V. BV Functions & Sets of Finite Perimeter

Definitions

1. $f \in L^1(U)$ has bounded variation in U if

$$\sup\left\{ \int_U f \operatorname{div} \varphi \, dx \;\middle|\; \varphi \in C_c^1(U;\mathbb{R}^n),\ |\varphi| \le 1 \right\} < \infty$$

$BV(U)$ — The space of functions of bdd variation

2. An $\mathcal{L}^n$-measurable subset $E \subset \mathbb{R}^n$ has finite perimeter in U if $\chi_E \in BV(U)$

3. $f \in L^1_{loc}(U)$ has locally bounded variation in U if, $\forall$ open set $V \subset\subset U$,

$$\sup\left\{ \int_V f \operatorname{div} \varphi \, dx \;\middle|\; \varphi \in C_c^1(V;\mathbb{R}^n),\ |\varphi| \le 1 \right\} < \infty$$

$BV_{loc}(U)$ — The space of functions of locally bdd variation

4. An $\mathcal{L}^n$-measurable subset $E \subset \mathbb{R}^n$ has locally finite perimeter in U if

$$\chi_E \in BV_{loc}(U)$$

Ex 1 $f \in W^{1,1}_{loc}(U)$

$\hookrightarrow$ $\forall\ V \subset\subset U$, $\varphi \in C^1_c(V; \mathbb{R}^n)$ with $|\varphi| \le 1$

$$\left|\int_U f \operatorname{div} \varphi \, dx\right| = \left|-\int_U Df \cdot \varphi \, dx\right| \le \int_V |Df| \, dx < \infty$$

$\hookrightarrow$ $f \in BV_{loc}(U)$

$\hookrightarrow$ $W^{1,1}_{loc}(U) \subset BV_{loc}(U)$

Similarly $W^{1,1}(U) \subset BV(U)$.

In particular. $W^{1,p}_{loc}(U) \subset BV_{loc}(U)$, $\forall 1 \le p \le \infty$

Ex 2 $\begin{cases} E \subset \mathbb{R}^n & \text{Smooth, open subset} \\ \mathcal{H}^{n-1}(\partial E \cap K) < \infty & \forall\ K \subset\subset U \end{cases}$

$\hookrightarrow$ $\forall\ \varphi \in C^1_c(V; \mathbb{R}^n)$, $V \subset\subset U$

$$\int_E \operatorname{div} \varphi \, dx = \int_{\partial E \cap V} \varphi \cdot \nu \, d\mathcal{H}^{n-1} \quad \text{(Gauss-Green formula)}$$

The outward unit normal on ∂E

$\hookrightarrow$ $\left|\int_E \operatorname{div} \varphi \, dx\right| \le \mathcal{H}^{n-1}(\partial E \cap V) < \infty \Rightarrow \chi_E \in BV_{loc}(U)$.

$\hookrightarrow$ E has locally finite perimeter in U $\Leftrightarrow$

But $\boxed{\chi_E \notin W^{1,1}_{loc}(U).}$

$\hookrightarrow$ $W^{1,1}_{loc}(U) \subsetneqq BV_{loc}(U)$. $W^{1,1}(U) \subsetneqq BV(U)$.

Structure Thm for $BV_{loc}(U)$ functions

$f \in BV_{loc}(U)$

$\hookrightarrow$ $\exists$ $\begin{cases} \text{a Radon measure } \mu \text{ on } U \\ \text{a } \mu\text{-measurable function } \sigma : U \to \mathbb{R}^n \end{cases}$

s.t.

$$\begin{cases} |\sigma(x)| = 1 & \mu\text{-a.e} \\ \int_U f \operatorname{div} \varphi \, dx = -\int_U \varphi \cdot \sigma \, d\mu, & \forall \varphi \in C_c^1(U; \mathbb{R}^n) \end{cases}$$

$\hookrightarrow$ The weak 1st partial derivatives of a BV function are Radon measures

Notation

(i) $f \in BV_{loc}(U)$: Write

$$\|Df\| \triangleq \mu$$

$$[Df] \triangleq \|Df\| \llcorner \sigma = \mu \llcorner \sigma$$

$\hookrightarrow$
$$\int_U f \operatorname{div} \varphi \, dx = -\int_U \varphi \cdot \sigma \, d\|Df\|$$
$$= -\int_U \varphi \cdot d[Df]$$
$$\forall \varphi \in C_c^1(U; \mathbb{R}^n)$$

We write
$$\mu^i := \|Df\| \llcorner \sigma^i \qquad i=1,2,\cdots,n \quad \sigma=(\sigma^1,\cdots,\sigma^n)$$

Lebesgue's Decomposition Thm

↳ $$\mu^i = \mu^i_{ac} + \mu^i_s \qquad \begin{cases} \mu^i_{ac} \ll \mathcal{L}^n \\ \mu^i_s \perp \mathcal{L}^n \end{cases}$$

↳ $\mu^i_{ac} = \mathcal{L}^n \llcorner g_i$ for some function $f_i \in L^1_{loc}(U)$ $\quad i=1,2,\cdots,n.$

Write
$$Df :\triangleq \left(\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n}\right) \triangleq (g_1,\cdots,g_n)$$
$$[Df]_{ac} \triangleq (\mu^1_{ac},\cdots,\mu^n_{ac}) = \mathcal{L}^n \llcorner Df$$
$$[Df]_s \triangleq (\mu^1_s,\cdots,\mu^n_s)$$

↳ $$[Df] = [Df]_{ac} + [Df]_s = \mathcal{L}^n \llcorner Df + [Df]_s.$$

↳ $Df \in L^1_{loc}(U;\mathbb{R}^n)$ is the density of the absolutely continuous part of $[Df]$.

