II. Hausdorff Measures

Lower dimensional measures on $\mathbb{R}^n$, which allow us to measure certain "very small" subsets of $\mathbb{R}^n$

II-1 Hausdorff Measures. Hausdorff Dimension

Definitions: Let $A \subset \mathbb{R}^n$, $0 \le s < \infty$, $0 < \delta \le \infty$.

(i) $$\mathcal{H}^s_\delta(A) \triangleq \inf\left\{ \sum_{j=1}^{\infty} \underbrace{\alpha(s)}_{= \frac{\pi^{\frac{s}{2}}}{\Gamma(\frac{s}{2}+1)}} \left(\frac{\operatorname{diam} C_j}{2}\right)^s \,\middle|\, A \subset \bigcup_{j=1}^{\infty} C_j,\ \operatorname{diam} C_j \le \delta \right\}$$

* $s =$ integer, $C_j =$ ball $\Rightarrow$ $\alpha(s)\left(\frac{\operatorname{diam} C_j}{2}\right)^s =$ volume of s-D ball

* The Gamma function for $0 < s < \infty$

$$\Gamma(s) = \int_0^\infty e^{-x} x^{s-1}\, dx$$

$$(\Gamma(2) = 1)$$

* $\mathcal{H}^s_\delta \nearrow$ $\delta \to 0$

(ii) $$\mathcal{H}^s(A) \triangleq \lim_{\delta \to 0} \mathcal{H}^s_\delta(A) = \sup_{\delta > 0} \mathcal{H}^s_\delta(A)$$

is called s-D Hausdorff measure on $\mathbb{R}^n$

Remarks

(i) Requiring $\delta \to 0$ forces the coverings to "follow the local geometry" of the set A

(ii) Observe

$$\mathcal{L}^n(B(x,r)) = \alpha(n)\, r^n \qquad \forall\, B(x,r) \subset \mathbb{R}^n$$

$\xrightarrow{s=k \text{ integer}}$ $\mathcal{H}^k$ = "k-D Surface Area" on nice sets.

This is the reason we include the normalizing constant $\alpha(s)$ in the definition.

Thm $\mathcal{H}^s$ $(0 \le s < \infty)$ is a Borel regular measure

* $\mathcal{H}^s$ is not a Radon measure if $0 \le s < n$, since $\mathbb{R}^n$ is not σ-finite w.r.t. $\mathcal{H}^s$

$\boxed{\text{Exercise 16}}$

Basic Properties of Hausdorff Measure

(i) $\mathcal{H}^0$ is counting measure

$$\alpha(0)=1 \implies \mathcal{H}^0(\{a\})=1 \qquad \forall a\in\mathbb{R}^n$$

(ii) $\mathcal{H}^1=\mathcal{L}^1$ on $\mathbb{R}^1$

Choose $A\subset\mathbb{R}$, $\delta>0$

$$\mathcal{L}^1(A)=\inf\left\{\sum_{j=1}^{\infty}\operatorname{diam}C_j \;\middle|\; A\subset\bigcup_{j=1}^{\infty}C_j\right\}$$

$$\le\inf\left\{\sum_{j=1}^{\infty}\operatorname{diam}C_j \;\middle|\; A\subset\bigcup_{j=1}^{\infty}C_j,\ \operatorname{diam}C_j\le\delta\right\}$$

$$=\mathcal{H}^1_\delta(A).$$

On the other hand, set $I_k\triangleq[k\delta,(k+1)\delta]$ $(k=\cdots,-1,0,1,\cdots)$

Then
$$\begin{cases}\operatorname{diam}(C_j\cap I_k)\le\delta\\ \sum_{k=-\infty}^{\infty}\operatorname{diam}(C_j\cap I_k)\le\operatorname{diam}C_j\end{cases}$$

$$\mathcal{L}^1(A)=\inf\left\{\sum_{j=1}^{\infty}\operatorname{diam}C_j \;\middle|\; A\subset\bigcup_{j=1}^{\infty}C_j\right\}$$

$$\ge\inf\left\{\sum_{j=1}^{\infty}\sum_{k=-\infty}^{\infty}\operatorname{diam}(C_j\cap I_k) \;\middle|\; A\subset\bigcup_{j=1}^{\infty}C_j\right\}$$

$$\ge\mathcal{H}^1_\delta(A)$$

$$\implies \mathcal{L}^1=\mathcal{H}^1_\delta \qquad \forall\delta>0.$$

$$\implies \mathcal{L}^1=\mathcal{H}^1 \quad \text{on } \mathbb{R}^1$$

Basic Properties (Conti.)

