$\boxed{\text{I-5}}$ Product Measures

Fubini's Theorem, Lebesgue Measure

X, Y are sets

Product Measures

μ ~ a measure on X

ν ~ a measure on Y

Define the product measure of μ and ν

$$\mu \times \nu : \begin{cases} 2^{X \times Y} \longrightarrow [0, \infty] \\ S \longrightarrow (\mu \times \nu)(S) \end{cases}$$

$$(\mu \times \nu)(S) \overset{\Delta}{=} \boxed{\inf \left\{ \sum_{i=1}^{\infty} \mu(A_i)\nu(B_i) \right\}}$$

The infimum is taken over all collections

of $\begin{cases} \mu\text{-measurable sets } A_i \subset X \\ \nu\text{-measurable sets } B_i \subset X \end{cases}$

$i = 1, 2, \cdots$

s.t.

$$S \subset \bigcup_{i=1}^{\infty} (A_i \times B_i)$$

Fubini's Thm $\begin{cases} \mu \sim \text{a measure on } X \\ \nu \sim \text{a measure on } Y. \end{cases}$

$\Rightarrow$

(i) $\mu \times \nu$ is a regular measure on $X \times Y$, even if μ and ν are not regular.

(ii) If $\begin{cases} A \subset X \text{ is } \mu\text{-measurable} \\ B \subset Y \text{ is } \nu\text{-measurable} \end{cases}$

$\hookrightarrow \begin{cases} A \times B \text{ is } (\mu \times \nu)\text{-measurable} \\ (\mu \times \nu)(A \times B) = \mu(A)\mu(B) \end{cases}$

(iii) If $S \subset X \times Y$ is σ-finite w.r.t. $\mu \times \nu$,

$\hookrightarrow$

$$\begin{cases} S_y \triangleq \{x \mid (x, y) \in S\} \text{ is } \mu\text{-measurable for } \nu \text{ a.e. } y \\ S_x = \{y \mid (x, y) \in S\} \text{ is } \nu\text{-measurable for } \mu \text{ a.e. } y \\ \mu(S_y) \text{ is } \nu\text{-integrable} \\ \nu(S_x) \text{ is } \mu\text{-integrable} \\ (\mu \times \nu)(S) = \int_Y \mu(S_y)\, d\nu(y) = \int_X \nu(S_x)\, d\mu(x) \end{cases}$$

Fubini's Thm (Conti)

(iv). If $\begin{cases} f \text{ is } (\mu\times\nu)\text{-integrable} \\ f \text{ is } \sigma\text{-finite w.r.t. } \mu\times\nu \end{cases}$

(in particular, if f is $(\mu\times\nu)$-summable)

$\hookrightarrow$ The mapping

$$\begin{cases} y \longmapsto \int_X f(x,y)\,d\mu(x) \text{ is } \nu\text{-integrable} \\ x \longmapsto \int_Y f(x,y)\,d\nu(y) \text{ is } \mu\text{-integrable} \end{cases}$$

and

$$\int_{X\times Y} f\,d(\mu\times\nu) = \int_Y \left(\int_X f(x,y)\,d\mu(x)\right) d\nu(y)$$

$$= \int_X \left(\int_Y f(x,y)\,d\nu(y)\right) d\mu(x)$$

* We will study the Coarea Formula, which is a kind of "curvilinear" version of Fubini's Thm.

Exercise 8

1-D Lebesgue Measure $\mathcal{L}^1$ on $\mathbb{R}^1$: $\forall A \subset \mathbb{R}$

$$\mathcal{L}^1(A) \triangleq \inf \left\{ \sum_{i=1}^{\infty} \operatorname{diam}(C_i) \,\middle|\, A \subset \bigcup_{i=1}^{\infty} C_i,\ C_i \subset \mathbb{R} \right\}$$

n-D Lebesgue Measure $\mathcal{L}^n$ on $\mathbb{R}^n$

$$\mathcal{L}^n = \mathcal{L}^{n-1} \times \mathcal{L}^1 = \underbrace{\mathcal{L}^1 \times \mathcal{L}^1 \times \cdots \times \mathcal{L}^1}_{n \text{ times}}$$

$$= \mathcal{L}^{n-k} \times \mathcal{L}^k \quad \text{for each } k \in \{1, \cdots, n-1\}$$

$\hookrightarrow$ All the usual facts about $\mathcal{L}^n$.

