I. Review: Basic Measure Theory

Modern Measure Theory

C. Jordan (1893) — Jordan Measure Theory

E. Borel (1895) — A new method for studying measures of point sets

↳ Denumerable Additivity of measures.

E. Baire — Discontinuous Real function Theory

↳ Lebesgue (1902). — The idea on Measure & Integration

Lebesgue Measure $\mathcal{L}^n$ over $\mathbb{R}^n$

Caratheodory (1914) — Define measures by Set Covering

↳ Hausdorff Measure

Hausdorff (1919)

Besicovith, et al: Properties of Hausdorff measures.

↓

Groundwork

$\boxed{\text{I-1}}$. Measures

$X \subset \mathbb{R}^n$ — a set

2^X — Collection of Subsets of X

Measures: A mapping $\mu : 2^X \to [0, \infty]$ is called a measure on X if

(i) $\mu(\phi) = 0$

(ii) $\mu(A) \leq \sum_{k=1}^{\infty} \mu(A_k)$

whenever $A \subset \bigcup_{k=1}^{\infty} A_k$

* μ is an outer measure in most texts.

↳ If $\begin{cases} \mu \text{ is a measure on } X \\ A \subset B \subset X \end{cases}$

$\Longrightarrow \quad \mu(A) \leq \mu(B)$

μ restricted to A: $\mu \llcorner A$

$$(\mu \llcorner A)(B) = \mu(A \cap B) \qquad \forall B \subset X$$

μ-Measurable $A \subset X$ is μ-measurable if, $\forall B \subset X$,

$$\boxed{\mu(B) = \mu(B \cap A) + \mu(B - A)}$$

(i) If $\mu(A) = 0 \Rightarrow A$ is μ-measurable

(ii) A is μ-measurable $\Longleftrightarrow$ $X - A$ is μ-measurable

(iii) If A is any subset of X $\Rightarrow$ Any μ-measurable set is also $\mu \llcorner A$ - measurable

Exercise 1

Properties of Measurable Sets

$\{A_k\}_{k=1}^{\infty}$ — A Sequence of μ-Measurable Sets

(i) $\bigcup_{k=1}^{\infty} A_k$, $\bigcap_{k=1}^{\infty} A_k$ are μ-measurable

(ii) If $\{A_k\}_{k=1}^{\infty}$ are disjoint, i.e., $A_k \cap A_l = \phi$, $k \neq l$

$\hookrightarrow$ $$\mu\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} \mu(A_k)$$

(iii) If $A_1 \subset \cdots \subset A_k \subset A_{k+1} \subset \cdots$

$\hookrightarrow$ $$\lim_{k\to\infty} \mu(A_k) = \mu\left(\bigcup_{k=1}^{\infty} A_k\right)$$

(iv) If $A_1 \supset \cdots \supset A_k \supset A_{k+1} \cdots$

and $\mu(A_1) < \infty$

$\hookrightarrow$ $$\lim_{k\to\infty} \mu(A_k) = \mu\left(\bigcap_{k=1}^{\infty} A_k\right)$$

Exercise 2

σ- Algebra A collection of subsets $\mathcal{A} \subset 2^X$ satisfies

(i) $\phi, X \in \mathcal{A}$

(ii) $A \in \mathcal{A} \Longrightarrow X - A \in \mathcal{A}$

(iii) $A_k \in \mathcal{A}, \; k = 1, 2, \cdots$

$$\Longrightarrow \bigcup_{k=1}^{\infty} A_k \in \mathcal{A}$$

↳ The collection of all μ-measurable subsets of X forms a σ-algebra.

σ- Finite w.r.t. μ $A \subset X$ is σ-finite w.r.t. μ

if
$$\begin{cases} A = \bigcup_{k=1}^{\infty} B_k \\ B_k \text{ is } \mu\text{-measurable} \\ \mu(B_k) < \infty \text{ for } k = 1, 2, \cdots \end{cases}$$

Borel σ- Algebra The smallest σ-algebra of $\mathbb{R}^n$ containing the open subsets of $\mathbb{R}^n$.

