II-3 Densities

Thm $\begin{cases} 0 < s < n \\ E \subset \mathbb{R}^n, \text{ is } \mathcal{H}^s\text{-measurable}, \ \mathcal{H}^s(E) < \infty \end{cases}$

$\Rightarrow$ (i) $$\lim_{r \to 0} \frac{\mathcal{H}^s(B(x,r) \cap E)}{\alpha(s) r^s} = 0 \text{ for } \mathcal{H}^s\text{-a.e. } x \in \mathbb{R}^n - E$$

(ii) $$\frac{1}{2^s} \le \limsup_{r \to 0} \frac{\mathcal{H}^s(B(x,r) \cap E)}{\alpha(s) r^s} \le 1$$

for $\mathcal{H}^s$-a.e. $x \in E$

Rm

(i) It is possible to have

$$\begin{cases} \limsup_{r \to 0} \dfrac{\mathcal{H}^s(B(x,r) \cap E)}{\alpha(s) r^s} < 1 \\ \liminf_{r \to 0} \dfrac{\mathcal{H}^s(B(x,r) \cap E)}{\alpha(s) r^s} = 0 \end{cases}$$

for $\mathcal{H}^s$-a.e. $x \in E$ even if $0 < \mathcal{H}^s(E) < \infty$

Exercise 18: Examples

(ii) $\forall E \subset \mathbb{R}^n$ $\mathcal{L}^n$-measurable

$$\lim_{r \to 0} \frac{|B(x,r) \cap E|}{\underset{\substack{\| \\ \alpha(n) r^n}}{|B(x,r)|}} = \begin{cases} 1 & \text{for } \mathcal{L}^n\text{-a.e. } x \in E \\ 0 & \text{for } \mathcal{L}^n\text{-a.e. } x \in \mathbb{R}^n - E \end{cases}$$

Ideas of Proof.

(i). Fix $t>0$

$$A_t \triangleq \left\{ x \in \mathbb{R}^n - E \;\middle|\; \limsup_{r\to 0} \frac{\mathcal{H}^s(B(x,r)\cap E)}{\alpha(s) r^s} > t \right\}$$

Claim $\mathcal{H}^s(A_t) = 0$. $\forall\, t>0$ $\to$ (i)

$\because$ $\mathcal{H}^s \llcorner E$ is a Radon measure

$\hookrightarrow$ $\forall \varepsilon>0$, $\exists$ $K \subset\subset E$ s.t.

$$\mathcal{H}^s(E-K) \le \varepsilon.$$

Set $U \triangleq \mathbb{R}^n - K$ open

$\hookrightarrow$ $A_t \subset U$

Fix $\delta>0$.

$$\mathcal{F} \triangleq \left\{ B(x,r) \;\middle|\; B(x,r)\subset U,\ 0<r<\delta,\ \frac{\mathcal{H}^s(B(x,r)\cap E)}{\alpha(s) r^s} > t \right\}$$

$\xrightarrow{\text{Vitali CT}}$ $\exists \{B_i\}_{i=1}^{\infty} \subset \mathcal{F}$, $B_i \cap B_j = \phi$, $i \neq j$. s.t.

$$A_t \subset \bigcup_{i=1}^{\infty} \hat{B}_i, \qquad B_i = \overline{B(x_i, r_i)}$$

$$\hookrightarrow \mathcal{H}^s_{10\delta}(A_t) \le \sum_{i=1}^{\infty} \alpha(s)(5r_i)^s \le \frac{5^s}{t} \sum_{i=1}^{\infty} \mathcal{H}^s(B(x_i, r_i)\cap E)$$

$$\le \frac{5^s}{t}\mathcal{H}^s(U\cap E) = \frac{5^s}{t}\mathcal{H}^s(E-K) \le \frac{5^s}{t}\varepsilon$$

$\xrightarrow{\delta\to 0}$ $\mathcal{H}^s(A_t) \le 5^s t^{-1} \varepsilon \xrightarrow[\varepsilon\to 0]{} 0.$

(ii)-1 claim $\limsup_{r\to 0} \frac{\mathcal{H}^s(B(x,r)\cap E)}{\alpha(s)r^s} \le 1$

for $\mathcal{H}^s$-a.e. $x\in E$

Fix $\varepsilon>0$, $t>1$.

