Geometric Measures and Conservation Laws

Lecture Room C1
Weeks 1-4, Hilary Term 2020
Wednesdays 14:00-16:00

By Professor Gui-Qiang G. Chen

Geometric Measures

have contributed greatly to the development of

Conservation Laws, Partial Differential Equations, Calculus of Variations, Geometric Analysis, …

have a wide range of applications to

Differential Geometry/Topology,
Continuum Physics, Fluid Mechanics,
Stochastic Analysis, Dynamical Systems, ……

- An introduction to some facets of Geometric Measures and Conservation Laws, and related applications.
- Basic Analysis & PDE - the only essential prerequisites.
- However, some familiarity with basic measure theory, functional analysis, nonlinear PDEs, and differential geometry is desirable.

Topics:

1. Connections: Geometric Measures and Conservation Laws
2. Review: Basic Measure Theory
3. Hausdorff Measures
4. Area/Co-Area Formulas
5. BV Functions and Sets of Finite Perimeter
6. Theory of Divergence-Measure Fields and Connections with Conservation Laws
7. *Differentiability and Approximation
8. *Varifolds and Currents
9. *Further Connections with Nonlinear PDEs

The topics with * are optional, depending on the course development.

References:

1. **H. Federer**: **Geometric Measure Theory**, Springer-Verlag: Berlin,1996.
2. **L. C. Evans & R. F. Gariepy**: **Measure Theory and Fine Properties of Functions ,** CRC Press: Boca Raton, Florida, 1992.
3. **R. Hardt & L. Simon**: **Seminar on Geometric Measure Theory,** Birkhauser, 1986.
4. **L. Ambrosio, N. Fusco & D. Pallara: Functions of Bounded Variation and Free Discontinuity Problems**, Oxford Univ. Press, 2000.
5. **W. P. Ziemer**: **Weakly Differentiable Functions**, Springer: NY, 1989.
6. **C. M. Dafermos: Hyperbolic Conservation Laws in Continuum Physics,** 4th Edition, Springer-Verlag: Berlin, 2016.
7. **F. Morgan**: **Geometric Measure Theory: A Beginners Guide**, Academic Press: Boston, 1988.
8. **H. Whitney**: **Geometric Integration Theory**, Princeton Univ. Press, 1957
9. **L. C. Evans: Partial Differential Equations,** 2nd ed., AMS: Providence, RI, 2010.
10. **G.-Q. Chen**: **Some Lecture Notes**

Geometric Measure Theory

could be described as differential geometry, generalised through measure theory to deal with

Maps
Surfaces

that are not necessarily smooth.

DeGiorgi (1961), H. Federer (1969)

→ Almgren, Schoen-Simon, Bomberi, ...

Integration by Parts & Gauss-Green Theorem in Analysis

Integration by Parts (Taylor 1715): Let $f(y), g(y) \in C^1(\mathbb{R})$. Then

$$\int_a^b f(y)g'(y)\,\mathrm{d}y = (f(b)g(b) - f(a)g(a)) - \int_a^b f'(y)g(y)\,\mathrm{d}y \qquad \text{for any } a \le b.$$

The rule is shown via the fundamental theorem of calculus and the product rule for derivatives:

$$f(b)g(b) - f(a)g(a) = \int_a^b \frac{d}{dy}(f(y)g(y))\,\mathrm{d}y = \int_a^b f'(y)g(y)\,\mathrm{d}\mathbf{y} + \int_a^b f(y)g'(y)\,\mathrm{d}y.$$

Gauss-Green Theorem (Divergence Theorem): Let $\Omega \Subset \mathcal{D} \subset \mathbb{R}^N$ be compact and have a smooth boundary. If $\mathbf{F} \in C^1(\mathcal{D}; \mathbb{R}^N)$, then

$$\int_\Omega \varphi\, div\,\mathbf{F}\,\mathrm{d}\mathbf{y} = -\int_{\partial\Omega} \varphi\,\mathbf{F}\cdot\boldsymbol{\nu}\,\mathrm{d}S - \int_\Omega \mathbf{F}\cdot\nabla\varphi\,\mathrm{d}\mathbf{y} \qquad \text{for any } \varphi \in C^1(\mathbb{R}^N; \mathbb{R}^N),$$

