1,r}(\Omega)$ )

$\subset$  (Compact set of  $W_{loc}^{-1,p}(\Omega)$ )

for any  $1 < q \leq p < r < \infty$

## 2x2 Hyperbolic Systems of Conservation Laws

$$\begin{cases} U_t + f(u)_x = 0 & u \in \mathbb{R}^2 \\ U|_{t=0} = U_0(x) \end{cases}$$

Assume

- $\exists$  a strictly convex entropy  $\eta_x(u)$ ,  
 $\nabla^2 \eta_x(u) > 0$
- $\exists$  globally defined Riemann Invariants  
 $W = (W_1, W_2): \mathbb{R}^2 \rightarrow \mathbb{R}^2$  s.t.  
 $\nabla W_j(u) \parallel \ell_j(u)$

$\hookrightarrow$  If  $u \in C^1$ .

$$\partial_t W_j + \lambda_j(u(W)) \partial_x W_j = 0$$

# Entropy Equation

Entropy  $\eta(u)$ ,

Entropy Flux  $g(u)$

$$\nabla g(u) = \nabla \eta(u) \nabla f(u)$$

$$(\lambda_j \nabla \eta - \nabla g) \cdot r_j = 0$$

↳

$$g_{w_j} = \lambda_j \eta_{w_j}$$

$$\eta_{w_1 w_2} - \frac{\lambda_1 w_2}{\lambda_2 - \lambda_1} \eta_{w_1} + \frac{\lambda_2 w_1}{\lambda_2 - \lambda_1} \eta_{w_2} = 0$$

## Genuine Nonlinearity

$$\nabla \lambda_j(u) \cdot r_j(u) \neq 0, \quad j=1,2.$$

$$\Leftrightarrow \frac{\partial \lambda_j}{\partial w_j} \neq 0, \quad j=1,2,$$



# Method of Vanishing Viscosity

$$\begin{cases} u_t + f(u)_x = 0 & u \in \mathbb{R}^2 \\ u|_{t=0} = u_0(x) \in L^\infty \cap L^1(\mathbb{R}) \end{cases}$$

## Viscosity Approximation

$$\begin{cases} u_t + f(u)_x = \varepsilon u_{xx} \\ u|_{t=0} = u_0^\varepsilon(x) \longrightarrow u_0(x) \text{ a.e.} \end{cases}$$

$$\hookrightarrow u^\varepsilon = u^\varepsilon(t, x)$$

- Invariant Regions or  $L^p$  Estimates

$$\|u^\varepsilon\|_{L^\infty} \leq C \quad \text{or} \quad \|u^\varepsilon\|_{L^p} \leq C$$

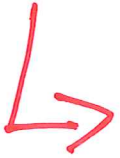
- Dissipation Estimate

$$\|\sqrt{\varepsilon} u_x^\varepsilon\|_{L^2} \leq C \quad \times \varepsilon.$$

# Dissipation Estimate

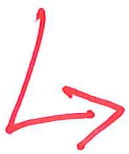
C-10

$$\nabla \bar{\gamma}_{*}^{(u^\varepsilon)} \left[ u_t^\varepsilon + f(u^\varepsilon)_x = \varepsilon u_{xx}^\varepsilon \right]$$



$$\varepsilon (u_x^\varepsilon)^T \nabla^2 \bar{\gamma}_{*}^{(u^\varepsilon)} u_x^\varepsilon \geq c_0 \varepsilon |u_x^\varepsilon|^2$$

$$= -\bar{\gamma}_{*}^{(u^\varepsilon)}{}_t - \bar{g}_{*}^{(u^\varepsilon)}{}_x + \varepsilon \bar{\gamma}_{*}^{(u^\varepsilon)}{}_{xx}$$



$$c_0 \int_0^T \int \varepsilon |u_x^\varepsilon|^2 dx dt$$

$$\leq \int \bar{\gamma}_{*}^{(u_0^\varepsilon)} dx - \int \bar{\gamma}_{*}^{(u^\varepsilon)}(T, x) dx.$$

$$\leq \int \bar{\gamma}_{*}^{(u_0^\varepsilon)} dx \leq C \varepsilon.$$

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$$\bar{\gamma}_{*}^{(u)} = \gamma_{*}^{(u)} - \gamma_{*}^{(0)} - \nabla \gamma^{(0)} u \geq c_0 > 0$$

$$\bar{g}_{*}^{(u)} = g_{*}^{(u)} - g_{*}^{(0)} - \nabla \gamma^{(0)} (f(u) - f(0)).$$