$$B_t \triangleq \left\{ x\in E \;\middle|\; \limsup_{r\to 0} \frac{\mathcal{H}^s(B(x,r)\cap E)}{\alpha(s)r^s} > t \right\}$$

?? $\mathcal{H}^s(B_t)=0 \quad \forall\, t>1 \quad \to$ claim

$\because \mathcal{H}^s \llcorner E$ is Radon

$\hookrightarrow \exists$ open set $U\supset B_t$ s.t.

$$\mathcal{H}^s(U\cap E) \le \mathcal{H}^s(B_t)+\varepsilon$$

Define

$$\mathcal{F} \triangleq \left\{ B(x,r) \;\middle|\; B(x,r)\subset U,\ 0<r<\delta,\ \frac{\mathcal{H}^s(B(x,r)\cap E)}{\alpha(s)r^s} > t \right\}.$$

$\xrightarrow[\text{VCT-Corollary}]{}$ $\exists \{B_i\}_{i=1}^\infty \subset \mathcal{F}$, $B_i\cap B_j=\phi$, $i\ne j$

$\underset{B_i(x_i,r_i)}{\shortparallel}$

s.t. $$B_t \subset \left(\bigcup_{i=1}^m B_i\right) \cup \left(\bigcup_{i=m+1}^\infty \widehat{B}_i\right), \quad \forall\, m=1,2\ldots$$

$\hookrightarrow$

$$\begin{aligned}\mathcal{H}^s_{10\delta}(B_t) &\le \sum_{i=1}^m \alpha(s) r_i^s + \sum_{i=m+1}^\infty \alpha(s)(5r_i)^s \\ &\le \frac{1}{t}\sum_{i=1}^m \mathcal{H}^s(B_i\cap E) + \frac{5^s}{t}\sum_{i=m+1}^\infty \mathcal{H}^s(B_i\cap E). \\ &\le \frac{1}{t}\mathcal{H}^s(U\cap E) + \frac{5^s}{t}\mathcal{H}^s\left(\bigcup_{i=m+1}^\infty (B_i\cap E)\right).\end{aligned}$$

$\forall\, m=1,2,\ldots$

$$\mathcal{H}^s_{10\delta}(B_t) \le \frac{1}{t}\mathcal{H}^s(U\cap E) + \frac{5^s}{t}\mathcal{H}^s\Big(\underbrace{\bigcup_{i=m+1}^{\infty}(B_i\cap E)}_{\xrightarrow{m\to\infty}\ 0}\Big)$$

$\hookrightarrow$ $$\mathcal{H}^s_{10\delta}(B_t) \le \frac{1}{t}\mathcal{H}^s(U\cap E) \le \frac{1}{t}\left(\mathcal{H}^s(B_t)+\varepsilon\right).$$

$\downarrow$ $\delta\to 0$ first $\qquad\qquad$ $\downarrow$ $\varepsilon\to 0$ second

$$\mathcal{H}^s(B_t) \le \frac{1}{t}\mathcal{H}^s(B_t) \qquad t>1$$

$$\mathcal{H}^s(B_t) \le \mathcal{H}^s(E) < \infty$$

$\hookrightarrow$ $$\mathcal{H}^s(B_t) = 0 \quad \forall\, t>1.$$

<u>(ii)-2</u> <u>claim</u> $$\limsup_{r\to 0}\frac{\mathcal{H}^s_\infty(B(x,r)\cap E)}{\alpha(s)r^s} \ge \frac{1}{2^s}$$

for $\mathcal{H}^s$-a.e. $x\in E$

$\hookrightarrow$

$$\limsup_{r\to 0}\frac{\mathcal{H}^s(B(x,r)\cap E)}{\alpha(s)r^s} \ge \limsup_{r\to 0}\frac{\mathcal{H}^s_\infty(B(x,r)\cap E)}{\alpha(s)r^s} \ge \frac{1}{2^s}.$$

$\Rightarrow$ $\square$

For $\delta > 0$, $\tau \in (0,1)$

$$E(\delta,\tau) \triangleq \left\{ x \in E \;\middle|\; \begin{array}{l} \mathcal{H}^s_\delta(C \cap E) \le \tau\, \alpha(s) \left(\frac{\operatorname{diam} C}{2}\right)^s, \\ \forall\, C \subset \mathbb{R}^n,\ x \in C,\ \operatorname{diam} C \le \delta \end{array} \right\}$$

