$\boxed{\text{II-3}}$ The Area Formula $\boxed{n \le m}$

Let $\begin{cases} f: \mathbb{R}^n \longrightarrow \mathbb{R}^m \text{ be Lipschitz.} \\ n \le m \end{cases}$

$\Rightarrow$ $\forall$ $\mathcal{L}^n$-measurable subset $A \subset \mathbb{R}^n$:

$$\int_A Jf\,dx = \int_{\mathbb{R}^m} \mathcal{H}^0(A \cap f^{-1}\{y\})\,d\mathcal{H}^n(y)$$

Basic Facts

1. $L: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ linear
   $\hookrightarrow$ $\mathcal{H}^n(L(A)) = [\![L]\!]\,\mathcal{L}^n(A)$

2. $\begin{bmatrix} f: \mathbb{R}^n \longrightarrow \mathbb{R}^m & \text{Lipschitz} \\ A \subset \mathbb{R}^n & \mathcal{L}^n\text{-measurable} \end{bmatrix}$

   $\hookrightarrow$ (i) $f(A)$ is $\mathcal{H}^n$-measurable

   (ii) The mapping $y \longmapsto \mathcal{H}^0(A \cap f^{-1}\{y\})$ is $\mathcal{H}^n$-measurable on $\mathbb{R}^n$.

   (iii) $\int_{\mathbb{R}^m} \mathcal{H}^0(A \cap f^{-1}\{y\})\,d\mathcal{H}^n \le (\mathrm{Lip}(f))^n \mathcal{L}^n(A)$

Rm: The mapping $y \longmapsto \mathcal{H}^0(A \cap f^{-1}\{y\})$ is called the multiplicity function

Basic Facts (Conti.)

3. $\begin{cases} B \triangleq \{x \mid Df(x) \text{ exists, } Jf(x) > 0\} \\ t > 1 \end{cases}$

$\Rightarrow \exists \{E_k\}_{k=1}^{\infty} \subset \mathbb{R}^n$ Borel subsets. s.t.

(i) $B = \bigcup_{k=1}^{\infty} E_k$

(ii) $f|_{E_k}$ is one-to-one, $k = 1, 2, \cdots$

(iii) $\forall k = 1, 2, \cdots,$ $\exists$ a symmetric automorphism $T_k : \mathbb{R}^n \to \mathbb{R}^n$

s.t.

$$\begin{cases} \operatorname{Lip}\left((f|_{E_k}) \circ T_k^{-1}\right) \leq t \\ \operatorname{Lip}\left(T_k \circ (f|_{E_k})^{-1}\right) \leq t \\ t^{-n} |\det T_k| \leq Jf|_{E_k} \leq t^n |\det T_k| \end{cases}$$

Exercise 21

Area formula

$\begin{cases} f: \mathbb{R}^n \longrightarrow \mathbb{R}^m & \text{Lipschitz} \\ n \le m \end{cases}$

$\Rightarrow$ $\forall$ $\mathcal{L}^n$-measurable subset $A \subset \mathbb{R}^n$,

$$\int_A Jf\,dx = \int_{\mathbb{R}^m} \mathcal{H}^0(A \cap f^{-1}\{y\})\,d\mathcal{H}^n(y).$$

Ideas of Proof:

1. Rademacher's Thm

↳ We may assume that $\begin{cases} Df(x) \\ Jf(x) \end{cases}$ exist for all $x \in A$.

We may also suppose $\mathcal{L}^n(A) < \infty$

2. Case 1 $\quad A \subset \{Jf > 0\}$

$\begin{cases} \text{Fix } t > 1 \\ \text{Choose Borel sets } \{E_k\}_{k=1}^\infty \text{ as in Fact 3} \end{cases}$

↳ We may assume that $\begin{cases} \{E_k\}_{k=1}^\infty \text{ are disjoint} \\ \{Jf > 0\} = \bigcup_{k=1}^\infty E_k \end{cases}$

$$B_k \triangleq \left\{ Q \;\middle|\; \begin{array}{l} Q = (a_1, b_1] \times \cdots \times (a_n, b_n],\ a_i = \frac{c_i}{k},\ b_i = \frac{c_i + 1}{k}, \\ c_i \text{ integers},\ i = 1, 2, \cdots n \end{array} \right\}$$