# $H^1$ -Compactness

$$\eta(u^\varepsilon)_t + \beta(u^\varepsilon)_x$$

$$= \varepsilon (\nabla \eta(u^\varepsilon) u_x^\varepsilon)_x - \varepsilon (u_x^\varepsilon)^T \nabla^2 \eta(u^\varepsilon) u_x^\varepsilon$$

$$= I_1^\varepsilon + I_2^\varepsilon$$

$$\bullet \|I_1^\varepsilon\|_{H^1(\Omega)} \leq \sqrt{\varepsilon} \|\sqrt{\varepsilon} u_x^\varepsilon\|_{L^2} \|\nabla \eta(u^\varepsilon)\|_{L^\infty} \leq \sqrt{\varepsilon} C \rightarrow 0$$

$$\bullet \|I_2^\varepsilon\|_{L^1(\Omega)} \leq \|\nabla^2 \eta(u^\varepsilon)\|_{L^\infty} \|\sqrt{\varepsilon} u_x^\varepsilon\|_{L^2}^2 \leq C$$

$\hookrightarrow I_2^\varepsilon$  compact in  $W^{1,\beta}(\Omega)$ ,  $1 < \beta < 2$

$\hookrightarrow I_1^\varepsilon + I_2^\varepsilon$  compact in  $W^{1,\beta}(\Omega)$ ,  $1 < \beta < 2$

But  $\eta(u^\varepsilon)_t + \beta(u^\varepsilon)_x$  bounded in  $W^{1,\infty}(\Omega)$

Lemma  $\rightarrow$

$$\eta(u^\varepsilon)_t + \beta(u^\varepsilon)_x$$

is compact in  $H^1_{loc}$

$$\forall (\eta, \beta) \in C^2$$

# Commutation Identity

for Young Measure  $\{\nu_{t,x}\}_{(t,x) \in \mathbb{R}_+^2}$

$$\begin{array}{c} \updownarrow \\ \{u^\varepsilon\}_{\varepsilon > 0} \end{array}$$

•  $\text{Supp } \nu_{t,x} \subset \subset \mathbb{R}^2$

• For any entropy pairs  $(\eta, \varrho)$ ,

$$(*) \quad \langle \nu_{t,x}, \begin{vmatrix} \eta_1 & \varrho_1 \\ \eta_2 & \varrho_2 \end{vmatrix} \rangle$$

$$= \begin{vmatrix} \langle \nu_{t,x}, \eta_1 \rangle & \langle \nu_{t,x}, \varrho_1 \rangle \\ \langle \nu_{t,x}, \eta_2 \rangle & \langle \nu_{t,x}, \varrho_2 \rangle \end{vmatrix} \quad \text{a.e. } (t,x)$$

$$\Rightarrow \nu_{t,x} = \delta_{u(t,x)} \quad ???$$

\* If  $f(u) = Au$  (linear)

↳ (\*) is trivial.

The imbalance of (\*) is enforced by the nonlinearity of  $f(u)$ .

