Analysis of PDE-3: Problem Set-1

29 April 2019

Instructions: Please submit your complete work by **3:00pm Monday, 13 May 2019**. Please work on these problems only by yourself.

1. (i) Assume that

$$\begin{cases} \mathbf{u}_k \rightharpoonup \mathbf{u} & \text{in } L^2(0,T;H_0^1(\Omega)), \\ \mathbf{u}_k' \rightharpoonup \mathbf{v} & \text{in } L^2(0,T;H^{-1}(\Omega)). \end{cases}$$

Prove that $\mathbf{v} = \mathbf{u}'$.

(ii) Suppose that H is a Hilbert space. Assume that

$$\mathbf{u}_k \rightharpoonup \mathbf{u} \qquad \text{in } L^2(0,T;H),$$

and, for $k = 1, 2, \ldots$,

$$\operatorname*{ess\,sup}_{0\le t\le T} \|\mathbf{u}_k(t)\|_H \le C$$

for some C independent of k. Prove

$$\operatorname*{ess\,sup}_{0\le t\le T} \|\mathbf{u}(t)\|_H \le C.$$

(iii) Assume that Ω is open, bounded, and $\partial\Omega$ is smooth. Suppose that $\mathbf{u} \in L^2(0,T;H^2(\Omega))$ with $\mathbf{u}' \in L^2(0,T;L^2(\Omega))$. Prove

(a) $\mathbf{u} \in C([0,T];H^1(\Omega))$ (after possibly being redefined on a set of measure zero).

(b) the following estimate holds:

$$\max_{0\le t\le T} \|\mathbf{u}\|_{H^1(\Omega)} \le C\big(\|\mathbf{u}\|_{L^2(0,T;H^2(\Omega))} + \|\mathbf{u}'\|_{L^2(0,T;L^2(\Omega))}\big)$$

where the constant C depends only on T and Ω.

2. Estimates for solutions in bounded regions for symmetric hyperbolic system. Consider the following hyperbolic system:

$$\mathbf{L}\mathbf{u} := \mathbf{B}_0(t,x)\mathbf{u}_t + \sum_{j=1}^{n} \mathbf{B}_j(t,x)\mathbf{u}_{x_j} + \mathbf{C}(t,x)\mathbf{u} = \mathbf{f}, \qquad \mathbf{u} \in \mathbb{R}^m,$$

where $\mathbf{B}_0, \mathbf{B}_1, \ldots, \mathbf{B}_n, \mathbf{C}$ are given $m \times m$ square matrices, and $\mathbf{f}$ a given m-vector function.

Definition. A hyper-surface $S = \{t = \phi(x)\}$ is called "space-like" with respect to the operator $\mathbf{L}$ if the matrix

$$\mathbf{B}_0(t,x) - \sum_{j=1}^{n} \phi_{x_j}(x)\mathbf{B}_j(t,x)$$

is positive definite for all $(t, x) \in S$.

(i) Write the wave equation $u_{tt} - c^2 \Delta u = 0$ (c is a constant) as a symmetric hyperbolic system. Show that the definition of "space-like" agrees with the following:

$$1 - c^2 \sum_{j=1}^{n} \phi_{x_j}^2 > 0.$$

(ii) For fixed positive a, T, and for $0 < \lambda < T$, consider the "truncated cone":

$$R_\lambda = \{(t, x) \,:\, |x| \leq \frac{a(T-t)}{T}, 0 \leq t \leq \lambda\}$$

bounded by the planes $t = 0$ and $t = \lambda$, and the conical surface:

$$S_\lambda = \{(t, x) \,:\, t = T - \frac{T}{a}|x|, 0 \leq t \leq \lambda\}.$$

We call R_λ space-like if S_λ is space-like, that is,

$$\mathbf{B}_0(t, x) + \frac{T}{a|x|} \sum_{j=1}^{n} x_j \mathbf{B}_j(t, x).$$

is positive definite for all $(t, x) \in S_\lambda$. Let $\mathbf{B}_0, \mathbf{B}_1, \ldots, \mathbf{B}_n$ be symmetric, $\mathbf{B}_0$ positive definite, and R_λ be space-like. Let $u \in C^1(R_\lambda)$ be a solution of

$$\begin{cases} \mathbf{Lu} = \mathbf{f} & \text{for } (t, x) \in R_\lambda, \\ \mathbf{u}|_{t=0} = \mathbf{g}(x) & \text{for } |x| \leq a. \end{cases} \tag{1}$$