* $f \in BV_{loc}(U)$ belongs to $W^{1,p}_{loc}(U)$

$\iff f \in L^p_{loc}(U)$, $\underline{[Df]_s = 0}$, $Df \in L^p_{loc}(U)$

(iii) E is a set of locally finite perimeter in U

$\hookrightarrow$ $f = \chi_E \in BV$

$\hookrightarrow$ $\begin{cases} \|\partial E\| \triangleq \mu \\ \nu_E \triangleq -\sigma \end{cases}$

$\hookrightarrow$ $\int_E \operatorname{div} \varphi \, dx = \int_U \varphi \cdot \nu_E \, d\|\partial E\|, \quad \forall \varphi \in C_c^1(U; \mathbb{R}^n)$

(iv) $\|Df\|$ is the variation measure of f

$\|\partial E\|$ is the perimeter measure of E

$\|\partial E\|(U)$ is the perimeter of E in U

(v) $f \in BV_{loc}(U) \cap L^1(U)$

$\hookrightarrow$ $f \in BV(U) \Longleftrightarrow \|Df\|(U) < \infty$

$$\|f\|_{BV(U)} \triangleq \|f\|_{L^1(U)} + \|Df\|(U)$$

$\hookrightarrow$ $BV(U)$ is a Banach space

(vi) $\forall V \subset\subset U$

$$\|Df\|(V) = \sup\left\{ \int_V f \operatorname{div} \varphi \, dx \;\middle|\; \varphi \in C_c^1(V; \mathbb{R}^n), |\varphi| \le 1 \right\}$$

$$\|\partial E\|(V) = \sup\left\{ \int_E \operatorname{div} \varphi \, dx \;\middle|\; \varphi \in C_c^1(V; \mathbb{R}^n), |\varphi| \le 1 \right\}$$

Approximation and Compactness

1. Lower Semicontinuity of Variation Measure

$$\begin{cases} f_k \in BV(U), & k=1,2,\cdots \\ f_k \longrightarrow f & \text{in } L^1_{loc}(U) \end{cases}$$

↳ $$\|Df\|(U) \leq \liminf_{k\to\infty} \|Df_k\|(U)$$

$\forall\, \varphi \in C_c^1(U;\mathbb{R}^n)$, $|\varphi| \leq 1$

$$\int_U f \operatorname{div}\varphi\, dx = \lim_{k\to\infty} \int_U f_k \operatorname{div}\varphi\, dx = -\lim_{k\to\infty} \int_U \varphi\cdot\sigma_k\, d\|Df_k\|$$

$$\leq \liminf_{k\to\infty} \|Df_k\|(U)$$

↳ $$\|Df\|(U) = \sup\left\{ \int_U f \operatorname{div}\varphi\, dx \;\middle|\; \varphi \in C_c^1(U;\mathbb{R}^n),\ |\varphi|\leq 1 \right\}$$

$$\leq \liminf_{k\to\infty} \|Df_k\|(U).$$

2. Local Approximation by Smooth Functions

$f \in BV(U) \implies \exists\ \{f_k\}_{k=1}^\infty \subset BV(U)\cap C^\infty(U)$ s.t.

$$\begin{cases} f_k \to f & \text{in } L^1(U) \\ \|Df_k\|(U) \to \|Df\|(U), & k\to\infty \\ \mu_k \longrightarrow \mu & (\mathcal{M}) \end{cases}$$

* $\mu_k(B) = \int_{B\cap U} Df_k\, dx$. $\mu(B) = \int_{B\cap U} d[Df]$ $\forall\ \underset{\text{Borel set}}{B} \subset \mathbb{R}^n$

* $\|D(f_k - f)\|(U) \xrightarrow{?} 0 \quad k\to\infty$

3. Compactness

Let $U \subset \mathbb{R}^n$ be open, bdd, with ∂U Lipschitz

Assume $\begin{cases} \{f_k\}_{k=1}^{\infty} \text{ is a sequence in } BV(U) \\ \sup_k \|f_k\|_{BV(U)} < \infty \end{cases}$

$\hookrightarrow$ $\exists \begin{cases} \{f_{k_j}\}_{j=1}^{\infty} \subset \{f_k\}_{k=1}^{\infty} \\ f \in BV(U) \end{cases}$

Such that

$$f_{k_j} \longrightarrow f \quad \text{in } L^1(U),\ j \to \infty$$

Choose $g_k \in C^\infty(\Omega)$ s.t. $\xrightarrow{②} \begin{cases} \int_U |f_k - g_k|\,dx < \frac{1}{k} \\ \sup_k \int_U |Dg_k|\,dx < \infty \end{cases}$

$\hookrightarrow$ $\|g_k\|_{W^{1,1}} \le C \ \cancel{\forall}\ k$

$\hookrightarrow$ $\exists \begin{cases} f \in L^1(U) \\ \{g_{k_j}\}_{j=1}^{\infty} \subset \{g_k\}_{k=1}^{\infty} \end{cases}$ s.t. $g_{k_j} \longrightarrow f$ in $L^1(U)$.