↳ $\mathbb{R}^n = \bigcup_{Q \in B_k} Q$

Set

$$F_j^i \triangleq E_j \cap Q_i \cap A, \quad Q_i \in \mathcal{B}_k$$

$$i, j = 1, 2, \cdots,$$

$\hookrightarrow$

$$\begin{cases} A \subset \bigcup_{i,j=1}^{\infty} F_j^i \\ F_j^i \text{ are disjoint} \end{cases}$$

Claim

$$\lim_{k\to\infty} \underbrace{\sum_{i,j=1}^{\infty} \mathcal{H}^n(f(F_j^i))}_{\displaystyle = \int_{\mathbb{R}^m} \boxed{g_k}\, d\mathcal{H}^n} = \int_{\mathbb{R}^m} \mathcal{H}^0(A\cap f^{-1}\{y\})\, d\mathcal{H}^n$$

$$\boxed{g_k} = \boxed{\sum_{i,j=1}^{\infty} \chi_{f(F_j^i)}}$$

The number of the sets $\{F_j^i\}$ s.t. $F_j^i \cap f^{-1}\{y\} \neq \phi$

Monotone increasing

$$\downarrow$$

$\mathcal{H}^0(A\cap f^{-1}\{y\})$ as $k\to\infty$

The Monotone Convergence Thm $\Rightarrow$ claim

Note

$$\mathcal{H}^n(f(F_j^i)) = \mathcal{H}^n(\underbrace{f|_{E_j}\circ T_j^{-1}}\circ T_j(F_j^i)) \overset{\text{Fact3}}{\leq} t^n \mathcal{L}^n(T_j(F_j^i))$$

$$\mathcal{L}^n(T_j(F_j^i)) = \mathcal{H}^n(T_j\circ (f|_{E_j})^{-1}\circ f(F_j^i)) \underset{\text{Fact3}}{\leq} t^n \mathcal{H}^n(f(F_j^i))$$

$\xrightarrow{\text{Facts 1\&3}}$

$$\begin{aligned}
t^{-2n}\mathcal{H}^n(f(F_j^i)) &\leq t^{-n}\mathcal{L}^n(T_j(F_j^i)) = t^{-n}|\det T_j|\,\mathcal{L}^n(F_j^i) \\
&\leq \int_{F_j^i} Jf\,dx \\
&\leq t^n |\det T_j|\,\mathcal{L}^n(F_j^i) \\
&= t^n \mathcal{L}^n(T_j(F_j^i)) \\
&\leq t^{2n}\mathcal{H}^n(f(F_j^i)).
\end{aligned}$$

$\Rightarrow$

$$t^{-2n}\sum_{i,j=1}^{\infty}\mathcal{H}^n(f(F_j^i)) \leq \int_A Jf\,dx \leq t^{2n}\sum_{i,j=1}^{\infty}\mathcal{H}^n(f(F_j^i)).$$

$\downarrow k\to\infty$ $\qquad\qquad$ $\downarrow k\to\infty$

$$t^{-2n}\int_{\mathbb{R}^m}\mathcal{H}^0(A\cap f^{-1}\{y\})\,d\mathcal{H}^n \qquad\qquad t^{2n}\int_{\mathbb{R}^m}\mathcal{H}^0(A\cap f^{-1}\{y\})\,d\mathcal{H}^n$$

$t\to 1$ $\rightarrow$ $\square$

3. Case 2 $A \subset \{Jf = 0\}$

Fix $\varepsilon > 0$

Factor $f = p \circ g$: $\begin{cases} g: \mathbb{R}^n \longrightarrow \mathbb{R}^m \times \mathbb{R}^n \\ \quad x \longrightarrow g(x) \triangleq (f(x), \varepsilon x) \\ p: \mathbb{R}^m \times \mathbb{R}^n \longrightarrow \mathbb{R}^m \\ \quad (y, z) \longrightarrow p(y,z) = y \end{cases}$