$\hookrightarrow$ If $\{C_j\}_{j=1}^\infty \subset \mathbb{R}^n$, $\operatorname{diam} C_j \le \delta$,

$C_j \cap E(\delta,\tau) \neq \phi$. $E(\delta,\tau) \subset \bigcup_{j=1}^\infty C_j$

$\hookrightarrow$

$$\begin{aligned} \mathcal{H}^s_\delta(E(\delta,\tau)) &\le \sum_{j=1}^\infty \mathcal{H}^s_\delta(C_j \cap E(\delta,\tau)) \\ &\le \sum_{j=1}^\infty \mathcal{H}^s_\delta(C_j \cap E) \\ &\le \tau \sum_{j=1}^\infty \alpha(s) \left(\frac{\operatorname{diam} C_j}{2}\right)^s \end{aligned}$$

$\hookrightarrow$ $\mathcal{H}^s_\delta(E(\delta,\tau)) \le \tau\, \mathcal{H}^s_\delta(E(\delta,\tau))$ $\le \mathcal{H}^s_\delta(E) \le \mathcal{H}^s(E) < \infty$

$\xrightarrow{\tau \in (0,1)}$ $\mathcal{H}^s_\delta(E(\delta,\tau)) = 0$

$\hookrightarrow$ $$\boxed{\mathcal{H}^s(E(\delta, 1-\delta)) = 0}$$

Set

$$D = \left\{ x \in E \;\middle|\; \limsup_{r \to 0} \frac{\mathcal{H}^s_\infty(B(x,r) \cap E)}{\alpha(s) r^s} < \frac{1}{2^s} \right\}$$

If we can show

$$\underline{\mathcal{H}^s(D) = 0} \quad \longrightarrow \quad \underline{\text{claim}}$$

$\forall x \in D$, $\exists \delta > 0$ s.t.

$$\frac{\mathcal{H}^s_\infty(B(x,r) \cap E)}{\alpha(s) r^s} \leq \frac{1-\delta}{2^s} \qquad \forall\, 0 < r \leq \delta \quad (*)$$

If $x \in C \cap D$, $\operatorname{diam} C \leq \delta$

$$\mathcal{H}^s_\delta(C \cap E) = \mathcal{H}^s_\infty(C \cap E) \leq \mathcal{H}^s_\infty(B(x, \operatorname{diam} C) \cap E)$$

$$\overset{(*)}{\leq} (1-\delta)\, \alpha(s) \left(\frac{\operatorname{diam} C}{2} \right)^s.$$

$\hookrightarrow \quad x \in E(\delta, 1-\delta)$

$$\Rightarrow \quad D \subset \bigcup_{k=1}^{\infty} E\left(\frac{1}{k}, 1 - \frac{1}{k}\right).$$

$\hookrightarrow \quad \mathcal{H}^s(D) = 0 \qquad \square$

$\boxed{\text{II-4}}$ Elementary Properties of Functions via Hausdorff Measures.

A. Lipschitz Mappings

$f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is called Lipschitz

if $\exists$ $C > 0$. s.t.

$$|f(x) - f(y)| \le C|x-y|, \quad \forall x, y \in \mathbb{R}^n$$

$$\mathrm{Lip}(f) \triangleq \sup\left\{ \frac{|f(x)-f(y)|}{|x-y|} \;\middle|\; x, y \in \mathbb{R}^n,\ x \neq y \right\}$$

Thm. $0 \le s < \infty$

(i) $\mathcal{H}^s(f(A)) \le (\mathrm{Lip}(f))^s \mathcal{H}^s(A), \quad \forall A \subset \mathbb{R}^n$

(ii) $\begin{cases} n > k \\ P: \mathbb{R}^n \longrightarrow \mathbb{R}^k \end{cases}$ the usual projection

$\hookrightarrow$ $\mathcal{H}^s(P(A)) \le \mathcal{H}^s(A)$

(i) $\Longrightarrow$ (ii)

$$\boxed{\mathrm{Lip}(P) = 1}$$

Proof of (i)