$\hookrightarrow$ $$f_{k_j} \longrightarrow f \quad \text{in } L^1(U)$$

$\xrightarrow{①}$ $f \in BV(U)$

Traces ~ The "boundary values" of f on ∂U

$\begin{cases} U \subset \mathbb{R}^n & \text{Open, bdd.} \\ \partial U & \text{Lipschitz} \end{cases}$

↳ The outer unit normal ν exists $\mathcal{H}^{n-1}$-a.e. on ∂U

Rademacher's Thm

↳ $\exists$ a bdd linear mapping

$$T: \ BV(U) \longrightarrow L^1(\partial U; \mathcal{H}^{n-1})$$

s.t.

$$\int_U f \operatorname{div} \varphi \, dx = -\int_U \varphi \cdot d[Df] + \int_{\partial U} (\varphi \cdot \nu) Tf \, d\mathcal{H}^{n-1}$$

$$\forall f \in BV(U), \ \varphi \in C^1(\mathbb{R}^n; \mathbb{R}^n)$$

Gauss-Green Formula

* Tf is uniquely defined up to sets of $\mathcal{H}^{n-1} \llcorner \partial U$ measure zero

↳ Tf is called the trace of f on ∂U

Traces (Conti.)

$f \in BV(U)$

↳ For $\mathcal{H}^{n-1}$-a.e. $x \in \partial U$

$$\lim_{r \to 0} \fint_{B(x,r)\cap U} |f - Tf(x)| \, dy = 0$$

↳ $$\boxed{Tf(x) = \lim_{r\to 0} \fint_{B(x,r)\cap U} f(y)\, dy}$$

* If $f \in BV(U) \cap C(\bar{U})$

↳ $Tf = f|_{\partial U}$ $\quad \mathcal{H}^{n-1}$-a.e.

Extensions

$$\begin{cases} U \subset \mathbb{R}^n & \text{open, bdd} \\ \partial U & \text{Lipschitz} \end{cases}$$

$$\begin{cases} f_1 \in BV(U) \\ f_2 \in BV(\mathbb{R}^n - \bar{U}) \end{cases}$$

Define

$$\bar{f}(x) \triangleq \begin{cases} f_1(x) & x \in U \\ f_2(x) & x \in \mathbb{R}^n - \bar{U} \end{cases}$$

$$\Longrightarrow \begin{cases} \bar{f} \in BV(\mathbb{R}^n) \\ \|D\bar{f}\|(\mathbb{R}^n) = \|Df_1\|(U) + \|Df_2\|(\mathbb{R}^n - \bar{U}) \\ \qquad + \int_{\partial U} |Tf_1 - Tf_2| \, d\mathcal{H}^{n-1} \end{cases}$$

$\hookrightarrow$ (i) The extension

$$Ef \triangleq \begin{cases} f & \text{on } U \\ 0 & \text{on } \mathbb{R}^n - U \end{cases} \in BV(\mathbb{R}^n).$$

provided $f \in BV(U)$

(ii) The set U has finite perimeter

& $\|\partial U\|(\mathbb{R}^n) = \mathcal{H}^{n-1}(\partial U)$

Coarea Formula for BV Functions

$$\begin{cases} f \in BV(U) \\ E_t \triangleq \{x \in U \mid f(x) > t\} \end{cases}$$

$\hookrightarrow$ (i) E_t has finite perimeter for $\mathcal{L}^1$-a.e. $t \in \mathbb{R}$

(ii) $\|Df\|(U) = \int_{-\infty}^{\infty} \|\partial E_t\|(U)\, dt$

Conversely, if $\begin{cases} f \in L^1(U) \\ \int_{-\infty}^{\infty} \|\partial E_t\|(U)\, dt < \infty \end{cases}$

$\hookrightarrow$ $f \in BV(U)$

Sobolev's and Poincaré's Inequalities for BV

(i) $\exists\, C_1 > 0$ s.t.

$$\|f\|_{L^{\frac{n}{n-1}}(\mathbb{R}^n)} \leq C_1 \|Df\|(\mathbb{R}^n), \quad \forall f \in BV(\mathbb{R}^n)$$

(ii) $\exists\, C_2 > 0$ s.t.

$$\|f - \underbrace{(f)_{x,r}}_{= \fint_{B(x,r)} f\,dy}\|_{L^{\frac{n}{n-1}}(B(x,r))} \leq C_2 \|Df\|(B(x,r))$$

$$\forall f \in BV_{loc}(\mathbb{R}^n), \ \forall B(x,r) \subset \mathbb{R}^n$$

(iii) $\forall\, 0 < \alpha \leq 1$, $\exists\, C_3 = C_3(\alpha)$ s.t.

$$\|f\|_{L^{\frac{n}{n-1}}(B(x,r))} \leq C_3 \|Df\|(B(x,r))$$

$\forall B(x,r) \subset \mathbb{R}^n$.