Claim $\exists\ C > 0$ s.t. $0 < Jg(x) \le C\varepsilon$, $\forall x \in A$

$\hookrightarrow$ $\because$ $p: \mathbb{R}^m \times \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is a projection

$$\mathcal{H}^n(f(A)) \le \mathcal{H}^n(g(A)) \le \int_{\mathbb{R}^{n+m}} \mathcal{H}^0(A \cap g^{-1}\{y,z\})\, d\mathcal{H}^n(y,z)$$

$$\| \quad \text{Case 1}$$

$$\int_A Jg(x)\, dx$$

$$\wedge|$$

$$\varepsilon C \mathcal{L}^n(A)$$

$$\downarrow \varepsilon \to 0$$

$$0.$$

$\hookrightarrow$

$$\mathcal{H}^n(f(A)) = 0.$$

$\longrightarrow$ $\operatorname{spt} \mathcal{H}^0(A \cap f^{-1}\{y\}) \subset f(A)$ $\longrightarrow$ $\int_{\mathbb{R}^n} \mathcal{H}^0(A \cap f^{-1}\{y\})\, d\mathcal{H}^n = 0$

$\square$

4. For general A.

Write

$$A = A_1 \cup A_2$$
$$A_1 \subset \{Jf > 0\}$$
$$A_2 \subset \{Jf = 0\}$$

Apply Cases 1 & 2 above

$\hookrightarrow$ Result

Exercise 22

Applications of the Area Formula

1. Change of Variables Formula

$$\begin{cases} f: \mathbb{R}^n \to \mathbb{R}^m \quad \text{Lipschitz} \\ n \le m \end{cases}$$

$\hookrightarrow$ $\forall$ $\mathcal{L}^n$-summable function $g: \mathbb{R}^n \to \mathbb{R}$

$$\int_{\mathbb{R}^n} g(x) Jf(x)\,dx = \int_{\mathbb{R}^m} \Big(\underbrace{\sum_{x \in f^{-1}\{y\}}}\, g(x)\Big)\, d\mathcal{H}^n(y)$$

By the Area Formula, $f^{-1}\{y\}$ is at most countable for $\mathcal{H}^n$-a.e. $y \in \mathbb{R}^m$

2. Length of a Curve ($n=1$, $m \ge 1$)

$f: \mathbb{R} \to \mathbb{R}^m$ Lipschitz, One-to-One

$\hookrightarrow$ $f = (f^1, \cdots, f^m)$. $Df = (\dot{f}^1, \cdots, \dot{f}^m)$

$$= |\dot{f}| \left(\frac{\dot{f}^1}{|\dot{f}|}, \cdots \frac{\dot{f}^m}{|\dot{f}|} \right)$$

$$Jf = |Df| = |\dot{f}|$$

For $-\infty < a < b < \infty$. $C \triangleq f([a,b]) \subset \mathbb{R}^m$. a curve

$$\mathcal{H}^1(C) = \text{"length" of } C = \int_a^b |\dot{f}|\,dt$$

3. Surface Area of a Graph ($n \geq 1$, $m = n+1$)

$g: \mathbb{R}^n \longrightarrow \mathbb{R}$ Lipschitz

$$f: \mathbb{R}^n \longrightarrow \mathbb{R}^{n+1}$$
$$x \longrightarrow f(x) \triangleq (x, g(x))$$

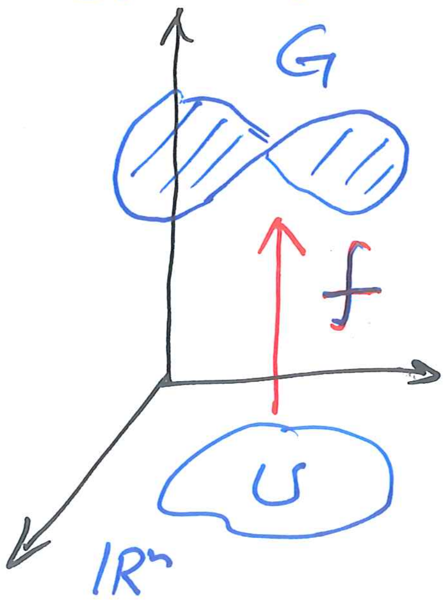
↳
$$Df = \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \\ \frac{\partial g}{\partial x_1} & \cdots & \frac{\partial g}{\partial x_n} \end{bmatrix}_{(n+1)\times n}$$

