Analysis of PDE-3: Problem Set-2

29 April 2019

Instructions: Please submit your complete work by **3:00pm Monday, 27 May 2019**. Please work on these problems only by yourself.

1. Consider the hyperbolic system

$$\mathbf{u}_t + \mathbf{A}\mathbf{u}_x = 0, \qquad \mathbf{u} \in \mathbb{R}^m. \tag{1}$$

(i) Show that, when $\mathbf{A}$ is a constant $m \times m$ matrix with m distinct real eigenvalues λ_j and corresponding right eigenvectors $\mathbf{r}_j$, $j = 1, \ldots, m$, then the following superposition of travelling waves:

$$\mathbf{u}(t,x) = \sum_{j=1}^{m} (\mathbf{l}_j \cdot \mathbf{u}_0)(x - \lambda_j t)\, \mathbf{r}_j,$$

is a unique C^1 solution of system (1) with the Cauchy data:

$$\mathbf{u}(0,x) = \mathbf{u}_0(x) \in C^1.$$

(ii) Let $\mathbf{A} = \mathbf{A}(\mathbf{u})$ have m distinct real eigenvalues $\lambda_j(\mathbf{u})$ and corresponding left and right eigenvectors $\mathbf{l}_j(\mathbf{u})$ and $\mathbf{r}_j(\mathbf{u})$, $j = 1, \ldots, m$, with $\mathbf{l}_i \cdot \mathbf{r}_j = 1$ when $i = j$ and 0 when $i \neq j$. Define the density of i–wave in $\mathbf{u}$:

$$u_x^i := \mathbf{l}_i(\mathbf{u}) \cdot \mathbf{u}_x.$$

Show that each u_x^i ($i = 1, \ldots, m$) is determined by the following evolution equation:

$$\left(u_x^i\right)_t + \left(\lambda_i u_x^i\right)_x = \sum_{j>k} (\lambda_j - \lambda_k)\left(\mathbf{l}_i \cdot [\mathbf{r}_j, \mathbf{r}_k]\right) u_x^j u_x^k,$$

where $[\mathbf{r}_j, \mathbf{r}_k]$ is the Lie bracket for vectors $\mathbf{r}_j$ and $\mathbf{r}_k$.

2. Consider the hyperbolic system of conservation laws:

$$\mathbf{u}_t + \mathbf{f}(\mathbf{u})_x = 0, \qquad \mathbf{u} \in \mathbb{R}^m, \tag{2}$$

with m linear independent right eigenvectors.

(i) State Lax's theorem for the existence and uniqueness of solutions of the general Riemann problem.

(ii) Describe in detail the construction of the front tracking approximate solutions for the general Cauchy problem for (2).

(iii) Describe in detail the construction of the Glimm approximate solutions for the general Cauchy problem for (2). Explain the reasons why the uniformly distributed random sequence is necessary for the scheme to converge to the right entropy solution.

[Hint: Please consult some references listed in the lecture notes]

3. Consider the following Euler equations in Lagrangian coordinates:

$$\begin{cases} v_t - u_x = 0, \\ u_t + p(v)_x = 0, \end{cases} \tag{3}$$

where u is the velocity, $v = \frac{1}{\rho} > 0$ with density ρ, and $p(v)$ is pressure:

$$p(v) = \frac{v^{-\gamma}}{\gamma}, \qquad \gamma > 1.$$

(i) Compute the eigenvalues, renomalised left and right eigenvectors, and show that system (3) is strictly hyperbolic when $v < \infty$.

(ii) Show that both fields of system (3) are genuinely nonlinear when $v < \infty$.

(iii) Compute the corresponding Riemann invariants $\mathbf{w} = (w_1, w_2)$ of system (3).

(iv) Derive the entropy equation for the entropy function in the (u, v)–coordinates, and in the coordinates of the Riemann invariants $\mathbf{w} = (w_1, w_2)$, respectively. Can you find a strictly convex entropy for system (3) when $v < \infty$?