$\forall f \in BV_{loc}(\mathbb{R}^n)$ satisfying $\dfrac{|B(x,r) \cap \{f = 0\}|}{|B(x,r)|} \geq \alpha$

Isoperimetric Inequality

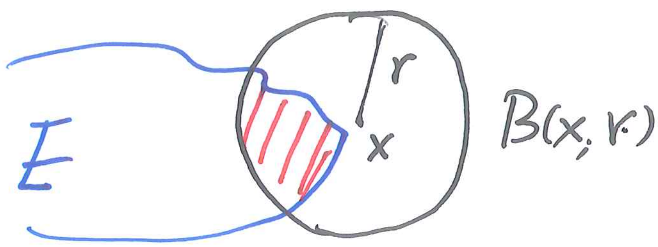
$$\begin{cases} E \subset \mathbb{R}^n \\ \|\partial E\|(\mathbb{R}^n) < \infty \end{cases}$$ Set of finite perimeter

$\hookrightarrow$ (i) $|E|^{1-\frac{1}{n}} \leq C_1 \|\partial E\|(\mathbb{R}^n)$ (Isoperimetric Ineq.)

(ii) $\forall B(x,r) \subset \mathbb{R}^n$:

$$\min\{|B(x,r)\cap E|, |B(x,r)-E|\}^{1-\frac{1}{n}} \leq C_2 \|\partial E\|(B(x,r))$$

(Relative Isoperimetric Ineq.)



The Reduced Boundary

$$\begin{cases} E \subset \mathbb{R}^n \\ \|\partial E\|(K) < \infty \qquad \forall\, K \subset\subset \mathbb{R}^n \end{cases}$$

Set of locally finite perimeter

$\hookrightarrow$ $$\int_E \operatorname{div} \varphi \, dx = \int_U \varphi \cdot \nu_E \, d\|\partial E\|$$

Definition $x \in \mathbb{R}^n$.

We say $x \in \partial^* E$, the reduced bdry of E, if

(i) $\|\partial E\|(B(x,r)) > 0 \qquad \forall\, r > 0$

(ii) $\lim\limits_{r \to 0} \fint_{B(x,r)} \nu_E \, d\|\partial E\| = \nu_E(x)$

(iii) $|\nu_E(x)| = 1$

Lebesgue-Besicovitch Differentiation Thm

$\hookrightarrow$ $$\|\partial E\|(\mathbb{R}^n - \partial^* E) = 0$$

The Reduced Boundary (Conti.)

↳

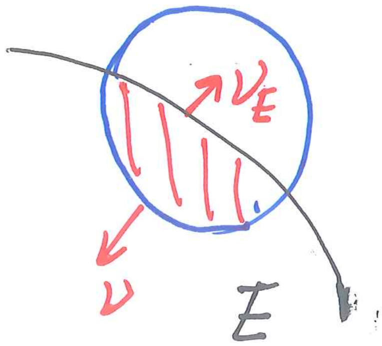
1. $\varphi \in C_c^1(\mathbb{R}^n; \mathbb{R}^n)$

$\forall\, x \in \mathbb{R}^n$

$$\int_{E\cap B(x,r)} \operatorname{div}\varphi \, dy = \int_{B(x,r)} \varphi\cdot\nu_E \, d\|\partial E\| + \int_{E\cap\partial B(x,r)} \varphi\cdot\nu \, d\mathcal{H}^{n-1}$$

for $\mathcal{L}^1$-a.e. $r>0$.

where ν is the outward unit normal to $\partial B(x,r)$



only, n

2. $\exists\, A_j>0,\ j=1,2,\cdots,5,$ s.t.

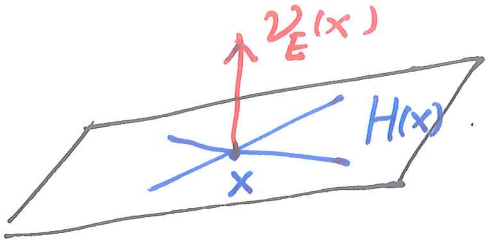
$\forall\, x\in\partial^* E$

$$\begin{cases} \liminf_{r\to 0} \dfrac{|B(x,r)\cap E|}{r^n} > A_1 > 0 \\ \liminf_{r\to 0} \dfrac{|B(x,r)-E|}{r^n} > A_2 > 0 \\ \liminf_{r\to 0} \dfrac{\|\partial E\|(B(x,r))}{r^{n-1}} > A_3 > 0 \\ \limsup_{r\to 0} \dfrac{\|\partial E\|(B(x,r))}{r^{n-1}} \le A_4 \\ \limsup_{r\to 0} \dfrac{\|\partial(E\cap B(x,r))\|(\mathbb{R}^n)}{r^{n-1}} \le A_5 \end{cases}$$