↳
$(Jf)^2 =$ sum of squares of $(n \times n)$-subdeterminants

$$= 1 + |Dg|^2$$

$\forall$ open set $U \subset \mathbb{R}^n$, define the graph of g over U:

$$G = G(g; U) \triangleq \{(x, g(x)) \mid x \in U\} \subset \mathbb{R}^{n+1}$$



↳
$\mathcal{H}^n(G) =$ "Surface area" of G

$$= \int_U (1 + |Dg|^2)^{1/2} dx$$

4. Surface Area of a Parametric Hypersurface ($n \geq 1$, $m = n+1$)

$f: \mathbb{R}^n \longrightarrow \mathbb{R}^{n+1}$ Lipschitz

One-to-one

$f = (f^1, \cdots, f^{n+1})$

$$Df = \begin{bmatrix} \frac{\partial f^1}{\partial x_1} & \cdots & \frac{\partial f^1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f^{n+1}}{\partial x_1} & \cdots & \frac{\partial f^{n+1}}{\partial x_n} \end{bmatrix}_{(n+1)\times n.}$$

↳ $(Jf)^2$ = sum of squares of $(n \times n)$-subdeterminants

$$= \sum_{k=1}^{n+1} \left[\frac{\partial(f^1, \cdots, f^{k-1}, f^{k+1}, \cdots f^{n+1})}{\partial(x_1, \cdots, x_n)} \right]^2$$

$\forall$ open set $U \subset \mathbb{R}^n$, write

$$S \triangleq f(U) \subset \mathbb{R}^{n+1}$$

↳ $\mathcal{H}^n(S)$ = "surface area" of S

$$= \int_U \left(\sum_{k=1}^{n+1} \left[\frac{\partial(f^1, \cdots, f^{k-1}, f^{k+1}, \cdots, f^{n+1})}{\partial(x_1, \cdots, x_n)} \right]^2 \right)^{\frac{1}{2}} dx$$

5. Submanifolds

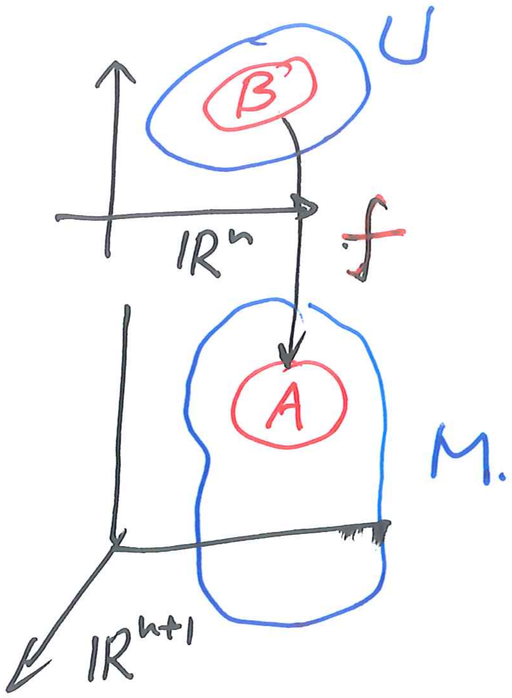
$M \subset \mathbb{R}^m$ Lipschitz. n-D embedded submanifold

$U \subset \mathbb{R}^n$

$f: U \to M$ is a chart for M

$A \subset f(U)$ Borel

$B = f^{-1}(A)$



$$g_{ij} \triangleq \frac{\partial f}{\partial x_i} \cdot \frac{\partial f}{\partial x_j}, \quad 1 \le i, j \le n$$