Proof of (\*)

$$\forall (\eta_i, g_i) \in C, \quad i=1, 2.$$

$$U^\varepsilon = (\eta_1(u^\varepsilon), g_1(u^\varepsilon), \eta_2(u^\varepsilon), g_2(u^\varepsilon)) \quad \text{uniformly bdd}$$

$$\hookrightarrow \exists \{\varepsilon_k\}_{k=1}^\infty, \text{ s.t. } \varepsilon_k \rightarrow 0 \text{ as } k \rightarrow \infty.$$

$$\bullet \quad U^{\varepsilon_k} \xrightarrow{*} (\langle U_{t,x}, \eta_1(\lambda) \rangle, \langle U_{t,x}, g_1(\lambda) \rangle, \langle U_{t,x}, \eta_2(\lambda) \rangle, \langle U_{t,x}, g_2(\lambda) \rangle)$$

||  
U(t,x)

$$\bullet \quad \begin{vmatrix} U_1^{\varepsilon_k} & U_2^{\varepsilon_k} \\ U_3^{\varepsilon_k} & U_4^{\varepsilon_k} \end{vmatrix} \xrightarrow{*} \langle U_{t,x}, \begin{vmatrix} \eta_1(\lambda) & g_1(\lambda) \\ \eta_2(\lambda) & g_2(\lambda) \end{vmatrix} \rangle$$

||

Div-Curl

$$\begin{vmatrix} U_1 & U_2 \\ U_3 & U_4 \end{vmatrix} = \begin{vmatrix} \langle U_{t,x}, \eta_1(\lambda) \rangle & \langle U_{t,x}, g_1(\lambda) \rangle \\ \langle U_{t,x}, \eta_2(\lambda) \rangle & \langle U_{t,x}, g_2(\lambda) \rangle \end{vmatrix}$$

# Reduction of the Young Measure: C-14

Scalar Conservation Laws:  $u^2 \xrightarrow{*} u, u \in \mathbb{R}$   
 $\hookrightarrow u(t,x) = \langle \mathcal{V}_{t,x}, \lambda \rangle$

$$\langle \mathcal{V}_{t,x}, \begin{vmatrix} \eta_1(\lambda) & \varrho_1(\lambda) \\ \eta_2(\lambda) & \varrho_2(\lambda) \end{vmatrix} \rangle = \begin{vmatrix} \langle \mathcal{V}_{t,x}, \eta_1(\lambda) \rangle & \langle \mathcal{V}_{t,x}, \varrho_1(\lambda) \rangle \\ \langle \mathcal{V}_{t,x}, \eta_2(\lambda) \rangle & \langle \mathcal{V}_{t,x}, \varrho_2(\lambda) \rangle \end{vmatrix}$$

Choose:  $(\eta_1(\lambda), \varrho_1(\lambda)) = (\lambda - u(t,x), f(\lambda) - f(u(t,x)))$   
 $(\eta_2(\lambda), \varrho_2(\lambda)) = (f(\lambda) - f(u(t,x)), \int_{u(t,x)}^{\lambda} (f'(s))^2 ds)$

$$\langle \mathcal{V}_{t,x}, \begin{vmatrix} \lambda - u & f(\lambda) - f(u) \\ f(\lambda) - f(u) & \int_u^{\lambda} (f'(s))^2 ds \end{vmatrix} \rangle$$

$$= \begin{vmatrix} \langle \mathcal{V}_{t,x}, \lambda - u \rangle & \langle \mathcal{V}_{t,x}, f(\lambda) - f(u) \rangle \\ \langle \mathcal{V}_{t,x}, f(\lambda) - f(u) \rangle & \langle \mathcal{V}_{t,x}, \int_u^{\lambda} (f'(s))^2 ds \rangle \end{vmatrix}$$

$$\langle \mathcal{V}_{t,x}, (\lambda - u) \int_u^{\lambda} (f'(s))^2 ds - (f(\lambda) - f(u))^2 \rangle + \langle \mathcal{V}_{t,x}, f(\lambda) - f(u) \rangle^2 = 0$$

$$\begin{aligned}
 & (\lambda - u) \int_u^\lambda (f'(s))^2 ds - (f(\lambda) - f(u))^2 \\
 &= (\lambda - u) \int_u^\lambda \left( f'(s) - \frac{1}{\lambda - u} \int_u^\lambda f'(z) dz \right)^2 ds \geq 0.
 \end{aligned}$$