Blow-up

Notation Fix $x \in \partial^* E$

Hyperplane $H(x) := \{ y \in \mathbb{R}^n \mid \nu_E(x) \cdot (y - x) = 0 \}$

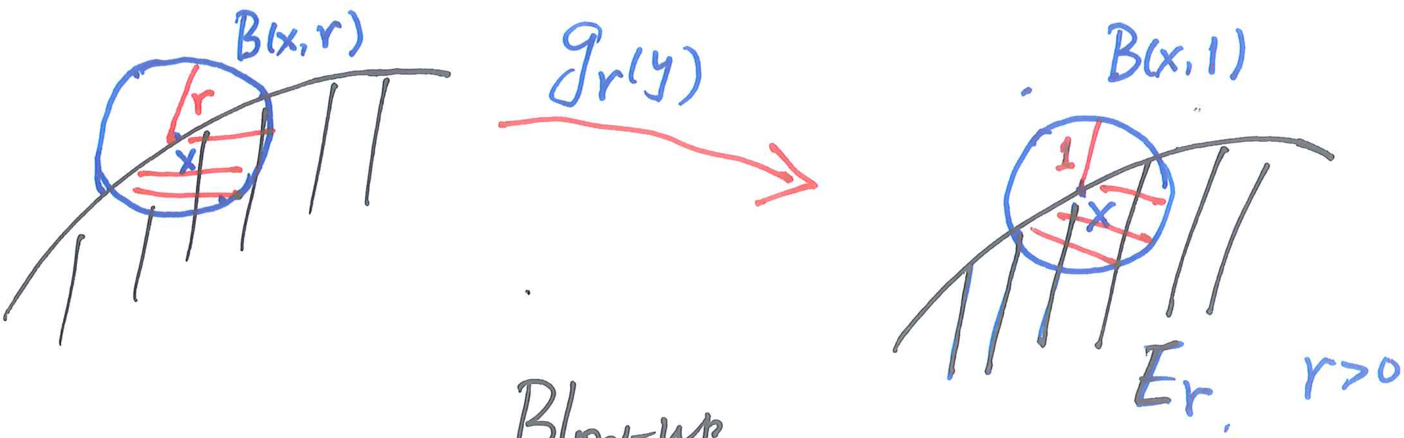


Half-Spaces

$$H^{\pm}(x) := \{ y \in \mathbb{R}^n \mid \pm \nu_E(x) \cdot (y - x) \geq 0 \}$$

$$E_r := \{ y \in \mathbb{R}^n \mid r(y - x) + x \in E \}, \quad r > 0.$$

$\hookrightarrow$ $y \in E \cap B(x, r) \iff g_r(y) := x + \frac{y - x}{r} \in E_r \cap B(x, 1)$



Blow-up

Thm Blow-up of Reduced Boundary

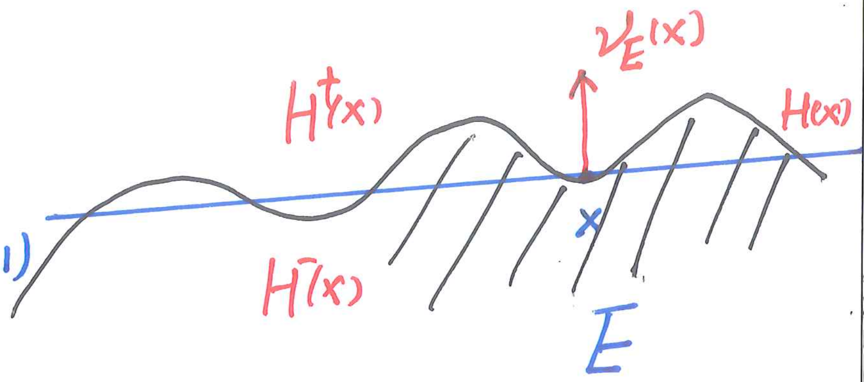
$x \in \partial^* E$

↳ $\chi_{E_r} \longrightarrow \chi_{H^-(x)}$ in $L^1_{loc}(\mathbb{R}^n)$ as $r \to 0$.

↳ For small enough $r > 0$. $E \cap B(x,r)$ approximately equals the half ball $H^-(x) \cap B(x,r)$.

Ideas of Proof

W.O.L.G. we may assume



$$\begin{cases} x = 0, \ \nu_E(0) = e_n = (0, \cdots, 0, 1) \\ H(0) = \{ y \in \mathbb{R}^n \mid y_n = 0 \} \\ H^+(0) = \{ y \in \mathbb{R}^n \mid y_n \geq 0 \} \\ H^-(0) = \{ y \in \mathbb{R}^n \mid y_n \leq 0 \} \end{cases}$$

1. Choose any sequence $r_k \to 0$. It suffices to show that $\exists \ \{s_j\}_{j=1}^\infty \subset \{r_k\}_{k=1}^\infty$ for which

$$\chi_{E_{s_j}} \longrightarrow \chi_{H^-(0)} \quad \text{in } L^1_{loc}(\mathbb{R}^n)$$

2. $\begin{cases} \text{Fix } L>0 \\ \text{Let } D_r := E_r \cap B(0,L), \quad g_r(y) = \frac{y}{r} \in E_r \end{cases}$

$y \in E$

$\forall\, \varphi \in C_c^1(\mathbb{R}^n;\mathbb{R}^n), \quad |\varphi|<1,$

$$\int_{D_r} \operatorname{div}\varphi \, dz = \frac{1}{r^{n-1}} \int_{E\cap B(0,rL)} \operatorname{div}(\varphi\circ g_r)\, dy$$

$$= \frac{1}{r^{n-1}} \int_{\mathbb{R}^n} (\varphi\circ g_r)\cdot \nu_{E\cap B(0,rL)} \, d\|\partial(E\cap B(0,rL))\|$$

$$\leq \frac{\|\partial(E\cap B(0,rL))\|(\mathbb{R}^n)}{r^{n-1}} \leq C < \infty \quad \forall r\in(0,1]$$