Set

$$E(\mu) = \int_{\sigma_\mu} \mathbf{u}^\top \mathbf{B}_0 \mathbf{u} \, dx,$$

where σ_μ is the cross section:

$$\sigma_\mu = \{(t, x) \,:\, (t, x) \in R_\lambda, t = \mu\}$$

of R_λ. Show that, for $0 < \mu < \lambda$,

$$E(\mu) \leq E(0) + \int_{R_\mu} (-\mathbf{u}^\top \mathbf{D} \mathbf{u} + 2\mathbf{u}^\top \mathbf{f}) dx dt,$$

where $\mathbf{D} = 2\mathbf{C} - \mathbf{B}_{0,t} - \sum_{j=1}^{n} \mathbf{B}_{j,x_j}$.

(iii) Show that there exists a constant K (depending only on upper bounds for the matrices $\mathbf{B}_0^{-1}, \mathbf{B}_0, \mathbf{B}_1, \ldots, \mathbf{B}_n, \mathbf{C}$ and their first derivatives in R_λ) such that

$$\int_{\sigma_\mu} |\mathbf{u}|^2 \, dx \leq K \left[\iint_{R_\mu} |\mathbf{f}|^2 dx dt + \int_{|x|<a} |\mathbf{g}|^2 dx \right]$$

for $0 \leq \mu \leq \lambda$ (This implies that $\mathbf{u}$ in R_λ is determined uniquely by the values of $\mathbf{f}$ in R_λ of $\mathbf{g}$ on $|x| \leq a$). [Hint: Estimate the forms $\xi^\top \mathbf{B}_0 \xi, \xi^\top \mathbf{D} \xi$, using $\inf_{|\xi|=1} \xi^\top \mathbf{B}_0 \xi = (\sup_{|x|=1} \xi^\top \mathbf{B}_0^{-1} \xi)^{-1}$; For $\phi, \phi', \psi, \psi' \geq 0$, the inequality $\phi'(\mu) \leq \gamma\big(\phi(\mu) + \psi(\mu) + \phi'(0)\big)$ implies the "Gronwall lemma": $\phi'(\mu) \leq \gamma e^{\gamma\mu}\big(\psi(\mu) + \phi'(0)\big)$.]

(iv) For $\mathbf{g} = \mathbf{g}(x), \mathbf{f} = \mathbf{f}(t,x)$ and an integer $k \geq 0$, define

$$\|\mathbf{g}\|_k = \sqrt{\sum_{|\alpha|\leq k} \int_{|x|\leq a} |D^\alpha \mathbf{g}|^2 dx}$$

$$\|\mathbf{f}\|_k = \sqrt{\sum_{|\alpha|\leq k} \int_{\sigma_\mu} |D^\alpha \mathbf{f}|^2 dx}$$

Show that there exists a constant K_k depending on upper bounds for the matrices $\mathbf{B}_0^{-1}, \mathbf{B}_0, \mathbf{B}_1, \ldots, \mathbf{B}_n, \mathbf{C}$ and their derivatives of orders $\leq k+1$ such that, for $0 < \mu < \lambda$,

$$\|\mathbf{u}(\mu)\|_k^2 \leq K_k \left(\int_0^\mu \|\mathbf{f}(\gamma)\|_k^2 d\gamma + \|\mathbf{g}\|_k^2 \right).$$

[Hint: Show that, for $|\alpha \leq k$, we have $\mathbf{L} D^\alpha \mathbf{u} = D^\alpha \mathbf{f} + \mathbf{L}_\alpha$, where $\mathbf{L}_\alpha$ is an operator of order $\leq k$. Apply Gronwall's lemma with $\phi(\mu) = \int_0^\mu \|\mathbf{u}\|_k^2 d\gamma$ and $\psi(\mu) = \int_0^\mu \|\mathbf{f}(\gamma)\|_k^2 d\gamma$.]