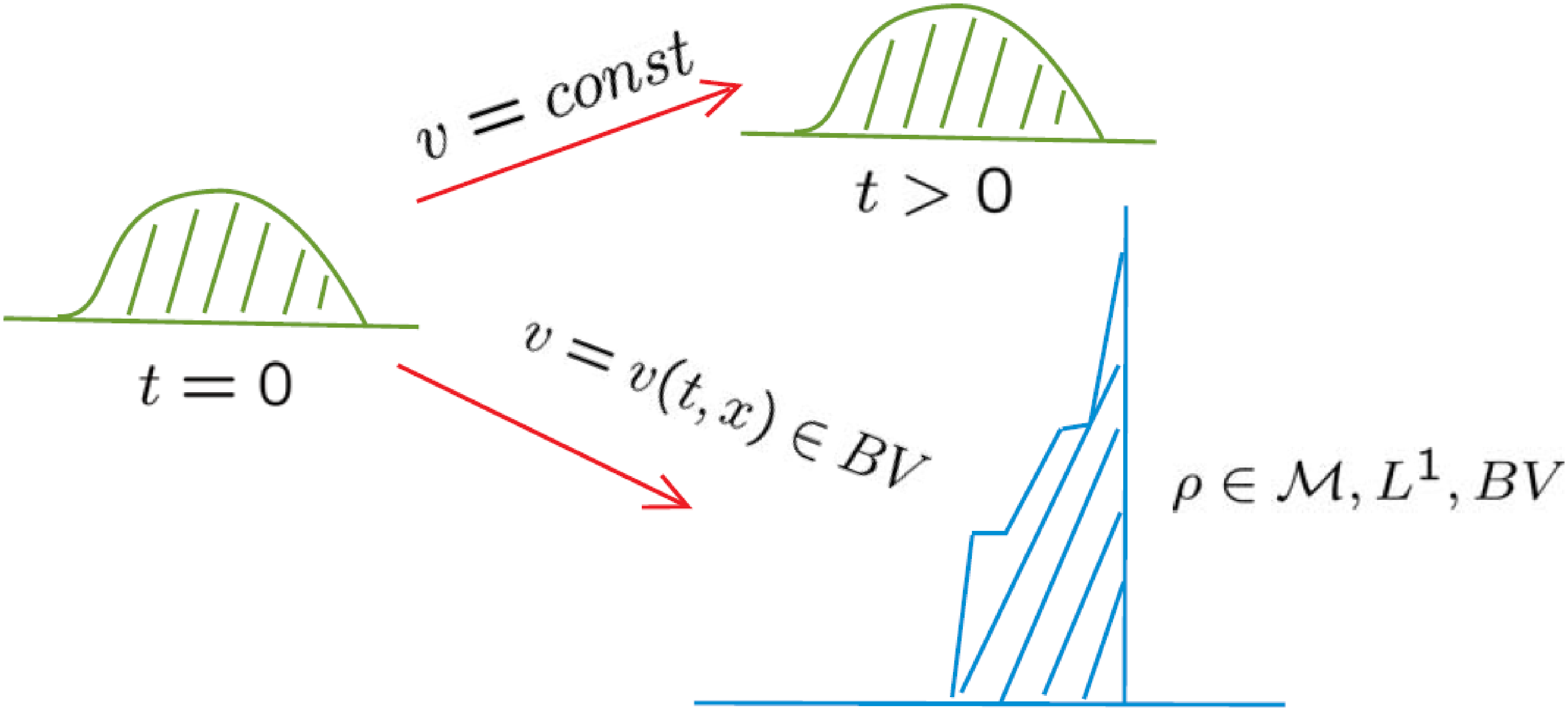
where $\boldsymbol{\nu}$ is the unit interior normal on $\partial\Omega$ to Ω and dS is the surface measure (Carl Friedrich Gauss in 1813, George Green in 1825).