Property of the Reduced Bdry

$\hookrightarrow$

$$\|\partial D_r\|(\mathbb{R}^n) \leq C<\infty, \qquad 0<r\leq 1.$$

Also $\|\chi_{D_r}\|_{L^1(\mathbb{R}^n)} = \mathcal{L}^n(D_r) \leq \mathcal{L}^n(B(0,L)) < \infty, \quad r>0.$

$\hookrightarrow$

$$\|\chi_{D_r}\|_{BV(\mathbb{R}^n)} \leq C<\infty \qquad \forall\, 0<r\leq 1$$

$\xrightarrow[\text{Thm}]{\text{Compactness}} \exists \begin{cases} \{s_j\}_{j=1}^\infty \subset \{r_k\}_{k=1}^\infty \\ f\in BV_{loc}(\mathbb{R}^n) \end{cases}$ s.t.

$$\chi_{E_j} \longrightarrow f \quad \text{in} \begin{cases} L^1_{loc}(\mathbb{R}^n) \\ \mathcal{L}^n\text{-a.e.} \end{cases}$$

$E_j = E_{s_j}$

↳ $f(x) \in \{0, 1\}$ for $\mathcal{L}^n$-a.e. x.

↳ $\begin{cases} f = \chi_F & \mathcal{L}^n\text{-a.e} \\ F \subset \mathbb{R}^n \text{ has locally finite perimeter} \end{cases}$

$\Longrightarrow$ $\forall \varphi \in C_c^1(\mathbb{R}^n; \mathbb{R}^n)$

(*) $$\int_F \operatorname{div} \varphi \, dy = \int_{\mathbb{R}^n} \varphi \cdot \nu_E \, d\|\partial F\|$$

$\boxed{\text{Some } \|\partial F\|\text{-measurable function with } |\nu_F| = 1, \ \|\partial F\|\text{-a.e}}$

$\overset{?}{\hookrightarrow}$ $F = H^{-1}(0)$.

3. claim: $\nu_F = e_n$ $\|\partial F\|$-a.e.

$\forall \varphi \in C_c^1(\mathbb{R}^n; \mathbb{R}^n)$.

$$\int_{\mathbb{R}^n} \varphi \cdot \underbrace{\nu_j}_{\nu_{E_j}} \, d\|\partial E_j\| = \int_{E_j} \operatorname{div} \varphi \, dy \qquad j = 1, 2, \cdots$$

$\chi_{E_j} \to \chi_F$ in L^1_{loc}

$$\int_F \operatorname{div} \varphi \, dy \overset{(*)}{=} \int_{\mathbb{R}^n} \varphi \cdot \nu_E \, \|\partial F\|.$$

↳ $\boxed{\nu_j \|\partial E_j\| \longrightarrow \nu_F \|\partial F\|. \qquad (\mathcal{M})}$

$\hookrightarrow$ For each $L>0$ for which $\|\partial F\|(\partial B(0,L))=0$

i.e. for all but at most countably many $L>0$,

$$(**) \quad \int_{B(0,L)} \nu_j \, d\|\partial E_j\| \longrightarrow \int_{B(0,L)} \nu_F \, d\|\partial F\|.$$

On the other hand. $\forall \varphi \in C_c^1(\mathbb{R}^n;\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} \varphi\cdot\nu_j \, d\|\partial E_j\| = \frac{1}{s_j^{n-1}} \int_{\mathbb{R}^n} (\varphi\circ g_{s_j})\cdot \nu_E \, d\|\partial E\|.$$

$\hookrightarrow$

$$(***) \quad \begin{cases} \|\partial E_j\|(B(0,L)) = \dfrac{1}{s_j^{n-1}} \|\partial E\|(B(0,s_jL)) \\ \displaystyle\int_{B(0,L)} \nu_j \, d\|\partial E_j\| = \frac{1}{s_j^{n-1}} \int_{B(0,s_jL)} \nu_E \, d\|\partial E\| \end{cases}$$

$\hookrightarrow$

$$(****) \quad \lim_{j\to\infty} \fint_{B(0,L)} \nu_j \, d\|\partial E_j\| = \lim_{j\to\infty} \fint_{B(0,s_jL)} \nu_E \, d\|\partial E\|$$

$$= \nu_E(0) \qquad (0\in\partial^* E)$$

$$= e_n$$

If $\|\partial F\|(\partial B(0,L))=0$

$$\hookrightarrow \|\partial F\|(B(0,L)) \overset{\text{L.S.C.}}{\leq} \liminf_{j\to\infty} \|\partial E_j\|(B(0,L))$$

$$\overset{(****)}{=} \lim_{j\to\infty} \int_{B(0,L)} e_n\cdot\nu_j \, d\|\partial E_j\|$$

$$\overset{(**)}{=} \int_{B(0,L)} e_n\cdot\nu_F \, d\|\partial F\|$$

$\hookrightarrow \quad \|\partial F\|(B(0,L)) \le \int_{B(0,L)} e_n \cdot \nu_F \, d\|\partial F\|$

$\longrightarrow$ $|\nu_F| = 1 \quad \|\partial F\|$-a.e.