⇒

$$\langle \mathcal{V}_{t,x}, f(\lambda) - f(u) \rangle = 0$$

$$\langle \mathcal{V}_{t,x}, \underbrace{(\lambda - u) \int_u^\lambda \left( f'(s) - \frac{1}{\lambda - u} \int_u^\lambda f'(z) dz \right)^2 ds}_{\parallel} \rangle = 0$$

$$(\lambda - u) \int_u^\lambda \left( \int_u^\lambda f''(\lambda + \theta(s-z))(s-z) dz \right)^2 ds$$

⇒

- $\langle \mathcal{V}_{t,x}, f(\lambda) \rangle = f(u(t,x))$

- If  $f''(u) \geq 0$

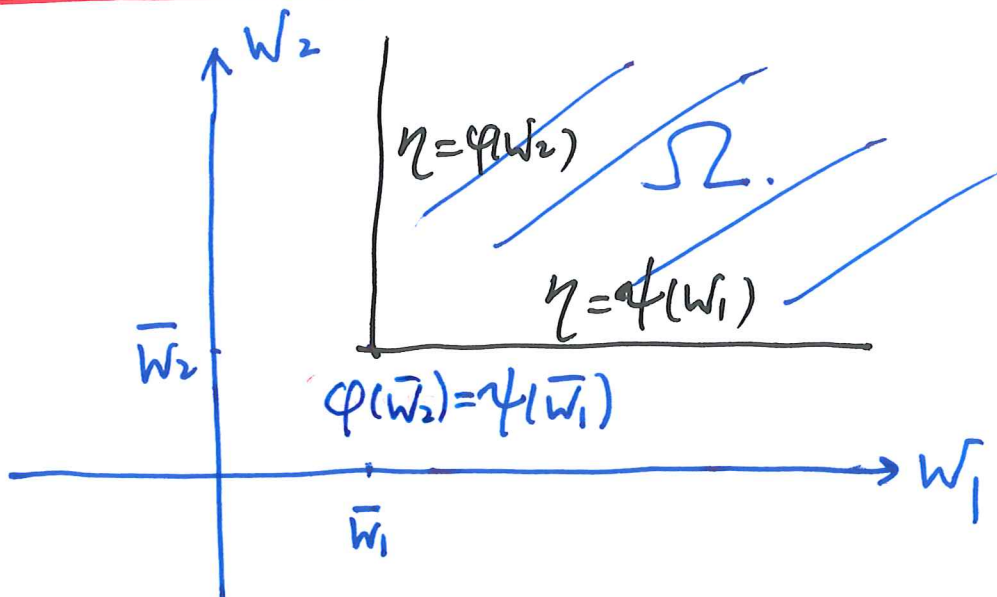
$$\hookrightarrow \boxed{\mathcal{V}_{t,x} = \delta_{u(t,x)}}$$

# The Goursat Entropy Pairs for $2 \times 2$ Hyperbolic Systems of Conservation Laws

$$\eta_{w_1 w_2} - \frac{\lambda_1 w_2}{\lambda_2 - \lambda_1} \eta_{w_1} + \frac{\lambda_2 w_1}{\lambda_2 - \lambda_1} \eta_{w_2} = 0 \quad (*)$$

$$\zeta_{w_j} = \lambda_j \eta_{w_j} \quad (**)$$

Goursat Problem for (\*)



Well-posed!