One of the Major Achievements of 20th Century in Mathematical Analysis:

Spaces of "Generalized" Functions: Sobolev Spaces, BV Space, $\cdots$

Calculus of "Generalized" Functions: Traces, Gauss-Green formula, $\cdots$

Transport Equation: $\partial_t \rho + \partial_x (v\rho) = 0$

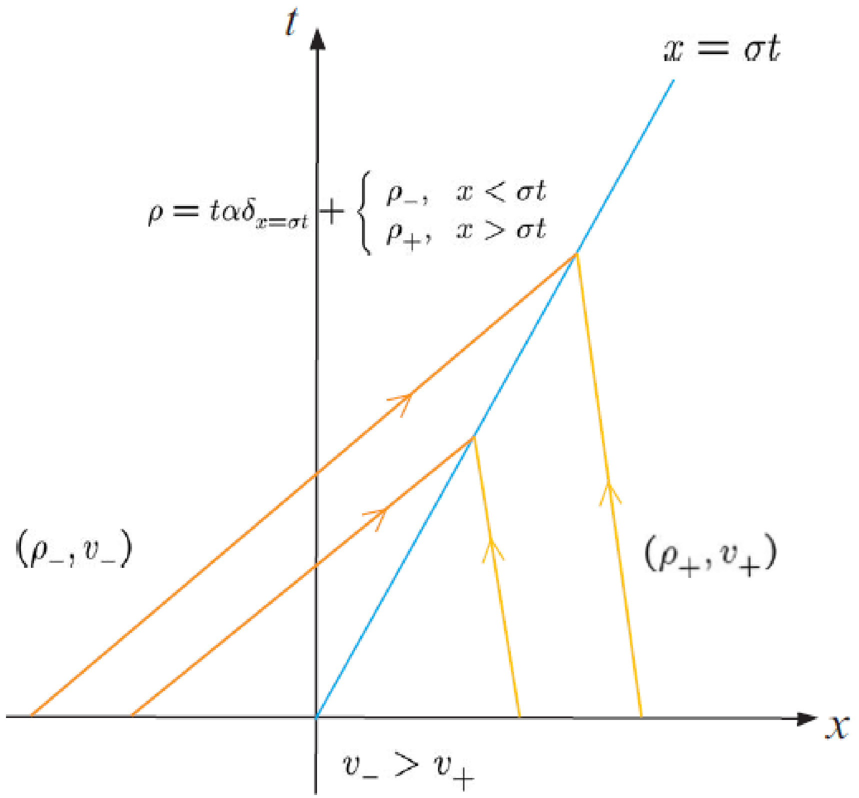


$$\int \mathrm{d}_x \rho(t,x) = \int \rho(0,x)\mathrm{d}x$$

* v - velocity, ρ - density

Pressureless Euler Equations

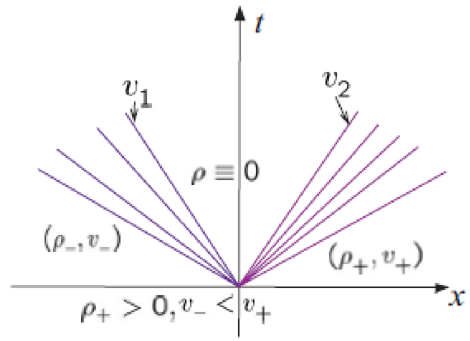
$$\partial_t \rho + \partial_x(\rho v) = 0, \qquad \partial_t(\rho v) + \partial_x(\rho v^2) = 0$$



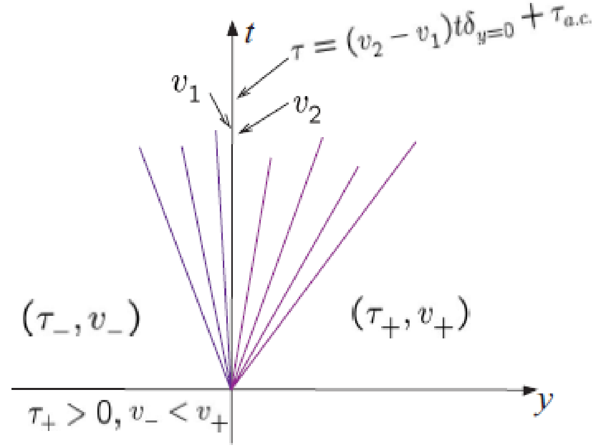
$$\alpha = \frac{1}{\sqrt{1+\sigma^2}}(\sigma[\rho] - [\rho v]) > 0, \qquad \sigma = \frac{\sqrt{\rho_+}v_+ + \sqrt{\rho_-}v_-}{\sqrt{\rho_+} + \sqrt{\rho_-}} \in (v_+, v_-)$$