$$\begin{cases} \nu_F = e_n & \|\partial F\|\text{-a.e.} \\ \|\partial F\|(B(0,L)) = \lim_{j\to\infty} \|\partial E_j\|(B(0,L)) \end{cases}$$

$\forall \quad \|\partial F\|(\partial B(0,L)) = 0$

4. Claim: F is a half-space

Step 3 $\longrightarrow$ $\forall \ \varphi \in C_c^1(\mathbb{R}^n;\mathbb{R}^n)$

$$\boxed{\int_F \operatorname{div}\varphi \, dz = \int_{\mathbb{R}^n} \varphi \cdot e_n \, d\|\partial F\|}$$

$\left[\begin{array}{l} \forall \varepsilon > 0 \\ f^\varepsilon := \eta_\varepsilon * \chi_F \end{array} \right. \Rightarrow \quad f^\varepsilon \in C^\infty(\mathbb{R}^n)$

the mollifier

$$\int_{\mathbb{R}^n} f^\varepsilon \operatorname{div}\varphi \, dz = \int_F \operatorname{div}(f^\varepsilon * \varphi)\, dz$$

$$= \int_{\mathbb{R}^n} \eta_\varepsilon * (\varphi \cdot e_n) \, d\|\partial F\|$$

$\parallel$

$$-\int_{\mathbb{R}^n} Df^\varepsilon \cdot \varphi \, dz$$

$\hookrightarrow$

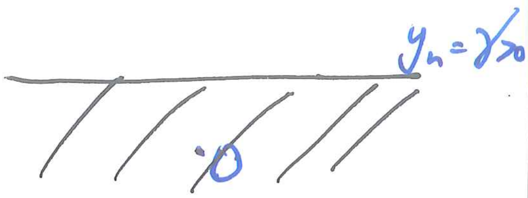
$$\begin{cases} \dfrac{\partial f^\varepsilon}{\partial z_i} = 0, & i = 1, 2, \cdots, n-1 \\ \dfrac{\partial f^\varepsilon}{\partial z_n} \le 0 \end{cases}$$

$\because$ $f_\varepsilon \longrightarrow \chi_F$ $\mathcal{L}^n$-a.e. as $\varepsilon \to 0$.

↳ Up to a set of $\mathcal{L}^n$-measure zero

$F = \{ y \in \mathbb{R}^n \mid y_n \le \gamma \}$ for some $\gamma \in \mathbb{R}$.

S $\boxed{F = H^-(0)} \iff \boxed{\gamma = 0}$



If $\gamma > 0$.

$\downarrow$ $\boxed{\chi_{E_j} \longrightarrow \chi_F \quad L^1_{loc}(\mathbb{R}^n)}$

$$\alpha(n)\gamma^n = \mathcal{L}^n(B(0,\gamma) \cap F)$$

$$= \lim_{j\to\infty} \mathcal{L}^n(B(0,\gamma) \cap E_j)$$

$$= \lim_{j\to\infty} \frac{\mathcal{L}^n(B(0,\gamma s_j) \cap E)}{s_j^n}$$

$$= \boxed{\lim_{r_j \to 0} \frac{\mathcal{L}^n(B(0,r_j) \cap E)}{\alpha(n) r_j^n}} \alpha(n)\gamma^n.$$

$\wedge$ 1. $\Longleftarrow$ $\boxed{\lim_{r_j\to 0} \frac{|B(0,r_j) - E|}{\alpha(n) r_j} > A_2 > 0}$

If $\gamma < 0$

$$\alpha(n)|\gamma|^n = \mathcal{L}^n(B(0,|\gamma|) - F) = \lim_{j\to\infty} \mathcal{L}^n(B(0,|\gamma|) - E_j)$$

$$= \boxed{\lim_{j\to\infty} \frac{|B(0,|\gamma| s_j) - E|}{\alpha(n)(|\gamma| s_j)^n}} \alpha(n)\gamma^n$$

$< 1.$

$\boxed{\text{Contradiction}}$

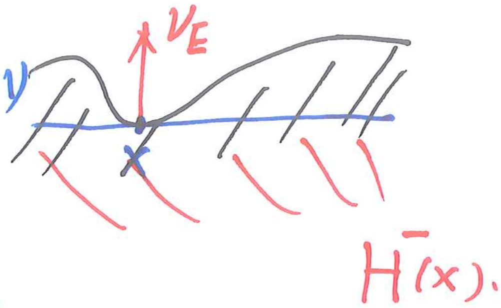
Thm $x\in\partial^*E$

↳ $$\chi_{E_r} \xrightarrow[r\to 0]{} \chi_{H^-(x)} \text{ in } L^1_{loc}(\mathbb{R}^n)$$

$$E_r = \{y\in\mathbb{R}^n \mid x+r(y-x)\in E\} \qquad H^-(x) = \{y\in\mathbb{R}^n \mid \nu_E(x)\cdot(y-x)\le 0\}$$

$\Downarrow$

$$y\in E\cap B(x,r) \Leftrightarrow x+\frac{y-x}{r}\in E_r\cap B(x,1)$$



↳ $x\in\partial^*E$

$$\begin{cases} \lim\limits_{r\to 0}\dfrac{|B(x,r)\cap E\cap H^+(x)|}{r^n} = 0 & (*) \\ \lim\limits_{r\to 0}\dfrac{|(B(x,r)-E)\cap H^-(x)|}{r^n} = 0 \\ \lim\limits_{r\to 0}\dfrac{\|\partial E\|(B(x,r))}{\alpha(n-1)\,r^{n-1}} = 1 \end{cases}$$