<u>Notation</u>

$dx, dy \xrightarrow{\text{replace}} d\mathcal{L}^n$

$L^1(\mathbb{R}^n) \xrightarrow{\text{replace}} L^1(\mathbb{R}^n; \mathcal{L}^n)$

$- - - -$

$\boxed{\text{I-6}}$ Covering Thms

Vitali's Covering Thm

$B \stackrel{\Delta}{=} \overline{B(x,r)} \subset \mathbb{R}^n$ closed ball

$\hat{B} = \overline{B(x, 5r)}$ Enlarged closed ball
5 times

Cover: A collection $\mathcal{F}$ of closed balls in $\mathbb{R}^n$ is a cover of a set $A \subset \mathbb{R}^n$, if

$$A \subset \bigcup_{B \in \mathcal{F}} B$$

Finite Cover: $\mathcal{F}$ is a finite cover of A.

if, in addition, $\forall x \in A$,

$$\inf\{\operatorname{diam} B \mid x \in B,\ B \in \mathcal{F}\} = 0$$

Vitali's Covering Thm

$\mathcal{F} \sim$ Any collection of nondegenerate closed balls in $\mathbb{R}^n$. with

$$\sup\{\operatorname{diam}(B) \mid B \in \mathcal{F}\} < \infty$$

$\hookrightarrow$ $\exists$ a countable family $\mathcal{G}$ of disjoint balls in $\mathcal{F}$ such that

$$\boxed{\bigcup_{B\in\mathcal{F}} B \subset \bigcup_{B\in\mathcal{G}} \widehat{B}}$$

$\Rightarrow$

1. If $\begin{cases} \mathcal{F} \text{ is a finite cover of } A \text{ by closed balls} \\ \sup\{\operatorname{diam}(B) \mid B\in\mathcal{F}\} < \infty \end{cases}$

   $\hookrightarrow$ $\exists$ a countable family $\mathcal{G}$ of disjoint balls in $\mathcal{F}$ such that, $\forall \{B_1, \cdots, B_m\} \subset \mathcal{F}$, $m<\infty$,

$$\boxed{A - \bigcup_{k=1}^{m} B_k \subset \bigcup_{B\in\mathcal{G}-\{B_1,\cdots,B_m\}} \widehat{B}}$$

2. $U \subset \mathbb{R}^m$ open, $\delta > 0$.

   $\hookrightarrow$ $\exists$ a countable collection $\mathcal{G}$ of disjoint balls in U such that

$$\begin{cases} \operatorname{diam}(B) \le \delta, & \forall \text{ all } B\in\mathcal{G} \\ \mathcal{L}^n\left(U \setminus \bigcup_{B\in\mathcal{G}} B\right) = 0 \end{cases}$$

μ ??

Exercise 9

Besicovitch's Covering Theorm

$\exists$ a constant N_n $\underset{\text{only}}{\sim}$ n, with the Property:

If (i) $\mathcal{F}$ is any collection of nondegenerate closed balls in $\mathbb{R}^n$ with

$$\sup\{\operatorname{diam}(B) \mid B \in \mathcal{F}\} < \infty$$

(ii) A is the set of centers of balls in $\mathcal{F}$.

$\Rightarrow$ $\exists$ $\mathcal{G}_1, \cdots, \mathcal{G}_{N_n} \subset \mathcal{F}$ such that

$\mathcal{G}_i$, $i=1,\cdots,N_n$, are countable collections of disjoint balls in $\mathcal{F}$.

$$\boxed{A \subset \bigcup_{i=1}^{N_n} \bigcup_{B \in \mathcal{G}_i} B}$$

* $\mu(\hat{B})$ $\cancel{\sim}$ $\mu(B)$

$\mathcal{L}^n(\hat{B}) = 5^n \mathcal{L}^n(B)$

Corollary of BCT

$\mu \sim$ Borel measure on $\mathbb{R}^n$

$\mathcal{F} \sim$ any collection of nondegenerate closed balls.

$A \sim$ Set of Centers of the balls in $\mathcal{F}$

with

$$\begin{cases} \mu(A) < \infty \\ \inf\{r \mid B(a,r) \in \mathcal{F}\} = 0 \quad \forall a \in A \end{cases}$$

$\Rightarrow$ $\forall$ open $U \subset \mathbb{R}^n$,

$\exists$ a countable Collection $\mathcal{G}$ of disjoint balls in $\mathcal{F}$. Such that

$$\begin{cases} \bigcup_{B \in \mathcal{G}} B \subset U \\ \mu\left((A \cap U) \setminus \bigcup_{B \in \mathcal{G}} B\right) = 0 \end{cases}$$

* "Fill up" U with a countable collection of disjoint balls in such a way that the remainder has μ-measure zero

* The set A need not be μ-measurable here.