(iii) $\mathcal{H}^s \equiv 0$ on $\mathbb{R}^n$ for all $s > n$.

Fix an integer $m \geq 1$.

The unit cube $Q \subset \mathbb{R}^n$ can be decomposed into m^n cubes with $\begin{cases} \text{Side } \frac{1}{m} \\ \text{diam. } \frac{\sqrt{n}}{m} \end{cases}$

$$\Rightarrow \mathcal{H}^s_{\frac{\sqrt{n}}{m}}(Q) \leq \sum_{i=1}^{m^n} \alpha(s) \left(\frac{\sqrt{n}}{m}\right)^s$$

$$= \alpha(s) n^{\frac{s}{2}} m^{n-s}$$

$$\xrightarrow[m \to \infty]{} 0 \quad \text{if } s > n$$

$$\Rightarrow \quad \mathcal{H}^s(Q) = 0$$

$$\Rightarrow \quad \mathcal{H}^s(\mathbb{R}^n) = 0$$

(iv) $\mathcal{H}^s(\lambda A) = \lambda^s \mathcal{H}^s(A) \qquad \forall \lambda > 0, A \subset \mathbb{R}^n$

(v) $\mathcal{H}^s(L(A)) = \mathcal{H}^s(A) \qquad \forall A \subset \mathbb{R}^n.$

for each affine isometry $L : \mathbb{R}^n \to \mathbb{R}^n$.

Basic Properties (Conti.)

(vi) Suppose $\begin{cases} A \subset \mathbb{R}^n \\ \mathcal{H}^s_\delta(A) = 0 \text{ for some } 0 < \delta \le \infty \end{cases}$

$\Rightarrow \mathcal{H}^s(A) = 0$

$s=0$ obvious

$s>0$. Fix $\varepsilon > 0$

$\Rightarrow \exists$ sets $\{C_j\}_{j=1}^\infty$ s.t. $\begin{cases} A \subset \bigcup_{j=1}^\infty C_j \\ \sum_{j=1}^\infty \alpha(s) \left(\frac{\operatorname{diam} C_j}{2} \right)^s \le \varepsilon \\ \operatorname{diam} C_j \le \delta \end{cases}$

In particular, for each i

$$\operatorname{diam} C_i \le 2 \left(\frac{\varepsilon}{\alpha(s)} \right)^{1/s} \triangleq \delta(\varepsilon) \to 0 \text{ as } \varepsilon \to 0.$$

$\Rightarrow \mathcal{H}^s_{\delta(\varepsilon)}(A) < \varepsilon \to 0$

$\Downarrow$

$\mathcal{H}^s(A) = 0$

Basic Properties (Conti.)

(vii') Let $A \subset \mathbb{R}^n$, $0 \le s < t < \infty$

(α) If $\mathcal{H}^s(A) < \infty \Rightarrow \mathcal{H}^t(A) = 0$

(β) If $\mathcal{H}^t(A) > 0 \Rightarrow \mathcal{H}^s(A) = \infty$

Let $\mathcal{H}^s(A) < \infty$, $\delta > 0$

$\Rightarrow \exists \{C_j\}_{j=1}^{\infty}$ s.t. $\operatorname{diam}(C_j) \le \delta$, $A \subset \bigcup_{j=1}^{\infty} C_j$

$$\sum_{j=1}^{\infty} \alpha(s) \left(\frac{\operatorname{diam} C_j}{2} \right)^s \le \mathcal{H}^s_\delta(A) + 1 \le \mathcal{H}^s(A) + 1$$