(v). Employ the entropy conditions to determine the shock curves and the rarefaction wave curves in the (u, v)–coordinates.

4. Consider the Cauchy problem for the following scalar conservation laws:

$$\begin{cases} u_t + f(u)_x = 0, \\ u|_{t=0} = u_0(x) \in L^1 \cap L^\infty(\mathbb{R}), \end{cases} \tag{4}$$

where $f \in C^2(\mathbb{R}_+ \times \mathbb{R}; \mathbb{R})$. To obtain entropy solutions of problem (1), we use the method of vanishing viscosity to construct global viscosity approximate solutions $u^\varepsilon = u^\varepsilon(t, x)$ that are solutions of the following Cauchy problem:

$$\begin{cases} u_t + f(u)_x = \varepsilon u_{xx}, \\ u|_{t=0} = u_0^\varepsilon(x) \to u_0(x) \qquad \text{a.e.} \end{cases} \tag{5}$$

where $\|u_0^\varepsilon\|_{L^1 \cap L^\infty(\mathbb{R})} \leq \|u_0\|_{L^1 \cap L^\infty(\mathbb{R})}$.

(i) Show that

$$\|u^\varepsilon\|_{L^\infty} \leq \|u_0\|_{L^\infty}.$$

(ii) There exists a constant $C > 0$ independent of $\varepsilon > 0$ such that

$$\|\sqrt{\varepsilon} u_x^\varepsilon\|_{L^2} \leq C.$$

(iii) For any C^2 entropy function $\eta(u)$ and entropy flux $q(u) = \int^u \eta'(v) f'(v) dv$, show that

$$\eta(u^\varepsilon)_t + q(u^\varepsilon)_x \qquad \text{are compact in } H^{-1}_{loc}.$$

(iv) Let $\{\nu_{t,x}\}_{(t,x)\in\mathbb{R}_+\times\mathbb{R}}$ be the family of the Young measures determined by the sequence $\{u^\varepsilon\}_{\varepsilon>0}$. Apply the Div-Curl lemma to prove that $\{\nu_{t,x}\}_{(t,x)\in\mathbb{R}_+\times\mathbb{R}}$ satisfies

$$\begin{aligned}
&\langle \nu_{t,x}, \eta_1(\lambda)q_2(\lambda) - \eta_2(\lambda)q_1(\lambda)\rangle \\
&= \langle \nu_{t,x}, \eta_1(\lambda)\rangle\langle \nu_{t,x}, q_2(\lambda)\rangle - \langle \nu_{t,x}, \eta_2(\lambda)\rangle\langle \nu_{t,x}, q_1(\lambda)\rangle
\end{aligned}$$

(v) Show that f is weakly continuous subsequentially with respect to the sequence u^ε:

$$\langle \nu_{t,x}, f(\lambda)\rangle = f(\langle \nu_{t,x}, \lambda\rangle).$$

In addition, if $f''(u) > 0$, show that the support of $\nu_{t,x}$ is only single point almost everywhere:

$$\nu_{t,x}(\lambda) = \delta_{w^*-\lim u^\varepsilon}(\lambda).$$

5. Assume u is an entropy solution of the scalar conservation law:

$$u_t + f(u)_x = 0, \qquad u \in \mathbb{R},$$

(that is, u satisfies the entropy inequality in the sense of distribution for any convex entropy pair), and u is smooth on either side of a curve $\{x = s(t)\}$.

(i) Prove that, along this curve, the left hand limit u_l and right hand limit u_r of u satisfying the relation:

$$f(v) \geq \frac{f(u_r) - f(u_l)}{u_r - u_l}(v - u_r) + f(u_r) \qquad \text{if } u_l \leq v \leq u_r,$$

and

$$f(v) \leq \frac{f(u_r) - f(u_l)}{u_r - u_l}(v - u_r) + f(u_r) \qquad \text{if } u_r \leq v \leq u_l.$$

This is called *Condition (E)*.

(ii) What does Condition (E) implies if $f(u)$ is uniformly convex?