Exercise 10

$\boxed{\text{I-7}}$ Differentiation of Radon Measures

$\begin{cases}\mu \\ \nu\end{cases}$ Radon measures on $\mathbb{R}^n$

Definitions

1. $\forall\, x \in \mathbb{R}^n$

$$\overline{D}_\mu \nu(x) \triangleq \begin{cases} \limsup\limits_{r\to 0} \dfrac{\nu(B(x,r))}{\mu(B(x,r))} & \text{if } \mu(B(x,r))>0 \quad r>0 \\ +\infty & \text{if } \mu(B(x,r))=0 \text{ for some small } r>0 \end{cases}$$

$$\underline{D}_\mu \nu(x) \triangleq \begin{cases} \liminf\limits_{r\to 0} \dfrac{\nu(B(x,r))}{\mu(B(x,r))} & \text{if } \mu(B(x,r))>0 \quad \forall \text{ small } r>0 \\ +\infty & \text{if } \mu(B(x,r))=0 \text{ for some small } r>0 \end{cases}$$

2. If $\overline{D}_\mu \nu(x) = \underline{D}_\mu \nu(x) < +\infty$, We say that

ν is differentiable w.r.t. μ at $x\in\mathbb{R}^n$

and write

$$D_\mu \nu(x) \triangleq \overline{D}_\mu \nu(x) = \underline{D}_\mu \nu(x)$$

$D_\mu \nu$ is the derivative of ν w.r.t. μ

is also called the density of ν w.r.t. μ

Questions

? When does $D_\mu \nu$ exist ?

? When can ν be recovered by integrating $D_\mu \nu$?

Properties of $\underline{D}_\mu\nu$, $\overline{D}_\mu\nu$, $D_\mu\nu$

(i) Fix $0<\alpha<\infty$. Then

$$A\subset\{x\in\mathbb{R}^n \mid \underline{D}_\mu\nu(x)<\alpha\} \Rightarrow \nu(A)\le\alpha\mu(A)$$

$$A\subset\{x\in\mathbb{R}^n \mid \overline{D}_\mu\nu(x)\ge\alpha\} \Rightarrow \nu(A)\ge\alpha\mu(A)$$

* The set A need not be μ- or ν-measurable here

(ii) $D_\mu\nu$ $\begin{cases}\text{exists}\\ \text{is finite } \mu\text{-a.e.}\\ \text{is } \mu\text{-measurable}\end{cases}$

$\boxed{\text{Exercise 11}}$

Definitions

1. Absolutely Continuous: ν is absolutely continuous w.r.t μ, written. $\nu \ll \mu$.
   provided $\mu(A)=0 \Rightarrow \nu(A)=0$, $\forall A\subset\mathbb{R}^n$

2. Mutually Singular ν and μ are mutually singular written $\nu\perp\mu$ if $\exists$ a Borel subset $B\subset\mathbb{R}^n$ such that

$$\mu(\mathbb{R}^n-B)=\nu(B)=0$$

Differentiation Theorem for Radon Measures

Let $\nu \ll \mu$.

↳ $$\boxed{\nu(A) = \int_A D_\mu \nu \, d\mu}$$

$\forall$ μ-measurable sets $A \subset \mathbb{R}^n$.

* This is a version of the Radon-Nikodym Thm

↳ (i) ν has a density w.r.t. μ

(ii) This density $D_\mu \nu$ can be computed by "differentiating" ν w.r.t. μ.

↳ Fundamental Theorem of Calculus for Radon Measures on $\mathbb{R}^n$

Lebesgue Decomposition Theorem

$\left\{ \begin{array}{l} \mu \\ \nu \end{array} \right.$ are Radon measures

↳ (i) $\boxed{\nu = \nu_{ac} + \nu_s}$ with $\left\{ \begin{array}{l} \nu_{ac} \ll \mu \\ \nu_s \perp \mu \end{array} \right.$

where ν_{ac} – absolutely continuous part w.r.t. μ

ν_s – singular part w.r.t. μ

(ii) $\boxed{D_\mu \nu = D_\mu \nu_{ac}. \quad D_\mu \nu_s = 0 \quad \mu\text{-a.e.}}$