Certain classes of measures that admit good approximations of various types.

Regular: μ on X is regular if, $\forall A \subset X$, $\exists$ a μ-measurable set B s.t.

$$A \subset B, \quad \mu(A) = \mu(B)$$

Borel: μ on $\mathbb{R}^n$ is called Borel if every Borel set is μ-measurable

Borel Regular μ on $\mathbb{R}^n$ is Borel regular if

$$\begin{cases} \mu \text{ is Borel} \\ \forall A \subset \mathbb{R}^n, \exists \text{ Borel set } B \text{ s.t.} \\ \quad A \subset B, \quad \mu(A) = \mu(B) \end{cases}$$

Radon Measure

$$\begin{cases} \mu \text{ on } \mathbb{R}^n \text{ is Borel regular} \\ \mu(K) < \infty \quad \forall K \subset\subset \mathbb{R}^n \end{cases}$$

Properties

1. $\begin{cases} \mu - \text{regular measure on } X \\ A_1 \subset \cdots \subset A_k \subset A_{k+1} \cdots \end{cases}$

$$\Longrightarrow \lim_{k\to\infty} \mu(A_k) = \mu\left(\bigcup_{k=1}^{\infty} A_k\right)$$

* $\{A_k\}_{k=1}^{\infty}$ need not be μ-measurable

2. $\begin{cases} \mu - \text{Borel regular on } \mathbb{R}^n \\ A \subset \mathbb{R}^n - \mu\text{-measurable} \\ \mu(A) < \infty \end{cases}$

$\Longrightarrow$ $\mu \llcorner A$ is a Radon Measure

* If A is a Borel Set $\rightarrow$ $\begin{cases} \mu \llcorner A \text{ is} \\ \text{Borel regular} \\ \text{even if } \mu(A) = \infty \end{cases}$

3. $\begin{cases} \mu - \text{Borel on } \mathbb{R}^n \\ B - \text{Borel set} \end{cases}$

$\Longrightarrow$

(i) If $\mu(B) < \infty \rightarrow \forall \varepsilon > 0, \exists$ closed set C s.t. $\begin{cases} C \subset B \\ \mu(B - C) < \varepsilon \end{cases}$

(ii) If μ is Radon $\rightarrow \forall \varepsilon > 0, \exists$ open set U s.t. $\begin{cases} B \subset U \\ \mu(U - B) < \varepsilon \end{cases}$

Properties (Conti)

4. μ — Radon measure on $\mathbb{R}^n$.

$\Rightarrow$

(i) $\forall A \subset \mathbb{R}^n$,

$$\mu(A) = \inf\{\mu(U) \mid A \subset U,\ U \text{ open}\}$$

(ii) $\forall$ μ-measurable set $A \subset \mathbb{R}^n$,

$$\mu(A) = \sup\{\mu(K) \mid K \subset A,\ K \text{ compact}\}$$

* For (i), A is not required to be μ-measurable

5. Caratheodory's Criterion: μ is a measure on $\mathbb{R}^n$

If $\mu(A \cup B) = \mu(A) + \mu(B)$

$\forall A, B \subset \mathbb{R}^n$, $\mathrm{dist}(A, B) > 0$

$\hookrightarrow$ μ is a Borel measure.

Exercise 3

$\boxed{\text{I-2}}$ Measurable Functions

$$\begin{cases} X & \text{— a set} \\ Y & \text{— topological space} \\ \mu & \text{— a measure on } X \end{cases}$$

μ-measurable A function $f: X \to Y$ is called μ-measurable if, $\forall$ open $U \subset Y$,

$f^{-1}(U)$ is μ-measurable

↳ If $f: X \to Y$ is μ-measurable

↳ $f^{-1}(B)$ is μ-measurable for each Borel set $B \subset Y$

σ-finite w.r.t. μ A function $f: X \to [-\infty, \infty]$ is σ-finite w.r.t. μ if

$$\begin{cases} f \text{ is } \mu\text{-measurable} \\ \{x \mid f(x) \neq 0\} \text{ is } \sigma\text{-finite w.r.t. } \mu \end{cases}$$