# Goursat Entropy Pairs

$\exists$  two families of entropy pairs:

$$\left\{ \begin{aligned} \eta_a(w) &= I_1(w) a(w_1) + \int_{\bar{w}_1}^{w_1} J_1(\beta; w) a(\beta) d\beta \\ \theta_a(w) &= K_1(w) a(w_1) + \int_{\bar{w}_1}^{w_1} L_1(\beta; w) a(\beta) d\beta \end{aligned} \right.$$

$$\left\{ \begin{aligned} \eta_b(w) &= I_2(w) b(w_2) + \int_{\bar{w}_2}^{w_2} J_2(\beta; w) b(\beta) d\beta \\ \theta_b(w) &= K_2(w) b(w_2) + \int_{\bar{w}_2}^{w_2} L_2(\beta; w) b(\beta) d\beta \end{aligned} \right.$$

where  $(I_i, J_i, K_i, L_i)$ ,  $i=1,2$ , are unique smooth functions and independent of  $\bar{w}_1$  and  $\bar{w}_2$ :

$$\left\{ \begin{aligned} I_i(w) &> 0, & \left\{ \begin{aligned} I_1(w_1, \bar{w}_2) &= 1 \\ J_1(\beta; w_1, \bar{w}_2) &= 0 \end{aligned} \right. & \left\{ \begin{aligned} I_2(\bar{w}_1, w_2) &= 1 \\ J_2(\beta; \bar{w}_1, w_2) &= 0 \end{aligned} \right. \\ K_i &= \lambda_i I_i & & & \\ \frac{\partial K_i(w)}{\partial w_i} + L_i(w_i; w) &= \lambda_i(w) \left( \frac{\partial I_i(w)}{\partial w_i} + J_i(w_i; w) \right) & & & \\ \frac{\partial L_i(\beta; w)}{\partial w_i} &= \lambda_i(w) \frac{\partial J_i(\beta; w)}{\partial w_i} & & & \\ \frac{\partial K_i(w)}{\partial w_j} &= \lambda_j(w) \frac{\partial I_i(w)}{\partial w_j}, \quad i \neq j & & & \\ \frac{\partial L_i(\beta; w)}{\partial w_j} &= \lambda_j(w) \frac{\partial J_i(\beta; w)}{\partial w_j} & & & \end{aligned} \right.$$

# Reduction of the Young Measure

Thm. If  $\frac{\partial \lambda_j}{\partial w_j} \neq 0, j=1, 2$  (Genuinely Nonlinear)

$\Rightarrow \mathcal{V}_{t,x} = \delta_{U(t,x)}$

Proof. If  $\mathcal{V}_{t,x} \neq \delta_{U(t,x)}$ , we denote

$[w_1^-, w_1^+] \times [w_2^-, w_2^+]$  the smallest rectangle containing  $\text{supp } \mathcal{V}_{t,x}$ .

1. Claim. If  $w_1^- < w_1^+$ , then  $\exists C_1(t,x)$  s.t.

$\langle \nu, \eta_a \rangle = C_1 \langle \nu, \eta_a \rangle$

$\forall a \in C, a(w_1) = 0$  when  $\begin{cases} w_1 \geq \bar{w}_1 \\ \text{or} \\ w_1 \leq \bar{w}_1 \end{cases}$  for  $\bar{w}_1 \in (w_1^-, w_1^+)$

• Choose  $\begin{cases} a_0(w_1) = (w_1 - w_1^*)_+, & w_1^* \geq \bar{w}_1, |w_1^+ - w_1^*| \ll 1 \\ a(w_1) = 0, & w_1 \geq \bar{w}_1 \end{cases}$

$\hookrightarrow \begin{cases} \eta_{a_0}(w) > 0 & \forall w \in \{\bar{w}_1 < w_1 < w_1^+\} \cap \text{supp } \nu \\ \eta_{a_0} \eta_a - \eta_a \eta_{a_0} \equiv 0 \end{cases}$

$\hookrightarrow \langle \nu, \eta_{a_0} \rangle \langle \nu, \eta_a \rangle = \langle \nu, \eta_a \rangle \langle \nu, \eta_{a_0} \rangle$

$$\begin{aligned} \hookrightarrow \langle v, \varrho_a \rangle &= C_1(\bar{w}_1) \langle v, \eta_a \rangle \\ &\quad \forall a \in C, \quad a(w_1) = 0, \quad w_1 \geq \bar{w}_1. \end{aligned}$$

$$\left[ C_1(\bar{w}_1) = \frac{\langle v, \varrho_{a_0} \rangle}{\langle v, \eta_{a_0} \rangle} \right]$$

Similarly,  $\forall a \in C, \quad a(w_1) = 0$  when  $w_1 \leq \bar{w}_1$ ,

$$\langle v, \varrho_a \rangle = C_1(\bar{w}_1) \langle v, \eta_a \rangle$$

• claim  $C_1(\bar{w}_1) \not\rightarrow \bar{w}_1$ .