↳ $$\boxed{\nu(A) = \int_A D_\mu \nu \, d\mu + \nu_s(A)}$$

$\forall$ Borel set $A \subset \mathbb{R}^n$

$\boxed{\text{Exercise 12}}$

I-8 Lebesgue Points; Approximate Continuity

Notation: The average of f over the set E w.r.t μ by

$$⨍_E f\,d\mu \triangleq \frac{1}{\mu(E)} \int_E f\,d\mu$$

Lebesgue-Besicovitch Differentiation Theorem

$$\begin{cases} \mu \sim \text{a Radon measure on } \mathbb{R}^n \\ f \in L^1_{loc}(\mathbb{R}^n, \mu) \end{cases}$$

$$\Rightarrow \boxed{\lim_{r\to 0} ⨍_{B(x,r)} f\,d\mu = f(x) \qquad \mu\text{-a.e. } x \in \mathbb{R}^n}$$

Corollaries

1. $$\begin{cases} \mu \sim \text{a Radon measure on } \mathbb{R}^n \\ f \in L^p_{loc}(\mathbb{R}^n, \mu) \qquad 1 \le p < \infty \end{cases}$$

$$\hookrightarrow \boxed{\lim_{r\to 0} ⨍_{B(x,r)} |f - f(x)|^p\,d\mu = 0} \qquad (*)$$

$$\boxed{\text{for } \mu\text{-a.e. } x \in \mathbb{R}^n}$$

* Lebesgue point of f w.r.t. μ:

The pt $x \in \mathbb{R}^n$ for which $(*)$ holds.

Corollaries of LBD Thm (Conti.)

2. If $f \in L^p_{loc}$ for some $1 \le p < \infty$ then

$$\lim_{B \downarrow \{x\}} \fint_B |f - f(x)|^p \, dy = 0$$

for $\mathcal{L}^n$-a.e. $x \in \mathbb{R}^n$

where the limit is taken over all balls B containing x as $\operatorname{diam}(B) \to 0$.

* The point is that the balls need not be centered at x when $\mu = \mathcal{L}^n$.

3. Let $E \subset \mathbb{R}^n$ be $\mathcal{L}^n$-measurable

$\hookrightarrow$

$$\begin{cases} \lim_{r \to 0} \dfrac{\mathcal{L}^n(\overline{B(x,r)} \cap E)}{\mathcal{L}^n(\overline{B(x,r)})} = 1 & \text{for } \mathcal{L}^n\text{-a.e. } x \in E \\ \lim_{r \to 0} \dfrac{\mathcal{L}^n(\overline{B(x,r)} \cap E)}{\mathcal{L}^n(\overline{B(x,r)})} = 0 & \text{for } \mathcal{L}^n\text{-a.e. } x \in \mathbb{R}^n \setminus E \end{cases}$$

$\nearrow$

($f = \chi_E$, $\mu = \mathcal{L}^n$)

$\boxed{\text{Exercise 13}}$

Definitions

1. $E \subset \mathbb{R}^n$

Point of density 1 for E: $x \in \mathbb{R}^n$ such that

$$\lim_{r \to 0} \frac{\mathcal{L}^n(\overline{B(x,r)} \cap E)}{\mathcal{L}^n(B(x,r))} = 1$$

Point of density 0 for E: $x \in \mathbb{R}^n$ such that

$$\lim_{r \to 0} \frac{\mathcal{L}^n(\overline{B(x,r)} \cap E)}{\mathcal{L}^n(B(x,r))} = 0.$$

Measure-theoretical Interior of E:

$\overset{\triangle}{=}$

$\boxed{\text{The set of points of density } 1 \text{ of } E}$

Measure-theoretical Exterior of E

$\overset{\triangle}{=}$

$\boxed{\text{The set of points of density } 0 \text{ of } E}$

? Measure-theoretical Boundary of E ?

Definitions (Conti.)

2. $f \in L^1_{loc}(\mathbb{R}^n)$

$$f^*(x) \triangleq \begin{cases} \lim_{r\to 0} ⨍_{B(x,r)} f(y)\,dy & (*) \text{ if the limit exists} \\ 0 & \text{otherwise} \end{cases}$$

is the precise representative of f

* $f, g \in L^1_{loc}(\mathbb{R}^n)$ with $f = g$ $\mathcal{L}^n$-a.e

$\Longrightarrow \quad f^* = g^* \quad \forall\, x \in \mathbb{R}^n$

* $\lim_{r\to 0} ⨍_{B(x,r)} f(y)\,dy$ exists $\mathcal{L}^n$-a.e.