$$g = \det(g_{ij}).$$

$$(g_{ij}) = (Df)^* \circ Df$$

$\hookrightarrow$ $Jf = g^{1/2}$

$\hookrightarrow$

$$\mathcal{H}^n(A) = \text{"volume" of } A \text{ in } M$$

$$= \int_B g^{1/2}\, dx$$

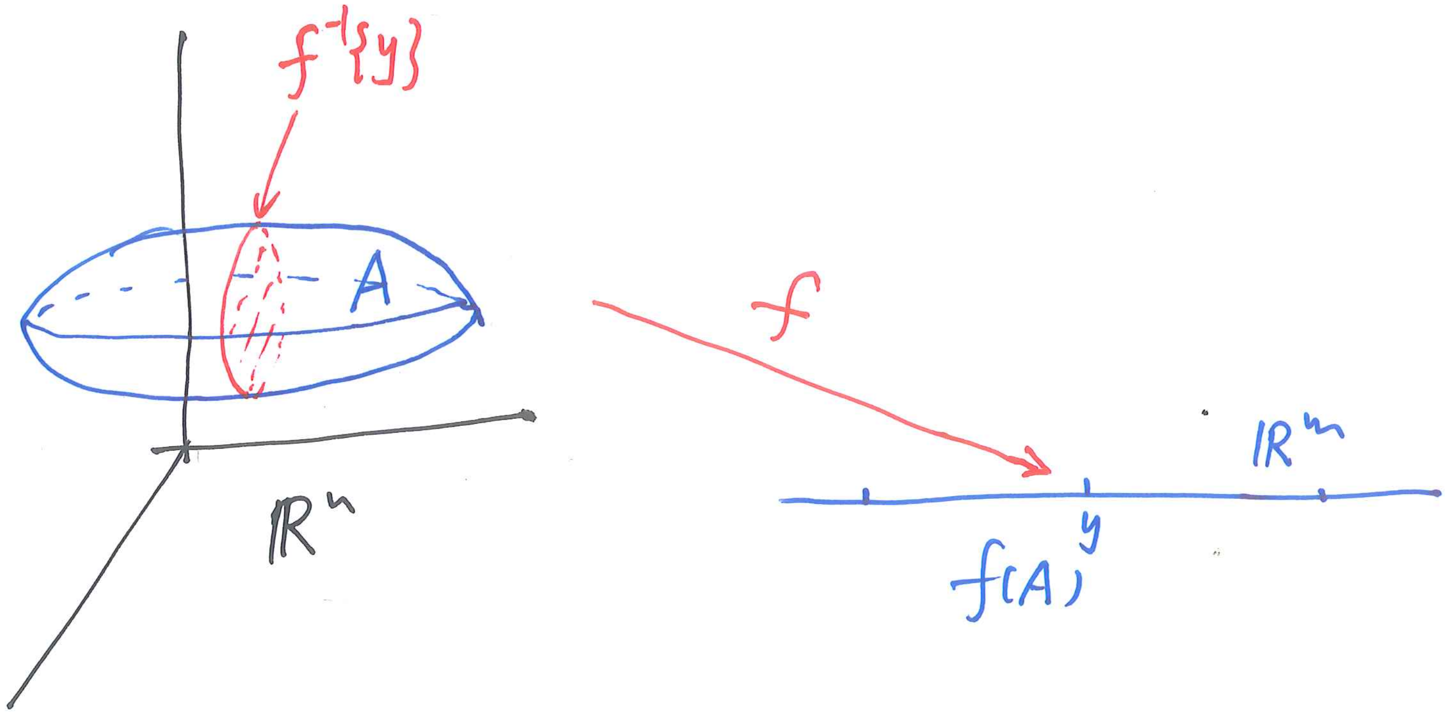
III-4 The Coarea Formula ($n \geq m$)

Let $\begin{cases} f: \mathbb{R}^n \to \mathbb{R}^m \text{ be Lipschitz.} \\ n \geq m \end{cases}$

$\Longrightarrow$ $\forall$ $\mathcal{L}^n$-measurable set $A \subset \mathbb{R}^n$:

$$\int_A Jf\,dx = \int_{\mathbb{R}^m} \mathcal{H}^{n-m}(A \cap f^{-1}\{y\})\,dy$$

* The Coarea formula is a kind of "curvilinear" generalization of Fubini's Thm.



Exercise 23

* Apply the Coarea Formula to $A = \{Jf = 0\}$.

↳ $\mathcal{H}^{n-m}\{\{Jf=0\} \cap f^{-1}\{y\}\} = 0$; $\mathcal{L}^m$-a.e. $y \in \mathbb{R}^m$

↑

f is required to be Lipschitz

This is a weak version/variant of the Morse-Sard Theorem:

$$\{Jf = 0\} \cap f^{-1}\{y\} = \phi \qquad \mathcal{L}^m\text{-a.e. } y$$

Provided $f \in C^k(\mathbb{R}^n; \mathbb{R}^m)$ for $k = 1 + n - m$.