For any  $\tilde{w}_1 \in (w_1^-, w_1^+)$ , choose

$$\left\{ \begin{array}{l} a_1(w_1) = 0 \quad w_1 \leq \tilde{w}_1 \\ a_2(w_1) = 0 \quad w_1 \geq \bar{w}_1 \end{array} \right. \quad \text{for } \tilde{w}_1 < \bar{w}_1$$

$$\left\{ \begin{array}{l} a_1(w_1) = 0 \quad w_1 \geq \tilde{w}_1 \\ a_2(w_1) = 0 \quad w_1 \leq \bar{w}_1 \end{array} \right. \quad \text{for } \tilde{w}_1 > \bar{w}_1$$

$$\Rightarrow \eta_{a_1} \varrho_{a_2} - \eta_{a_2} \varrho_{a_1} \equiv 0$$

$$\hookrightarrow C_1(\bar{w}_1) = C_1(\tilde{w}_1)$$

2. claim If  $w_2^- < w_2^+$ , then  $\exists C_2(t, x)$  s.t. C-20

$$\langle v, \eta_b \rangle = C_2 \langle v, \eta_b \rangle$$

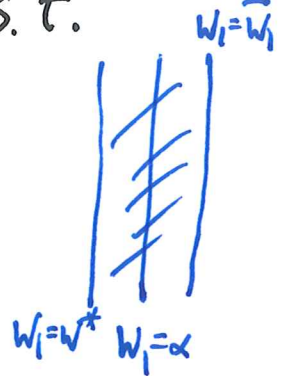
$$\forall b \in C, \quad b(w_2) = 0 \quad \text{when} \quad \begin{cases} w_2 \geq \bar{w}_2 \\ \text{or} \\ w_2 \leq \bar{w}_2 \end{cases} \quad \bar{w}_2 \in (w_2^-, w_2^+)$$

3.  $\forall \alpha \in (w_1^-, w_1^+)$ , choose  $w_1^*, \bar{w}_1$  s.t.

$$w_1^* < \alpha < \bar{w}_1, \quad \bar{w}_1 - w_1^* \ll 1$$

Choose  $(\eta_a, \eta_{a'})$ :  $a(w_1) = (w_1 - w_1^*)_+$

Choose  $(\eta_{\bar{a}}, \eta_{\bar{a}'})$ :  $\bar{a}(w_1) = (w_1 - \bar{w}_1)_-$



$$\begin{aligned} \langle v, \eta_a \eta_{\bar{a}'} - \eta_{\bar{a}} \eta_{a'} \rangle &= \langle v, \eta_a \rangle \langle v, \eta_{\bar{a}'} \rangle - \langle v, \eta_{\bar{a}} \rangle \langle v, \eta_{a'} \rangle \\ &= 0 \end{aligned}$$