Isentropic Euler Equations: Pressure Function $p(\rho) = \kappa\rho^\gamma$

$$\partial_t \rho + \partial_x(\rho v) = 0, \qquad \partial_t(\rho v) + \partial_x(\rho v^2 + p(\rho)) = 0$$



$$(t, x) \to (t, y) : \ y_t = \rho(t, x), y_x = -(\rho v)(t, x); \qquad \tau(t, y) = 1/\rho(t, x)$$



$$\partial_t \tau - \partial_y v = 0, \qquad \partial_t v + \partial_y p(1/\tau) = 0$$

Nonlinear Hyperbolic Conservation Laws

$$\partial_t \mathbf{u} + \nabla_{\mathbf{x}} \cdot \mathbf{f}(\mathbf{u}) = 0$$

$$\mathbf{u} = (u_1, \cdots, u_m)^\top, \ \ \mathbf{x} = (x_1, \cdots, x_d), \ \ \nabla_{\mathbf{x}} = (\partial_{x_1}, \cdots, \partial_{x_d})$$

$\mathbf{f} = (\mathbf{f}_1, \cdots, \mathbf{f}_d) : \ \mathbb{R}^m \to (\mathbb{R}^m)^d$ is a nonlinear mapping

$\mathbf{f}_i : \ \mathbb{R}^m \to \mathbb{R}^m$ for $i = 1, \cdots, d$

Connections and Applications:

- Fluid Mechanics and Related: Euler Equations and Related Equations
  Gas, shallow water, elastic body, reacting gas, plasma, ...
- Special Relativity: Relativistic Euler Equations and Related Equations
  General Relativity: Einstein Equations and Related Equations
- Differential Geometry: Isometric Embeddings, Nonsmooth Manifolds...
- ......

Challenges and Well-Posedness

$$\partial_t \mathbf{u} + \nabla_{\mathbf{x}} \cdot \mathbf{f}(\mathbf{u}) = 0$$

Challenges: Singularity $\longrightarrow$ Discontinuous/Singular Solutions

- Concentration, Cavitation, ...
- Shock Waves, Vortex Sheets, Vorticity Waves, Entropy Waves, ...
- Breaking and Focusing of Waves, ...
- ......

Entropy Solutions:

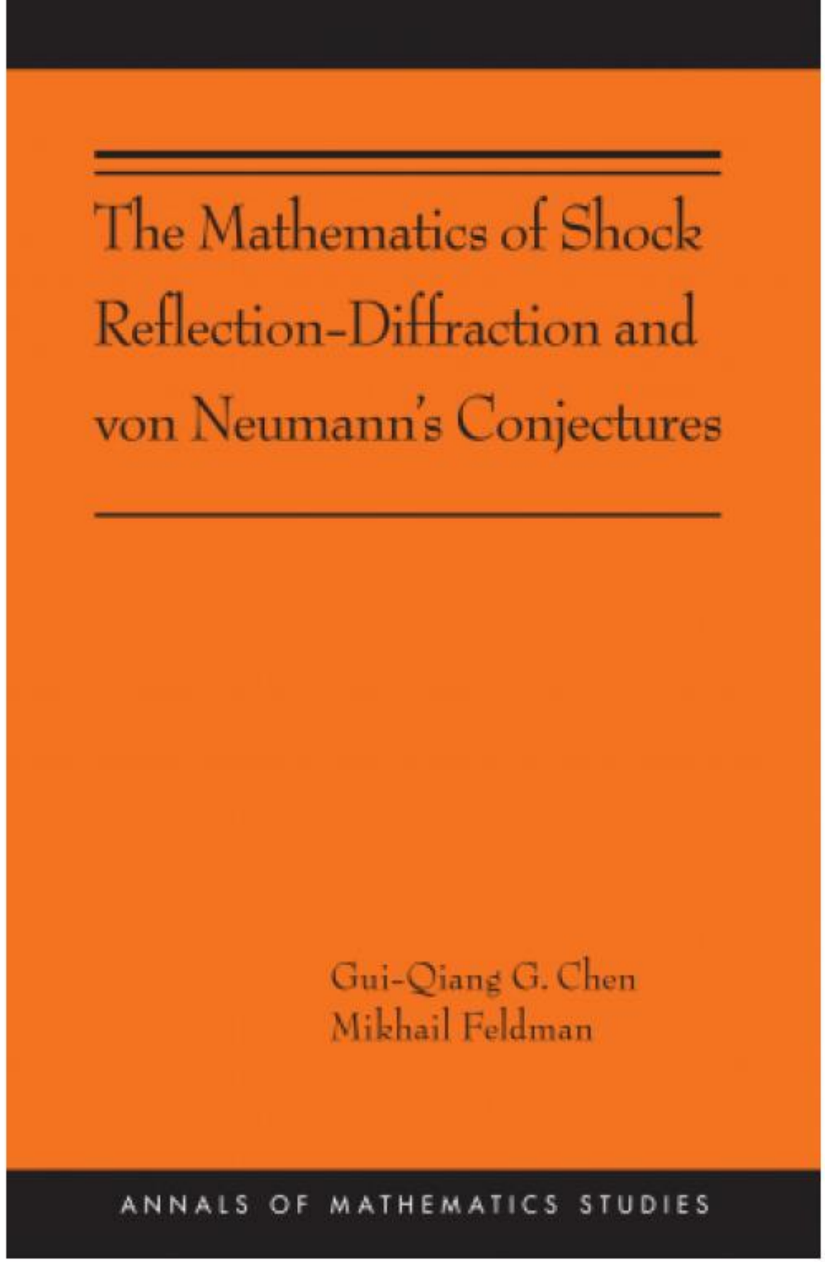
(i) $\mathbf{u}(t, \mathbf{x}) \in L^\infty, L^p, \mathcal{M}$;