It is possible for the limit $(*)$ to exist even if x is not a Lebesgue pt of f.

* If f is a Sobolev or BV function, then $f^* = f$ except possibly on a "very small" set of appropriate capacity or Hausdorff measure zero

Definitions (Approximate Limits, Approximate Continuity)

1. **Approximate Limit of $f: \mathbb{R}^n \to \mathbb{R}^m$ as $y \to x$**

written

$$\operatorname{ap}\lim_{y\to x} f(y) \triangleq l$$

if, $\forall \varepsilon > 0$

$$\lim_{r\to 0} \frac{\mathcal{L}^n(B(x,r)\cap\{|f-l|\geq\varepsilon\})}{\mathcal{L}^n(B(x,r))} = 0$$

$\Updownarrow$

The set $\{|f-l|>\varepsilon\}$ has density zero at x

* An approximate limit is unique $\to$ well-defined

2. **Approximate limsup of $f: \mathbb{R}^n \to \mathbb{R}$ as $y \to x$**

written

$$\operatorname{ap}\limsup_{y\to x} f(y) \triangleq l$$

$$\inf\left\{t\in\mathbb{R} \;\middle|\; \lim_{r\to 0}\frac{\mathcal{L}^n(B(x,r)\cap\{f>t\})}{\mathcal{L}^n(B(x,r))}=0\right\}$$

Approximate liminf of $f: \mathbb{R}^n \to \mathbb{R}$ as $y \to x$

written

$$\operatorname{ap}\liminf_{y\to x} f(y) \triangleq l$$

$$\sup\left\{t\in\mathbb{R} \;\middle|\; \lim_{r\to 0}\frac{\mathcal{L}^n(B(x,r)\cap\{f<t\})}{\mathcal{L}^n(B(x,r))}=0\right\}$$

3. **Approximate Continuity of $f: \mathbb{R}^n \to \mathbb{R}^m$ at $x \in \mathbb{R}^n$**

if

$$\boxed{\operatorname{ap}\lim_{y\to x} f(y) = f(x)}$$

Thm Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be $\mathcal{L}^n$-measurable
$\Rightarrow$ f is approximately continuous $\mathcal{L}^n$-a.e.

↳ A measurable function is "practically continuous at practically every point".
The converse is also true.
[see Federer §2.9.13]

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* If $f \in L^1_{loc}(\mathbb{R}^n)$, the proof is simple

$\forall \varepsilon > 0$

$$\frac{\mathcal{L}^n(B(x,r) \cap \{|f - f(x)| > \varepsilon\})}{\mathcal{L}^n(B(x,r))} \leq \frac{1}{\varepsilon} \fint_{B(x,r)} |f - f(x)|\, dy \xrightarrow{\mathcal{L}^n\text{-a.e. } x} 0$$

↳ Any Lebesgue point is a point of approximate continuity

? Approximate differentiability:

$\boxed{\text{I-9}}$ Riesz Representation Theorem

Let $L : C_c(\mathbb{R}^n; \mathbb{R}^m) \longrightarrow \mathbb{R}$ be a linear functional such that

$$\sup\{ L(f) \mid f \in C_c(\mathbb{R}^n; \mathbb{R}^m),\ |f| \leq 1,\ \operatorname{spt}(f) \subset K \} < \infty$$

for each $K \subset\subset \mathbb{R}^n$

$\Rightarrow \exists$ $\begin{cases} \text{Radon measure } \mu \text{ on } \mathbb{R}^n \\ \mu\text{-measurable function } \sigma : \mathbb{R}^n \longrightarrow \mathbb{R}^m \end{cases}$

Such that

(i) $|\sigma(x)| = 1$ $\qquad$ μ-a.e. x

(ii) $L(f) = \int_{\mathbb{R}^n} f \cdot \sigma \, d\mu$ $\qquad$ $\forall\, f \in C_c(\mathbb{R}^n; \mathbb{R}^m)$

* μ is called the variation measure, defined for each open set $V \subset \mathbb{R}^n$ by

$$\mu(V) \triangleq \sup\{ L(f) \mid f \in C_c(\mathbb{R}^n; \mathbb{R}^m),\ |f| \leq 1,\ \operatorname{spt}(f) \subset V \}$$

* Radon measures can characterize certain linear functionals.