Properties

(i) $f, g : X \longrightarrow \mathbb{R}$ are μ-measurable

$\Rightarrow \begin{cases} f+g,\ fg,\ |f|,\ \min(f,g),\ \max(f,g) \text{ are } \mu\text{-measurable} \\ \frac{f}{g} \text{ is } \mu\text{-measurable provided } g \neq 0 \text{ on } X \end{cases}$

(ii) $f_k : X \longrightarrow [-\infty, \infty]$ are μ-measurable, $k = 1, 2 \cdots$

$\Rightarrow \inf_{k \geq 1} f_k,\ \sup_{k \geq 1} f_k,\ \liminf_{k \to \infty} f_k,\ \limsup_{k \to \infty} f_k$

are also μ-measurable

(iii) $f : X \longrightarrow [0, \infty]$ is μ-measurable.

$\Rightarrow \exists$ μ-measurable sets $\{A_k\}_{k=1}^{\infty} \subset X$

s.t.

$$f = \sum_{k=1}^{\infty} \frac{1}{k} \chi_{A_k}$$

Exercise 4

$\boxed{\text{I-3}}$ Lusin's and Egoroff's Thms

Lusin's Thm

$$\begin{cases} \mu \text{ — Borel regular measure on } \mathbb{R}^n \\ f : \mathbb{R}^n \to \mathbb{R}^m \text{ — } \mu\text{-measurable} \\ A \subset \mathbb{R}^n \text{ — } \mu\text{-measurable} \\ \mu(A) < \infty \end{cases}$$

$\Rightarrow$ $\forall \varepsilon > 0$, $\exists$ a compact set $K \subset A$ s.t.

$$\begin{cases} \mu(A-K) < \varepsilon \\ f|K \text{ is continuous} \end{cases}$$

$\xRightarrow{\text{Extension Thm}}$ $\forall \varepsilon > 0$, $\exists$ a continuous function

$\bar{f} : \mathbb{R}^n \to \mathbb{R}^m$

s.t.

$$\mu\{x \in A \mid \bar{f}(x) \neq f(x)\} < \varepsilon$$

Exercise 5

μ a.e (Almost everywhere w.r.t. μ)

$\Updownarrow$

Except possibly on a set A with $\mu(A) = 0$

Egoroff's Thm

$$\begin{cases} \mu \text{ — a measure on } \mathbb{R}^n \\ f_k : \mathbb{R}^n \to \mathbb{R}^m,\ k = 1, 2, \cdots, \text{ are } \mu\text{-measurable} \\ A \subset \mathbb{R}^n \text{ — } \mu\text{-measurable with } \mu(A) < \infty \\ f_k \to g \quad \mu\text{-a.e. on } A \end{cases}$$

$\Longrightarrow$ $\forall \varepsilon > 0$, $\exists$ a μ-measurable set $B \subset A$ s.t.

$$\begin{cases} \mu(A - B) < \varepsilon \\ f_k \to g \text{ uniformly on } B \end{cases}$$

Exercise 6

$\boxed{\text{I-4}}$ Integrals and Limit Thms

Notation

$f^+ = \max(f, 0)$, $f^- = \max(-f, 0)$, $f = f^+ - f^-$

Simple function A function $g: X \to [-\infty, \infty]$ is called a simple function if the image of g is countable

Definitions (Integrals)

1. If $g \geq 0$, simple, μ-measurable function

   · $\int g \, d\mu \triangleq \sum_{0 \leq y \leq \infty} y \, \mu(g^{-1}\{y\})$

2. If $\begin{cases} g \text{ is a simple, } \mu\text{-measurable function} \\ \text{Either } \int g^+ d\mu < \infty \text{ or } \int g^- d\mu < \infty \end{cases}$

   · $\int g \, d\mu \triangleq \int g^+ d\mu - \int g^- d\mu$

   $= \sum_{-\infty \leq y \leq \infty} y \, \mu(g^{-1}\{y\})$

   · g is called a μ-integrable simple function

Definitions (Conti)

3. $f: X \longrightarrow [-\infty, \infty]$.