(ii) For any convex entropy pair $(\eta, \mathbf{q})$, $\partial_t \eta(\mathbf{u}) + \nabla_{\mathbf{x}} \cdot \mathbf{q}(\mathbf{u}) \leq 0 \ \mathcal{D}'$

as long as $(\eta(\mathbf{u}(t, \mathbf{x})), \mathbf{q}(\mathbf{u}(t, \mathbf{x}))) \in \mathcal{D}'$; that is, $(\eta, \mathbf{q}) := (\eta, q_1, \ldots, q_d)$ is a solution of $\nabla q_k(\mathbf{u}) = \nabla \eta(\mathbf{u}) \nabla \mathbf{f}_k(\mathbf{u}), 1 \leq k \leq d$.

Posed Spaces for Entropy Solutions ??

Candidates: L^∞, L^p, $\mathcal{M}$, ...



Chen-Feldman: Research Monograph, 832 pages
Annals of Mathematics Studies, 197, **Princeton Univ. Press**, 2018

Entropy Methods for the Analysis of Entropy Solutions of Multidimensional Conservation Laws?

A general mathematical framework may be derived from the theory of divergence-measure fields via the entropy methods, which are based on the Entropy Solutions:

(i) $\mathbf{u}(t,\mathbf{x}) \in \mathcal{M},\ L^\infty,\ L^p$;

(ii) $\forall$ convex entropy pair $(\eta, \mathbf{q})$ (i.e. $\nabla q_k(\mathbf{u}) = \nabla\eta(\mathbf{u})\nabla\mathbf{f}_k(\mathbf{u}), k = 1, ..., d$),

$$\partial_t \eta(\mathbf{u}) + \nabla_{\mathbf{x}} \cdot \mathbf{q}(\mathbf{u}) \le 0 \qquad \mathcal{D}'$$

as long as $(\eta(\mathbf{u}(t,\mathbf{x})), \mathbf{q}(\mathbf{u}(t,\mathbf{x}))) \in \mathcal{D}'$.

Existence of entropy solutions in L^p via Compensated Compactness

Isentropic Euler Equations, Equations of elastodynamics, $\cdots$

Schwartz's lemma $\Longrightarrow$ $\quad div_{(t,\mathbf{x})}(\eta(\mathbf{u}(t,\mathbf{x})), \mathbf{q}(\mathbf{u}(t,\mathbf{x}))) \in \mathcal{M}$

$\Longrightarrow$ The vector field $(\eta(\mathbf{u}(t,\mathbf{x})), \mathbf{q}(\mathbf{u}(t,\mathbf{x})))$ is a divergence-measure field

Divergence-Measure Fields over an Open Set $\mathcal{D} \subset \mathbb{R}^N$

- For $1 \le p \le \infty$, $\mathbf{F}$ is called a $\mathcal{DM}^p(\mathcal{D})$–field if $\mathbf{F} \in L^p(\mathcal{D})$ and

$$\|\mathbf{F}\|_{\mathcal{DM}^p(\mathcal{D})} := \|\mathbf{F}\|_{L^p(\mathcal{D};\mathbb{R}^N)} + \|div\,\mathbf{F}\|_{\mathcal{M}(\mathcal{D})} < \infty \tag{1}$$