Corollary of RP Theorem

$L : C_c^\infty(\mathbb{R}^n) \longrightarrow \mathbb{R}$ is linear and nonnegative.

i.e. $\quad L(f) \geq 0 \qquad \forall\, f \in C_c^\infty(\mathbb{R}^n),\ f \geq 0.$

$\Rightarrow$ $\exists$ a Radon measure μ on $\mathbb{R}^n$ s.t.

$$L(f) = \int_{\mathbb{R}^n} f\, d\mu \qquad \forall\, f \in C_c^\infty(\mathbb{R}^n).$$

$\boxed{\text{Exercise 14}}$

$\boxed{\text{I-10}}$ Weak Convergence / Weak Compactness } for Radon Measures

Thm Let μ, μ_k $(k=1,2,\cdots)$ be Radon measures on $\mathbb{R}^n$

The following three statements are equivalent

(i) $\displaystyle\lim_{k\to\infty}\int_{\mathbb{R}^n} f\,d\mu_k = \int_{\mathbb{R}^n} f\,d\mu \quad \forall f\in C_c(\mathbb{R}^n)$

(ii) $\begin{cases} \displaystyle\limsup_{k\to\infty} \mu_k(K) \le \mu(K) & \forall K \subset\subset \mathbb{R}^n \\ \displaystyle\mu(U) \le \liminf_{k\to\infty} \mu_k(U). & \forall \text{ open set } U\subset\mathbb{R}^n \end{cases}$

(iii) $\displaystyle\lim_{k\to\infty} \mu_k(B) = \mu(B) \quad \forall$ bdd Borel set $B\subset\mathbb{R}^n$ with $\mu(\partial B)=0$

Weak Convergence If (i)–(iii) hold, we say that the measures μ_k converge weakly to μ written

$$\mu_k \longrightarrow \mu$$

Thm (Weak Compactness for measures)

$\{\mu_k\}_{k=1}^{\infty}$ ~ a sequence of Radon measures on $\mathbb{R}^n$ with

$$\sup_k \mu_k(K) < \infty \qquad \forall K\subset\subset\mathbb{R}^n$$

$\Rightarrow \exists \begin{cases} \text{subsequence } \{\mu_{k_j}\}_{j=1}^{\infty} \\ \text{Radon measure } \mu \end{cases}$ s.t. $\mu_{k_j} \longrightarrow \mu$

$U \subset \mathbb{R}^n$ open. $1 \le p < \infty$

Def: A sequence $\{f_k\}_{k=1}^{\infty} \subset L^p(U)$ converges weakly to $f \in L^p(U)$, written $f_k \rightharpoonup f$ in $L^p(U)$ provided

$$\lim_{k\to\infty} \int_U f_k g \, dx = \int_U f g \, dx$$

$$\forall g \in L^q(U), \ \frac{1}{p}+\frac{1}{q}=1, \ 1<q\le\infty$$

Weak Compactness in L^p

$$\begin{cases} 1<p<\infty \\ \{f_k\}_{k=1}^{\infty} \sim \text{sequence of functions in } L^p(U) \text{ satisfying} \end{cases}$$

$$\sup_k \|f_k\|_{L^p(U)} < \infty$$

$$\Longrightarrow \exists \begin{cases} \{f_{k_j}\}_{j=1}^{\infty} \subset \{f_k\}_{k=1}^{\infty} \\ f \in L^p(U) \end{cases} \text{ s.t. } f_{k_j} \rightharpoonup f \text{ in } L^p(U)$$

* The assertion is in general false for $p=1$

$U \subset \mathbb{R}^n$ open, bdd smooth

$\mathcal{M}(U) \sim$ Space of signed Radon measures on U with finite mass

Levi Metric Convergence A sequence $\{\mu_n\} \subset \mathcal{M}(U)$ is said to converge to a measure $\mu \in \mathcal{M}(U)$ in the Levi metric ρ, if $\forall \varepsilon > 0$, $\exists N > 0$, s.t. whenever $n \geq N$,

$$\boxed{\rho(\mu_n, \mu) < \varepsilon}$$

$$\Updownarrow$$

$\forall \delta > 0$, & a δ-n.b.h.d A_δ of A

$\hookrightarrow$
$$\begin{cases} \mu(A) \leq \nu(A_\delta) + \varepsilon \\ \nu(A) \leq \mu(A_\delta) + \varepsilon \end{cases}$$

Thm. The Levi metric Convergence

$$\Updownarrow$$

$$\boxed{\begin{array}{c}\text{The weak convergence for measures} \\ \mu_k \longrightarrow \mu \quad \text{weakly in } \mathcal{M}(U).\end{array}}$$

$\boxed{\text{Exercise 15}}$