We know that, on  $\{w_1 < w_1^*\} \cup \{w_1 > \bar{w}_1\}$ ,

$$\eta_a \eta_{\bar{a}'} - \eta_{\bar{a}} \eta_{a'} = 0$$

$$O_n \{ \underline{w}_1^* \leq w_1 \leq \bar{w}_1 \}.$$

$$\left\{ \begin{aligned} \eta_a(w) &= I_1(w)(w_1 - w_1^*) + \frac{1}{2} J_1(\alpha; w)(w_1 - w_1^*)^2 + O(|w_1 - w_1^*|^3) \\ g_a(w) &= K_1(w)(w_1 - w_1^*) + \frac{1}{2} L_1(\alpha; w)(w_1 - w_1^*)^2 + O(|w_1 - w_1^*|^3) \end{aligned} \right.$$

$$\left\{ \begin{aligned} \eta_{\bar{a}}(w) &= I_1(w)(w_1 - \bar{w}_1) + \frac{1}{2} J_1(\alpha; w)(w_1 - \bar{w}_1)^2 + O(|w_1 - \bar{w}_1|^3) \\ g_{\bar{a}}(w) &= K_1(w)(w_1 - \bar{w}_1) - \frac{1}{2} L_1(\alpha; w)(w_1 - \bar{w}_1)^2 + O(|w_1 - \bar{w}_1|^3) \end{aligned} \right.$$

$$(\eta_a g_{\bar{a}} - \eta_{\bar{a}} g_a)(w)$$

$$= \frac{1}{2} (\bar{w}_1 - w_1^*)(\bar{w}_1 - w_1)(w_1 - w_1^*) \left( I^2 \frac{\partial \lambda_1}{\partial w_1} \right) (\alpha, w_2) \\ + O((\bar{w}_1 - w_1^*)^2 (\bar{w}_1 - w_1)(w_1 - w_1^*))$$

$$\Rightarrow \langle \nu, (\bar{w}_1 - w_1)_+ (w_1 - w_1^*)_+ \left( \underbrace{I^2 \frac{\partial \lambda_1}{\partial w_1}}_{\neq 0} (\alpha, w_2) + O(|\bar{w}_1 - w_1^*|) \right) \rangle = 0$$

$$\Rightarrow \text{Supp } \nu \cap \{ w_1^* \leq w_1 \leq \bar{w}_1 \} = \emptyset \quad \forall \begin{matrix} w_1^* < \bar{w}_1 \\ |\bar{w}_1 - w_1^*| \ll 1. \end{matrix}$$

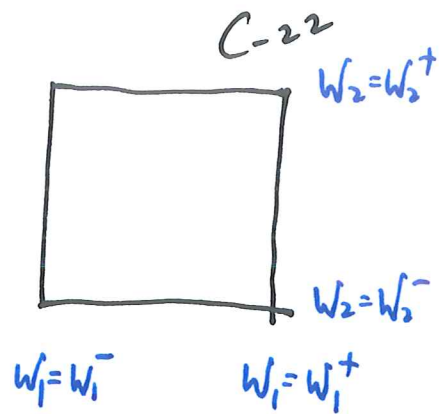
$$\hookrightarrow \text{Supp } \nu \cap \{ w_1^- < w_1 < w_1^+ \} = \emptyset$$

Similarly,

$$\hookrightarrow \text{Supp } \nu \cap \{ w_2^- < w_2 < w_2^+ \} = \emptyset$$

4. If

$$\text{Supp } \nu \cap (\{w_1 = w_1^\pm\} \cup \{w_2 = w_2^\pm\}) \neq \emptyset$$



for example

$$\text{Supp } \nu \cap \{w_1 = w_1^-\} \neq \emptyset$$

$$\Rightarrow \nu(\{w_1 = w_1^-\}) \neq 0$$

then we follow Step 3 to choose

$$\alpha = w_1^-, \quad \bar{w}_1 = w_1^- + \varepsilon, \quad w_1^* = w_1^- - \varepsilon.$$

to conclude

$$\text{Supp } \nu \cap \{w_1^- - \varepsilon \leq w_1 \leq w_1^- + \varepsilon\} = \emptyset$$

for sufficiently small  $\varepsilon > 0$

↳ Contradiction

5. Conclusion

$$\text{Supp } \nu \cap ([w_1^-, w_1^+] \times [w_2^-, w_2^+]) = \emptyset$$

↳ Contradiction

$$\Rightarrow \nu_{t,x} = \delta_{U(t,x)} \quad \text{Single point support.}$$