$$\Rightarrow \mathcal{H}^t_\delta(A) \le \sum_{j=1}^{\infty} \alpha(t) \left(\frac{\operatorname{diam} C_j}{2} \right)^t$$

$$= \frac{\alpha(t)}{\alpha(s)} 2^{s-t} \sum_{j=1}^{\infty} \alpha(s) \left(\frac{\operatorname{diam} C_j}{2} \right)^s (\operatorname{diam} C_j)^{t-s}$$

$$\le \frac{\alpha(t)}{\alpha(s)} 2^{s-t} \delta^{t-s} \left(H^s(A) + 1 \right)$$

$$\xrightarrow{t > s} 0 \qquad \delta \to 0$$

* Assertion (β) follows at once from (α)

Hausdorff Dimension of a Set $A \subset \mathbb{R}^n$

$$\mathcal{H}\text{-dim}(A) \triangleq \inf\{0 \le s < \infty \mid \mathcal{H}^s(A) = 0\}$$

↳

(i) $\mathcal{H}\text{-dim}(A) \le n$

(ii) Let $s = \mathcal{H}\text{-dim}(A)$.

↳ $$\mathcal{H}^t(A) = \begin{cases} 0 & \forall t > s \\ \infty & \forall t < s \\ \in (0,\infty), & \text{when } t = s \end{cases}$$

(iii) $\mathcal{H}\text{-dim}(A)$ need not be an integer

(iv) Even if $\begin{cases} \mathcal{H}\text{-dim}(A) = k & \text{is an integer} \\ 0 < \mathcal{H}^k(A) < \infty \end{cases}$

A need not be a "k-dimensional surface" in any sense.

e.g. Extremely complicated Cantor-like subsets $A \subset \mathbb{R}^n$ with $0 < \mathcal{H}^k(A) < \infty$

<u>Falconer</u>: Fractal Geometry. Wiley, New York, 1990.

$\boxed{\text{II-2}}$ Isodiametric Inequality; $\mathcal{H}^n = \mathcal{L}^n$ on $\mathbb{R}^n$

- $\mathcal{L}^n$ is defined as the n-fold product of 1-D Lebesgue measure $\mathcal{L}^1$

$$\mathcal{L}^n(A) = \inf\left\{ \sum_{i=1}^{\infty} \mathcal{L}^n(Q_i) \,\middle|\, Q_i \text{ cubes}, A \subset \bigcup_{i=1}^{\infty} Q_i \right\}$$

- $\mathcal{H}^n(A)$ is computed in terms of arbitrary coverings of small diameter.

$\Longrightarrow$ $\mathcal{H}^n = \mathcal{L}^n$ on $\mathbb{R}^n$.

Lemma Let $f : \mathbb{R}^n \to [0, \infty]$ be $\mathcal{L}^n$-measurable

$\hookrightarrow$ The region "under the graph of f":

$$A \triangleq \{(x, y) \mid x \in \mathbb{R}^n, y \in \mathbb{R}, 0 \leq y \leq f(x)\}$$

is $\mathcal{L}^{n+1}$-measurable.

$\boxed{\text{Exercise 17}}$

Steiner Symmetrization

Fix $a, b \in \mathbb{R}^n$; $|a|=1$

Define

$L_b^a \triangleq \{b+ta \mid t \in \mathbb{R}\}$ The line through b in the direction a

$P_a \triangleq \{x \in \mathbb{R}^n \mid x \cdot a = 0\}$ The plane through the origin perpendicular to a.

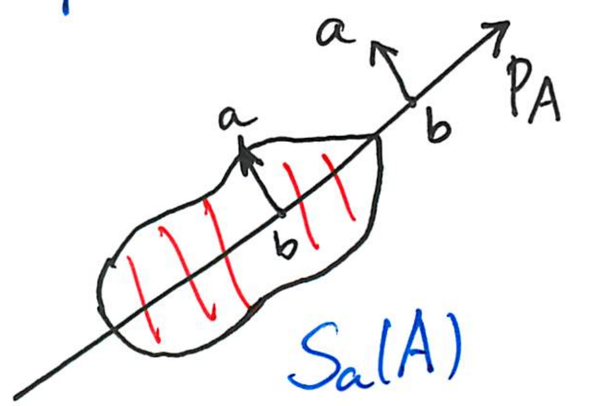
Choose $a \in \mathbb{R}^n$, $|a|=1$

Let $A \subset \mathbb{R}^n$

Steiner Symmetrization of A w.r.t. the plane P_a:

$$S_a(A) \triangleq \bigcup_{\substack{b \in P_a \\ A \cap L_b^a \neq \phi}} \left\{ b+ta \;\middle|\; |t| \leq \tfrac{1}{2}\mathcal{H}^1(A \cap L_b^a) \right\}$$

A



$S_a(A)$

$\Longrightarrow$

(i) $\operatorname{diam} S_a(A) \leq \operatorname{diam} A$

(ii) If A is $\mathcal{L}^n$-measurable, then $\begin{cases} \text{so is } S_a(A) \\ \mathcal{L}^n(S_a(A)) = \mathcal{L}^n(A) \end{cases}$

Isodiametric Inequality

$\forall$ all sets $A \subset \mathbb{R}^n$

$$\mathcal{L}^n(A) \leq \alpha(n) \left(\frac{\operatorname{diam} A}{2} \right)^n$$

* A is not necessarily contained in a ball of diameter $\operatorname{diam} A$

Ideas of the Proof W.O.L.G. we assume $\operatorname{diam} A < \infty$

Let $\{e_1, \cdots, e_n\}$ be the standard basis for $\mathbb{R}^n$.

Define $A_1 \triangleq S_{e_1}(A)$, $A_2 \triangleq S_{e_2}(A_1)$ ---

$A_n \triangleq S_{e_n}(A_{n-1}) \triangleq A^*$.

Claim 1. A^* is symmetric w.r.t. the origin.

Claim 2 $\mathcal{L}^n(A^*) \leq \alpha(n) \left(\frac{\operatorname{diam} A^*}{2} \right)^n$

$\forall x \in A^* \longrightarrow -x \in A^* \longrightarrow \operatorname{diam} A^* \geq 2|x|$.

$\hookrightarrow A^* \subset B(0, \frac{\operatorname{diam} A^*}{2})$.

$\hookrightarrow$
$$\mathcal{L}^n(A^*) \leq \mathcal{L}^n\left(B(0, \frac{\operatorname{diam} A^*}{2})\right) = \alpha(n) \left(\frac{\operatorname{diam} A^*}{2} \right)^n$$

Claim 3. $\mathcal{L}^n(A) \leq \alpha(n)\left(\frac{\operatorname{diam} A}{2}\right)^n$

$\bar{A}$ is $\mathcal{L}^n$-measurable.

$\hookrightarrow \begin{cases} \mathcal{L}^n((\bar{A})^*) = \mathcal{L}^n(\bar{A}) \\ \operatorname{diam}(\bar{A})^* \leq \operatorname{diam} \bar{A} \end{cases}$

$$\hookrightarrow \mathcal{L}^n(A) \leq \mathcal{L}^n(\bar{A}) = \mathcal{L}^n((\bar{A})^*)$$

$$\overset{\text{claim 2}}{\leq} \alpha(n)\left(\frac{\operatorname{diam}(\bar{A})^*}{2}\right)^n$$

$$\leq \alpha(n)\left(\frac{\operatorname{diam} \bar{A}}{2}\right)^n$$

$$= \alpha(n)\left(\frac{\operatorname{diam} A}{2}\right)^n$$

Thm $\mathcal{H}^n = \mathcal{L}^n$ on $\mathbb{R}^n$

$*$ $\mathcal{H}^n_\delta = \mathcal{L}^n$, $\delta > 0$.