Fix $\delta > 0$

Choose $\{C_i\}_{i=1}^{\infty} \subset \mathbb{R}^n$ s.t. $\operatorname{diam} C_i \leq \delta$, $A \subset \bigcup_{i=1}^{\infty} C_i$

$$\Rightarrow \begin{cases} \operatorname{diam} f(C_i) \leq \operatorname{Lip}(f) \operatorname{diam} C_i \leq \operatorname{Lip}(f)\,\delta \\ f(A) \subset \bigcup_{i=1}^{\infty} f(C_i) \end{cases}$$

$$\Rightarrow \mathcal{H}^s_{\operatorname{Lip}(f)\delta}(f(A)) \leq \sum_{i=1}^{\infty} \alpha(s) \left(\frac{\operatorname{diam} f(C_i)}{2} \right)^s$$

$$\leq (\operatorname{Lip}(f))^s \sum_{i=1}^{\infty} \alpha(s) \left(\frac{\operatorname{diam} C_i}{2} \right)^s$$

Take infima over all such sets $\{C_i\}_{i=1}^{\infty}$

$$\hookrightarrow \mathcal{H}^s_{\operatorname{Lip}(f)\delta}(f(A)) \leq (\operatorname{Lip}(f))^s \mathcal{H}^s_\delta(A)$$

$\downarrow \delta \to 0$ $\qquad\qquad$ $\downarrow \delta \to 0$

$$\mathcal{H}^s(f(A)) \leq (\operatorname{Lip}(f))^s \mathcal{H}^s(A)$$

B. Graphs of Lipschitz functions

$f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$, $A \subset \mathbb{R}^n$

The graphs of f over A.

$$G(f; A) \triangleq \{(x, f(x)) \mid x \in A\} \subset \mathbb{R}^n \times \mathbb{R}^m = \mathbb{R}^{n+m}$$

Thm Assume $f: \mathbb{R}^n \to \mathbb{R}^m$, $\mathcal{L}^n(A) > 0$

$\hookrightarrow$ (i) $\mathcal{H}\text{-dim}(G(f; A)) \geq n$.

(ii) If f is Lipschitz,

$\hookrightarrow$ $\boxed{\mathcal{H}\text{-dim}(G(f; A)) = n}$

* $\mathcal{H}^n(G(f; A))$ can be computed via the Area formula according to the usual rules of Calculus.

Proof.

(i) Let $P: \mathbb{R}^{n+m} \to \mathbb{R}^n$ be the projection

$$\hookrightarrow \mathcal{H}^n(G(f;A)) \geq \mathcal{H}^n(A) = \mathcal{L}^n(A) > 0.$$

$$\mathcal{H}^n(G(f;A)) \geq \mathcal{H}^n(P(G(f;A))) = \mathcal{H}^n(A)$$

$$\hookrightarrow \mathcal{H}\text{-}\dim(G(f;A)) \geq n$$

(ii) $Q \subset \mathbb{R}^n$ unit cube (side length 1)

$$Q = \bigcup_{j=1}^{k^n} Q_j, \quad \operatorname{diam} Q_j = \frac{\sqrt{n}}{k}, \qquad \text{side length of } Q_j = \frac{1}{k}.$$

$$a_j^i = \min_{x \in Q_j} f^i(x), \quad b_j^i = \max_{x \in Q_j} f^i(x), \quad \begin{matrix} i = 1, \cdots, m \\ j = 1, \cdots, k^n \end{matrix}$$

$$\hookrightarrow |b_j^i - a_j^i| \underset{\uparrow}{\leq} \operatorname{Lip}(f) \operatorname{diam} Q_j = \operatorname{Lip}(f) \frac{\sqrt{n}}{k}$$

f is Lipschitz

$$\hookrightarrow \{(x, f(x)) \mid x \in Q_j \cap A\} \subset C_j \triangleq Q_j \times \prod_{i=1}^m (a_j^i, b_j^i).$$

$$\hookrightarrow \boxed{G(f; A \cap Q) \subset \bigcup_{j=1}^{k^n} C_j} \qquad \operatorname{diam} C_j \leq \frac{C}{k}$$

$$\hookrightarrow \mathcal{H}^n_{\frac{C}{k}}(G(f; A \cap Q)) \leq \sum_{j=1}^{k^n} \alpha(n) \left(\frac{\operatorname{diam} C_j}{2}\right)^n$$

$$\leq k^n \alpha(n) \left(\frac{C}{2k}\right)^n \leq \alpha(n) \left(\frac{C}{2}\right)^n$$

$$\Big\downarrow k \to \infty$$

$$\mathcal{H}^n(G(f; A \cap Q)) < \infty$$

$$\hookrightarrow \mathcal{H}\text{-}\dim(G(f; A \cap Q)) \leq n \longrightarrow \mathcal{H}\text{-}\dim(G(f;A)) \leq n$$