Applications of the Coarea Formula

1. Change of Variables Formula:

$$\begin{cases} f: \mathbb{R}^n \longrightarrow \mathbb{R}^m \quad \text{Lipschitz} \\ n \geq m \end{cases}$$

$\Longrightarrow$ $\forall$ $\mathcal{L}^n$-summable function $g: \mathbb{R}^n \to \mathbb{R}$,

$$\begin{cases} g|_{f^{-1}\{y\}} \text{ is } \mathcal{H}^{n-m}\text{-summable for } \mathcal{L}^m\text{-a.e. } y \\ \int_{\mathbb{R}^n} g(x)\, Jf(x)\, dx = \int_{\mathbb{R}^m} \left[\int_{f^{-1}\{y\}} g \, d\mathcal{H}^{n-m} \right] dy \end{cases}$$

* for each $y \in \mathbb{R}^m$, $f^{-1}\{y\}$ is closed

$\hookrightarrow$ $\mathcal{H}^{n-m}$-measurable

- $g = g^+ - g^-$.
- W.O.L.G. we assume $g \geq 0$.

$g = \sum_{i=1}^{\infty} \frac{1}{i} \chi_{A_i}$ appropriate $\mathcal{L}^n$-measurable set $\{A_i\}_{i=1}^{\infty}$.

$$\xrightarrow[\text{Convergence Thm}]{\text{Monotone}} \int_{\mathbb{R}^n} g\, Jf\, dx = \sum_{i=1}^{\infty} \frac{1}{i} \int_{A_i} Jf\, dx$$

$$= \sum_{i=1}^{\infty} \frac{1}{i} \int_{\mathbb{R}^m} \mathcal{H}^{n-m}(A_i \cap f^{-1}\{y\})\, dy$$

$$= \int_{\mathbb{R}^m} \sum_{i=1}^{\infty} \frac{1}{i} \mathcal{H}^{n-m}(A_i \cap f^{-1}\{y\})\, dy$$

$$= \int_{\mathbb{R}^m} \left[\int_{f^{-1}\{y\}} g\, d\mathcal{H}^{n-m} \right] dy$$

2. Polar Coordinates

$g : \mathbb{R}^n \longrightarrow \mathbb{R}$ $\quad \mathcal{L}^n$-summable

$\hookrightarrow$ $$\int_{\mathbb{R}^n} g \, dx = \int_0^\infty \left(\int_{\partial B(0,r)} g \, d\mathcal{H}^{n-1} \right) dr$$

$\hookrightarrow$ $$\frac{d}{dr}\left(\int_{B(0,r)} g \, dx \right) = \int_{\partial B(0,r)} g \, d\mathcal{H}^{n-1}$$

$\mathcal{L}^1$-a.e. $r > 0$

$$\boxed{f(x) = |x|, \quad Df(x) = \frac{x}{|x|} ; \quad Jf(x) = 1. \quad x \neq 0}$$

3. Level sets.

① $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ Lipschitz

$\hookrightarrow$ $$\boxed{\int_{\mathbb{R}^n} \underbrace{|Df|}_{\substack{\| \\ Jf}} \, dx = \int_{-\infty}^{\infty} \mathcal{H}^{n-1}(\{f = t\}) \, dt}$$

3. Level sets (Conti)

② $\begin{cases} f: \mathbb{R}^n \longrightarrow \mathbb{R} \text{ Lipschitz} \\ \operatorname{ess\,inf} |Df| > 0 \\ g: \mathbb{R}^n \longrightarrow \mathbb{R} \quad \mathcal{L}^n\text{-summable} \end{cases}$

$$\Longrightarrow \int_{\{f>t\}} g\,dx = \int_t^{\infty} \left(\int_{\{f=s\}} \frac{g}{|Df|}\, d\mathcal{H}^{n-1} \right) ds$$

$$\hookrightarrow \frac{d}{dt}\left(\int_{\{f>t\}} g\,dx \right) = -\int_{\{f=t\}} \frac{g}{|Df|}\, d\mathcal{H}^{n-1}$$

$\mathcal{L}^1$-a.e. t.

$Jf = |Df|$

$E_t \triangleq \{f>t\}$

$$\int_{\{f>t\}} g\,dx = \int_{\mathbb{R}^n} \underbrace{\chi_{E_t} \frac{g}{|Df|}} Jf\,dx$$

$$= \int_{-\infty}^{\infty} \left(\int_{\partial E_s} \frac{g}{|Df|} \chi_{E_t}\, d\mathcal{H}^{n-1} \right) ds$$

$$= \int_t^{\infty} \left(\int_{\partial E_s} \frac{g}{|Df|}\, d\mathcal{H}^{n-1} \right) ds$$