Ideas of Proof

1. $\mathcal{L}^n(A) \le \mathcal{H}^n(A)$ for all $A \subset \mathbb{R}^n$

Fix $\delta > 0$. Choose sets $\{C_j\}_{j=1}^\infty$ s.t. $A \subset \bigcup_{j=1}^\infty C_j$, $\operatorname{diam} C_j \le \delta$

$$\xrightarrow[\text{Ineq.}]{\text{Isodiametric}} \mathcal{L}^n(A) \le \sum_{j=1}^\infty \mathcal{L}^n(C_j) \le \sum_{j=1}^\infty \alpha(n)\left(\frac{\operatorname{diam} C_j}{2}\right)^n$$

Taking infima;

$\hookrightarrow$ $\mathcal{L}^n(A) \le \mathcal{H}^n_\delta(A)$ $\forall \delta > 0$.

2. $\forall A \subset \mathbb{R}^n$, $\forall \delta > 0$

$$\mathcal{L}^n(A) = \inf\left\{ \sum_{i=1}^\infty \mathcal{L}^n(Q_i) \,\middle|\, Q_i \text{ cubes, } A \subset \bigcup_{i=1}^\infty Q_i,\ \operatorname{diam} Q_i \le \delta \right\}$$

$\uparrow$ parallel to the coordinate axes in $\mathbb{R}^n$

3. $\mathcal{H}^n_\delta \ll \mathcal{L}^n$

$\forall Q \subset \mathbb{R}^n$, $\alpha(n)\left(\frac{\operatorname{diam} Q}{2}\right)^n = \alpha(n)\left(\frac{\sqrt{n}}{2}\right)^n \mathcal{L}^n(Q)$.

$$\hookrightarrow \mathcal{H}^n_\delta(A) \le \inf\left\{ \sum_{i=1}^\infty \alpha(n)\left(\frac{\operatorname{diam} Q_i}{2}\right)^n \,\middle|\, \begin{array}{l} Q_i \text{ cubes, } \operatorname{diam} Q_i \le \delta \\ A \subset \bigcup_{i=1}^\infty Q_i \end{array} \right\}$$

$$= \alpha(n)\left(\frac{\sqrt{n}}{2}\right)^n \mathcal{L}^n(A)$$

4. $\mathcal{H}^n_\delta(A) \le \mathcal{L}^n(A), \quad \forall A \subset \mathbb{R}^n$

Fix $\delta > 0$, $\varepsilon > 0$.

We select cubes $\{Q_i\}_{i=1}^\infty$ s.t. $\begin{cases} A \subset \bigcup_{i=1}^\infty Q_i, \ \operatorname{diam} Q_i < \delta \\ \sum_{i=1}^\infty \mathcal{L}^n(Q_i) \le \mathcal{L}^n(A) + \varepsilon \end{cases}$

$\xrightarrow[\text{Corollary}]{\text{Vitalli's Covering Thm}}$ $\forall i$, $\exists$ disjoint closed balls $\{B_k^i\}_{k=1}^\infty \subset Q_i^0$

s.t. $\begin{cases} \operatorname{diam} B_k^i \le \delta \\ \mathcal{L}^n(Q_i - \bigcup_{k=1}^\infty B_k^i) = \mathcal{L}^n(Q_i^0 - \bigcup_{k=1}^\infty B_k^i) = 0 \end{cases}$

$\xrightarrow{\text{Step 3}}$ $\mathcal{H}^n_\delta(Q_i - \bigcup_{k=1}^\infty B_k^i) = 0$

$$\hookrightarrow \mathcal{H}^n_\delta(A) \le \sum_{i=1}^\infty \mathcal{H}^n_\delta(Q_i) = \sum_{i=1}^\infty \mathcal{H}^n_\delta\left(\bigcup_{k=1}^\infty B_k^i\right)$$

$$\le \sum_{i=1}^\infty \sum_{k=1}^\infty \mathcal{H}^n_\delta(B_k^i)$$

$$\le \sum_{i=1}^\infty \sum_{k=1}^\infty \alpha(n) \left(\frac{\operatorname{diam} B_k^i}{2}\right)^n$$

$$= \sum_{i=1}^\infty \sum_{k=1}^\infty \mathcal{L}^n(B_k^i) = \sum_{i=1}^\infty \mathcal{L}^n\left(\bigcup_{k=1}^\infty B_k^i\right)$$

$$\le \mathcal{L}^n(A) + \varepsilon.$$