□

C. The Set where a Summable Function is Large

Thm Let $f \in L^1_{loc}(\mathbb{R}^n)$. $0 \le s < n$

$\Lambda_s \triangleq \{x \in \mathbb{R}^n \mid \limsup_{r\to 0} \frac{1}{r^s}\int_{B(x,r)} |f|\,dy > 0\}$

$\Longrightarrow \quad \mathcal{H}^s(\Lambda_s) = 0$

Proof. W.O.L.G. we assume $f \in L^1(\mathbb{R}^n)$

1. Lebesgue-Besicovitch Differentiation Thm

$\hookrightarrow \quad \lim_{r\to 0} ⨍_{B(x,r)} |f|\,dy = |f(x)| \quad \mathcal{L}^n\text{-a.e}$

$\hookrightarrow_{0\le s<n} \quad \lim_{r\to 0} \frac{1}{r^s}\int_{B(x,r)} |f|\,dy = 0 \quad \mathcal{L}^n\text{-a.e.}$

$\Longrightarrow \quad \mathcal{L}^n(\Lambda_s) = 0$

2. Fix $\varepsilon > 0$, $\delta > 0$, $\sigma > 0$

$f \in L^1(\mathbb{R}^n) \Longrightarrow \exists\, \eta > 0$, s.t. $\mathcal{L}^n(U) \le \eta$

$\hookrightarrow \int_U |f|\,dx < \sigma$

$\Lambda_s^\varepsilon \triangleq \{x \in \mathbb{R}^n \mid \limsup_{r\to 0} \frac{1}{r^s}\int_{B(x,r)} |f|\,dy > \varepsilon\}$

$\hookrightarrow \quad \mathcal{L}^n(\Lambda_s^\varepsilon) = 0$

$\hookrightarrow \quad \exists$ open $U \supset \Lambda_s^\varepsilon$ s.t $\mathcal{L}^n(U) < \eta$.

3. Set

$$\mathcal{J} \triangleq \left\{ B(x,r) \;\middle|\; \begin{array}{l} x \in \Lambda_s^\varepsilon,\ 0<r<\delta,\ B(x,r) \subset U \\ \int_{B(x,r)} |f|\,dy > \varepsilon r^s \end{array} \right\}$$

$$\xrightarrow{\text{Vitali CT}} \quad \exists \{B_j\}_{j=1}^\infty \subset \mathcal{J},\quad B_i \cap B_j = \phi,\ i \neq j$$

s.t. $\Lambda_s^\varepsilon \subset \bigcup_{j=1}^\infty \hat{B}_j$, $\hat{B}_j = \overline{B_j(x_j, r_j)}$

$$\hookrightarrow \mathcal{H}^s_{10\delta}(\Lambda_s^\varepsilon) \le \sum_{i=1}^\infty \alpha(s)(5r_i)^s$$

$$\le \frac{\alpha(s)5^s}{\varepsilon} \sum_{i=1}^\infty \int_{B_i} |f|\,dy$$

$\delta \to 0$

$$\mathcal{H}^s(\Lambda_s^\varepsilon) \le \frac{\alpha(s)5^s}{\varepsilon} \int_U |f|\,dy$$

$$\le \frac{\alpha(s)5^s}{\varepsilon} \sigma$$

$\sigma \to 0$

$$0$$

$$\hookrightarrow \boxed{\mathcal{H}^s(\Lambda_s^\varepsilon) = 0}$$

III. Area and Coarea Formulas

"change of variables" formulas for Lipschitz mappings $f: \mathbb{R}^n \to \mathbb{R}^m$

$m \geq n$ Area Formula

$m \leq n$ Coarea Formula

III-1 Lipschitz functions Rademacher's Thm.