- The field $\mathbf{F}$ is called a $\mathcal{DM}^{ext}(\mathcal{D})$–field if $\mathbf{F} \in \mathcal{M}(\mathcal{D})$ and

$$\|\mathbf{F}\|_{\mathcal{DM}^{ext}(\mathcal{D})} := \|(\mathbf{F}, div\,\mathbf{F})\|_{\mathcal{M}(\mathcal{D})} < \infty \tag{2}$$

- $\mathbf{F}$ is called a $\mathcal{DM}^p_{loc}(\mathcal{D})$ field if $\mathbf{F} \in \mathcal{DM}^p(\Omega)$ and $\mathbf{F}$ called a $\mathcal{DM}^{ext}_{loc}(\mathcal{D})$ if $\mathbf{F} \in \mathcal{DM}^{ext}(\Omega)$, for any open set $\Omega \Subset \mathcal{D}$

$\mathcal{DM}^p(\mathcal{D})$ and $\mathcal{DM}^{ext}(\mathcal{D})$ are **Banach spaces**, which are LARGER than the space of BV fields (they coincide when $N = 1$).

BV theory (esp. the Gauss-Green Formula and Traces) has significantly advanced our understanding of solutions of nonlinear PDEs and related problems in the calculus of variations, differential geometry,...

Goal: Develop a $\mathcal{DM}$ theory to analyze entropy solutions without BV for nonlinear conservation laws and related problems via entropy methods.

Examples

1: $\mathbf{F}(y_1, y_2) = (\sin(\frac{1}{y_1 - y_2}), -\sin(\frac{1}{y_1 - y_2}))$.

(i) $\mathbf{F} \in \mathcal{DM}^\infty(\mathbb{R}^2)$, while $F_j \notin BV(\mathbb{R}^2)$ for $j = 1, 2$;

(ii) $\mathbf{F}$ has an essential singularity at each point of $L = \{y_1 = y_2\}$.

$\Longrightarrow$ $\mathbf{F}$ has no trace on L in the classical sense.

2: Whitney 1957: $\mathbf{F}(y_1, y_2) = (\frac{y_1}{y_1^2 + y_2^2}, \frac{y_2}{y_1^2 + y_2^2}) \in \mathcal{DM}^1_{loc}(\mathbb{R}^2)$.

However, for $\Omega = \{\mathbf{y} : |\mathbf{y}| < 1, y_2 > 0\}$,

$$\int_\Omega div\, \mathbf{F}\, \mathrm{d}\mathbf{y} = 0 \neq -\int_{\partial\Omega} \mathbf{F} \cdot \boldsymbol{\nu}\, \mathrm{d}\mathcal{H}^1 = \pi \quad \text{(in the classical sense)},$$

where $\boldsymbol{\nu}$ is the interior unit normal on $\partial\Omega$ to Ω

$\Longrightarrow$ The classical Gauss-Green theorem fails for a $\mathcal{DM}$-field.

3: For any $\mu_i \in \mathcal{M}(\mathbb{R}), i = 1, 2,$ with finite total variation,

$$\mathbf{F}(y_1, y_2) = (\mu_1(y_2), \mu_2(y_1)) \in \mathcal{DM}^{ext}(\mathbb{R}^2)$$

A non-trivial example of such fields is provided by the Riemann solutions of the 1-D Isentropic Euler equations in Lagrangian coordinates for which the vacuum generally develops.

Axiomatic Foundation for Continuum Physics
Cauchy Flux (Cauchy 1823, 1827): Derivation for
– PDE Systems of Balance Laws/Conservation Laws

Physical Balance Laws/Conservation Laws:
Cauchy Flux & Production
(Discontinuities & Singularities)

$\Longleftrightarrow$

Gauss-Green Formula and Normal Trace
for $\mathcal{DM}$–Fields over General Open Sets

C–Torres-Ziemer: Gauss-Green Theorem for Weakly Differentiable Fields, Sets of Finite Perimeter, and Balance Laws, Comm. Pure Appl. Math. **62** (2009), 242–304.

C–Comi-Torres: Cauchy Fluxes and Gauss-Green Formulas for Divergence-Measure Fields over General Open Sets, Arch. Rational Mech. Anal. **233** (2019), 87–166.

*Noll, Gurtin-Martins, Ziemer, Šilhavý, $\cdots\cdots$