(v) Let $s = [\frac{n}{2}] + 1$, and let $k > 0$. Let $\mathbf{B}_0, \mathbf{B}_1, \ldots, \mathbf{B}_n, \mathbf{C} \in C^{k+s+1}(R_\lambda)$, $\mathbf{f} \in C^{k+s}(R_\lambda)$. Show that there exists a constant K_k (depending on upper bounds for $\mathbf{B}_0^{-1}, \mathbf{B}_0, \mathbf{B}_1, \ldots, \mathbf{B}_n, \mathbf{C}$ and their derivatives of orders $\leq k+s+1$ in R_λ) such that, for a solution $\mathbf{u} \in C^m(R_\lambda)$ of (1), the inequalities

$$|D^\alpha \mathbf{u}(t,x)| \leq K_k \left[\max_{|\beta|\leq k+s} \sup_{R_\lambda} |D^\beta \mathbf{f}(t,x)| + \max_{|\beta|\leq k+s} \sup_{|x|<a} |D^\beta \mathbf{g}(x)| \right]$$

hold for any $(t,x) \in R_\lambda, |\alpha| \leq k$.

3. Consider the following the Cauchy problem:

$$\begin{cases} u_t + f(u)_x + u = 0, \\ u|_{t=0} = u_0(x), \end{cases} \tag{2}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a given C^1 function and $u_0 \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$ is given initial data function.

Definition. A function $u \in L^\infty(\mathbb{R}_+ \times \mathbb{R})$ is called an entropy solution of Problem (2) provided that, for any convex entropy $\eta(u), \eta''(u) \geq 0$, and corresponding entropy flux $q(u) = \int^u \eta'(v) f'(v) dv$, and any non-negative test function $\psi \in C_0^\infty(\mathbb{R} \times \mathbb{R}), \psi \geq 0$,

$$\int_0^\infty \int_{-\infty}^\infty (\eta(u)\psi_t + q(u)\psi_x - \eta'(u)u\psi)\, dxdt + \int_{-\infty}^\infty \psi(0,x)\eta(u_0(x))dx \geq 0.$$

(i) Prove that problem (2) admits a global entropy solution $u \in C([0,\infty); L^1_{loc}(\mathbb{R}))$ via the vanishing viscosity method.

(ii) Let $u(t,\cdot), v(t,\cdot) \in C([0,\infty); L^1_{loc}(\mathbb{R}))$ be entropy solutions with the initial data functions $u_0, v_0 \in L^\infty(\mathbb{R})$, respectively. Prove in detail via the test function method that there exists $s > 0$ depending on $\|u_0\|_{L^\infty}$ and $\|v_0\|_{L^\infty}$ such that, for any $t > 0$ and $r > 0$,

$$\int_{-r}^{r} [u(t,x) - v(t,x)]^+ dx \le \int_{-r-st}^{r+st} [u_0(x) - v_0(x)]^+ dx,$$

and

$$\|u(t,\cdot) - v(t,\cdot)\|_{L^1(\mathbb{R})} \le e^{-t} \|u_0(\cdot) - v_0(\cdot)\|_{L^1(\mathbb{R})}. \tag{3}$$

(iii) If $u_0 \in BV_{loc}(\mathbb{R}) \cap L^\infty(\mathbb{R})$, then

$$u(t,x) \in BV_{loc}(\mathbb{R}_+ \times \mathbb{R}).$$

(iv) If equation in (2) is replaced by

$$u_t + f(u)_x + a(x)u = 0,$$

with $a \in C^1(\mathbb{R})$ and $|a(x)| \le a_0 < \infty$, can a similar L^1-stability estimate to (3) be obtained with e^{-t} replaced by another factor? If so, please prove your claim; otherwise, please provide your reason(s).