Globally Lipschitz $f: A \subset \mathbb{R}^n \longrightarrow \mathbb{R}^m$

$$|f(x)-f(y)| \leq C|x-y|, \quad \forall x, y \in A$$

$$\mathrm{Lip}(f) \triangleq \sup\left\{ \frac{|f(x)-f(y)|}{|x-y|} \,\middle|\, x, y \in A,\ x \neq y \right\}$$

Locally Lipschitz $f: A \subset \mathbb{R}^n \longrightarrow \mathbb{R}^m$

$\forall K \subset\subset A, \quad \exists C_K$ s.t.

$$|f(x)-f(y)| \leq C_K |x-y|, \quad \forall x, y \in K$$

Extension of Lipschitz Functions

$$\begin{cases} A \subset \mathbb{R}^n \\ f : A \to \mathbb{R}^m \quad \text{Lipschitz} \end{cases}$$

Kirszbraun's Thm [Federer, §2.10.43]

$\exists$ a Lipschitz function $\bar{f} : \mathbb{R}^n \to \mathbb{R}^m$

such that
$$\begin{cases} \bar{f} = f \quad \text{on } A \\ \mathrm{Lip}(\bar{f}) = \mathrm{Lip}(f) \end{cases}$$

Exercise 18

Differentiability $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable

at $x \in \mathbb{R}^n$ if $\exists$ a linear mapping

$L : \mathbb{R}^n \to \mathbb{R}^m$ s.t.

$$\boxed{\lim_{y \to x} \frac{|f(y) - f(x) - L(x-y)|}{|x-y|} = 0}$$

$$\Updownarrow$$

$$\boxed{f(y) = f(x) + L(y-x) + o(|y-x|) \text{ as } y \to x}$$

$\hookrightarrow$ $\boxed{L \triangleq Df(x)}$ the derivative of f at x

Rademacher's Thm

Let $f: \mathbb{R}^n \to \mathbb{R}^m$ be a locally Lipschitz function

$\hookrightarrow$ f is differentiable $\mathcal{L}^n$-a.e.

* The inequality $|f(x)-f(y)| \leq \mathrm{Lip}(f)|x-y|$ apparently says nothing about the possibility of locally approximating f by a linear map.

$\Longrightarrow$

(i) Let $\begin{cases} f: \mathbb{R}^n \to \mathbb{R}^m \text{ be locally Lipschitz} \\ Z \triangleq \{x \in \mathbb{R}^n \mid f(x) = 0\} \end{cases}$

$\hookrightarrow$ $Df(x) = 0 \quad \mathcal{L}^n\text{-a.e. } x \in Z$

(ii) Let $\begin{cases} f, g: \mathbb{R}^n \to \mathbb{R}^n \text{ be locally Lipschitz} \\ Y \triangleq \{x \in \mathbb{R}^n \mid g(f(x)) = x\} \end{cases}$

$\hookrightarrow$ $Dg(f(x))\,Df(x) = I \quad \mathcal{L}^n\text{-a.e. } x \in Y$

$\boxed{\text{Exercise 19}}$

$\boxed{\text{III-2}}$ Review: Linear Maps, Jacobians

Linear Algebra

Definitions

1. A Linear Map $O: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is orthogonal if $(Ox)\cdot(Oy) = x\cdot y \quad \forall x, y \in \mathbb{R}^n$

2. A Linear Map $S: \mathbb{R}^m \longrightarrow \mathbb{R}^n$ is symmetric if $x\cdot(Sy) = (Sx)\cdot y \quad \forall x, y \in \mathbb{R}^n$

3. A Linear Map $D: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is diagonal if $\exists\ d_1, \cdots, d_n \in \mathbb{R}$ s.t. $Dx = (d_1 x_1, \cdots, d_n x_n) \quad \forall x \in \mathbb{R}^n$

4. Let $A: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be linear. The adjoint of A is the linear map $A^*: \mathbb{R}^m \longrightarrow \mathbb{R}^n$ s.t.
$$x\cdot(A^*y) = (Ax)\cdot y \quad \forall x \in \mathbb{R}^n, y \in \mathbb{R}^m$$

Routine Facts from Linear Algebra

- $A^{**} = A$
- $(A \circ B)^* = B^* \circ A^*$
- $O^* = O^{-1}$ if $O : \mathbb{R}^n \to \mathbb{R}^n$ is orthogonal
- $S^* = S$ if $S : \mathbb{R}^n \to \mathbb{R}^n$ is symmetric
- If $S : \mathbb{R}^n \to \mathbb{R}^n$ is symmetric