Upper Integral

$$\int^* f \, d\mu \triangleq \inf\left\{ \int g \, d\mu \;\middle|\; \begin{array}{l} g \text{ is a } \mu\text{-integrable simple} \\ \text{function with } g \geq f \ \mu\text{-a.e.} \end{array} \right\}$$

Lower Integral

$$\int_* f \, d\mu \triangleq \sup\left\{ \int g \, d\mu \;\middle|\; \begin{array}{l} g \text{ is a } \mu\text{-integrable simple} \\ \text{function with } g \leq f \ \mu\text{-a.e.} \end{array} \right\}$$

μ-Integrable. A μ-measurable function is called **μ-integrable** if

$$\int^* f \, d\mu = \int_* f \, d\mu$$

$$\triangleq \ \boxed{\int f \, d\mu}$$

* $\int f \, d\mu = \pm\infty$ are allowed.

Definitions (Conti)

4. μ-Summable: $f: X \to [-\infty, \infty]$ is μ-summable

if $\begin{cases} f \text{ is } \mu\text{-integrable} \\ \int |f| d\mu < \infty \end{cases}$

Locally μ-Summable: $\forall K \subset\subset \mathbb{R}^n$

$f|_K$ is μ-summable

5. We say ν is a signed measure on $\mathbb{R}^n$

if $\begin{cases} \exists \text{ a Radon measure } \mu \text{ on } \mathbb{R}^n \\ \exists \text{ a locally } \mu\text{-summable function} \end{cases}$

$f: \mathbb{R}^n \longrightarrow [-\infty, \infty]$

$\Rightarrow \quad \forall K \subset\subset \mathbb{R}^n$

$$\boxed{\nu(K) = \int_K f \, d\mu} \iff \nu = \mu \llcorner f$$

$\mu \llcorner A = \mu \llcorner \chi_A$

- $L^1(X, \mu) \triangleq \{\mu\text{-Summable functions on } X\}$
  $L^1_{loc}(\mathbb{R}^n, \mu) \triangleq \{\text{Locally } \mu\text{-summable functions on } \mathbb{R}^n\}$

Limit Theorems

1. Fatou's Lemma $f_k : X \to [0,\infty]$ μ-measurable
$k = 1, 2, \cdots$

↳ $$\int \liminf_{k\to\infty} f_k \, d\mu \le \liminf_{k\to\infty} \int f_k \, d\mu$$

2. Monotone Convergence Thm

$$\begin{cases} f_k : X \to [0,\infty] & \mu\text{-measurable } k = 1, 2, \cdots \\ f_1 \le \cdots \le f_k \le f_{k+1} \le \cdots \end{cases}$$

↳ $$\int \lim_{k\to\infty} f_k \, d\mu = \lim_{k\to\infty} \int f_k \, d\mu$$

3. Dominated Convergence Thm

$$\begin{cases} g \sim \mu\text{-Summable} \\ f, \{f_k\}_{k=1}^{\infty} \sim \mu\text{-measurable} \\ |f_k| \le g \\ f_k \to f \quad \mu\text{-a.e. as } k \to \infty \end{cases}$$

↳ $$\lim_{k\to\infty} \int |f_k - f| \, d\mu = 0$$

Limit Theorems (Conti)

4. $\begin{cases} f,\ \{f_k\}_{k=1}^{\infty} \text{ are } \mu\text{-summable} \\ \lim_{k\to\infty} \int |f_k - f| \, d\mu = 0 \end{cases}$

$\hookrightarrow \exists \{f_{k_j}\}_{j=1}^{\infty}$ s.t.

$$f_{k_j} \longrightarrow f \quad \mu\text{-a.e.}$$

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* $\lim_{k\to\infty} \int |f_k - f| \, d\mu = 0$ does not necessarily imply

$$f_k \longrightarrow f \quad \mu\text{-a.e.}$$

Exercise 7