  $\hookrightarrow$ $\exists$ $\begin{cases} \text{an orthogonal map } O : \mathbb{R}^n \to \mathbb{R}^n \\ \text{a diagonal map } D : \mathbb{R}^n \to \mathbb{R}^n \end{cases}$

  s.t. $S = O \circ D \circ O^{-1}$
- If $O : \mathbb{R}^n \to \mathbb{R}^m$ is orthogonal

  $$\hookrightarrow \quad \begin{cases} n \leq m \\ O^* \circ O = I & \text{on } \mathbb{R}^n \\ O \circ O^* = I & \text{on } O(\mathbb{R}^n) \subset \mathbb{R}^m \end{cases}$$

Polar Decomposition

$L: \mathbb{R}^n \to \mathbb{R}^m$ a linear mapping

(i) If $n \le m$

$\hookrightarrow$ $\exists$ $\begin{cases} \text{a symmetric map} & S: \mathbb{R}^n \to \mathbb{R}^n \\ \text{an orthogonal map} & O: \mathbb{R}^n \to \mathbb{R}^m \end{cases}$

s.t. $L = O \circ S$

(ii) If $n \ge m$

$\hookrightarrow$ $\exists$ $\begin{cases} \text{a symmetric map} & S: \mathbb{R}^m \to \mathbb{R}^m \\ \text{an orthogonal map} & O: \mathbb{R}^m \to \mathbb{R}^n \end{cases}$

s.t. $L = S \circ O^*$

Jacobian of $L: \mathbb{R}^n \to \mathbb{R}^m$

(i) If $n \le m$.

$\hookrightarrow$ The Jacobian of L: $[\![L]\!] = |\det S|$

(ii) If $n \ge m$

$\hookrightarrow$ the Jacobian of L: $[\![L]\!] = |\det S|$.

Rms

① $[\![L]\!]$ ~~$\sim$~~ choices of O and S

② $[\![L]\!] = [\![L^*]\!]$

Thm

(i) If $n \le m$. $\to$ $[\![L]\!]^2 = \det(L^* \circ L)$

(ii) If $n \ge m$ $\to$ $[\![L]\!] = \det(L \circ L^*)$

Notations

(i) If $n \le m$, we define

$$\Lambda(m,n) \triangleq \{\lambda : \{1,\cdots,n\} \to \{1,\cdots,m\} \mid \lambda \text{ is increasing}\}$$

(ii) For each $\lambda \in \Lambda(m,n)$, we define $P_\lambda : \mathbb{R}^m \to \mathbb{R}^n$ by

$$P_\lambda(x_1,\cdots,x_m) \triangleq (x_{\lambda(1)},\cdots,x_{\lambda(n)})$$

$\hookrightarrow$ For each $\lambda \in \Lambda(m,n)$, $\exists$ an n-D subspace

$$S_\lambda \triangleq \operatorname{span}\{e_{\lambda(1)},\cdots,e_{\lambda(n)}\} < \mathbb{R}^m$$

s.t. P_λ is the projection of $\mathbb{R}^m$ onto S_λ.

Binet-Cauchy Formula

$$\begin{cases} n \le m \\ L : \mathbb{R}^n \to \mathbb{R}^m \text{ is linear} \end{cases}$$

$\hookrightarrow$ $$[\![L]\!]^2 = \sum_{\lambda \in \Lambda(m,n)} \big(\underbrace{\det(P_\lambda \circ L)}\big)^2$$

$\Updownarrow$

Determinants of each $(n\times n)$-submatrix of the $(m\times n)$-matrix representing L (w.r.t. the standard bases of $\mathbb{R}^n$ and $\mathbb{R}^m$).

Jacobians

$f = (f^1, \cdots, f^m) : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ Lipschitz

Rademacher's Thm

↳ f is differentiable $\mathcal{L}^n$-a.e.

↳ $Df(x)$ $\begin{cases} \text{exists} \\ \text{is a linear mapping: } \mathbb{R}^n \to \mathbb{R}^m \end{cases}$

for $\mathcal{L}^n$-a.e. $x \in \mathbb{R}^n$

$$Df = \begin{bmatrix} \frac{\partial f^1}{\partial x_1} & \cdots & \frac{\partial f^1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f^m}{\partial x_1} & \cdots & \frac{\partial f^m}{\partial x_n} \end{bmatrix}_{m \times n}.$$

The Jacobian of f is

$$Jf(x) \triangleq [\![Df(x)]\!] \qquad \mathcal{L}^n\text{-a.e. } x$$

Exercise 20