* The unit vector $\nu_E(x)$ for which $(*)$ holds. is called the <u>measure-theoretic unit outer normal to E at x</u>

Structure Thm for Sets of Finite Perimeter

$E \subset \mathbb{R}^n$ has locally finite perimeter

$\hookrightarrow$

(i) $\|\partial E\| = \mathcal{H}^{n-1} \llcorner \partial^* E$

(ii) $\partial^* E = \bigcup_{k=1}^{\infty} K_k \cup N$

$$\begin{cases} K_k \subset\subset \underbrace{S_k} \text{ — a } C^1\text{-hypersurface} \\ \|\partial E\|(N) = 0 \end{cases}$$

(iii) $\nu_E|_{S_k}$ is the normal to S_k, $k = 1, 2, \cdots$

$\hookrightarrow$ [A set of locally finite perimeter has "measure-theoretically a C^1-boundary

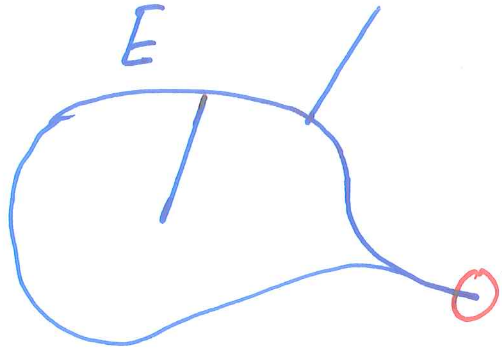
The Measure-theoretic Boundary

$E \subset \mathbb{R}^n$ a set of locally finite perimeter

$x \in \partial_* E$ ~ the measure-theoretic boundary of E

$\Updownarrow$

$$\begin{cases} \limsup_{r\to 0} \dfrac{|B(x,r)\cap E|}{r^n} > 0 \\ \limsup_{r\to 0} \dfrac{|B(x,r)-E|}{r^n} > 0 \end{cases}$$



Thm

(i) $\partial^* E \subset \partial_* E$

(ii) $\mathcal{H}^{n-1}(\partial_* E - \partial^* E) = 0$

* This is a refinement of the classical theorem

$E \subset \mathbb{R}^n$ is $\mathcal{L}^n$-measurable

$\hookrightarrow$

$$\begin{cases} \lim_{r\to 0} \dfrac{|B(x,r)\cap E|}{|B(x,r)|} = 1 & \mathcal{L}^n\text{-a.e. } x \in E \\ \lim_{r\to 0} \dfrac{|B(x,r)\cap E|}{|B(x,r)|} = 0 & \mathcal{L}^n\text{-a.e. } x \in \mathbb{R}^n - E \end{cases}$$

Gauss-Green Theorem

$E \subset \mathbb{R}^n$ locally finite perimeter

$\hookrightarrow$

(i) $\mathcal{H}^{n-1}(\partial_* E \cap \bar{K}) < \infty \qquad \forall\, K \subset\subset \mathbb{R}^n$

(ii) For $\mathcal{H}^{n-1}$-a.e. $x \in \partial_* E$,

$\hookrightarrow$ $\exists\, 1$ measure-theoretic unit outer normal $\nu_E(x)$ such that

$$(*) \qquad \boxed{\int_E \operatorname{div} \varphi \, dx = \int_{\partial_* E} \varphi \cdot \nu_E \, d\mathcal{H}^{n-1} \qquad \forall\, \varphi \in C_c^1(\mathbb{R}^n; \mathbb{R}^n)}$$

$$\boxed{\|\partial E\| = \mathcal{H}^{n-1} \llcorner \partial_* E}$$

$*$ $(*)$ holds for $E = U$, an open set with Lipschitz boundary

Pointwise Properties of BV Functions

$f \in BV(\mathbb{R}^n)$

Define

$$\mu(x) := \operatorname{ap}\limsup_{y\to x} f(y) = \inf\left\{ t \,\middle|\, \lim_{r\to 0} \frac{|B(x,r)\cap\{f>t\}|}{r^n} = 0 \right\}$$

$$\lambda(x) := \operatorname{ap}\liminf_{y\to x} f(y) = \sup\left\{ t \,\middle|\, \lim_{r\to 0} \frac{|B(x,r)\cap\{f<t\}|}{r^n} = 0 \right\}.$$

$\hookrightarrow$ $\begin{cases} -\infty \le \lambda(x) \le \mu(x) \le \infty & \forall x \in \mathbb{R}^n. \\ \lambda(x),\ \mu(x) \text{ are Borel measurable} \end{cases}$

$J := \{ x \in \mathbb{R}^n \mid \lambda(x) < \mu(x) \}$

the set of points at which the approximate limit of f does not exist.

$\hookrightarrow$ $\mathcal{L}^n(J) = 0$

$\Rightarrow$ (i) $\exists$ countably many C^1-hypersurfaces $\{S_k\}_{k=1}^\infty$

s.t. $$\mathcal{H}^{n-1}\left(J - \bigcup_{k=1}^\infty S_k\right) = 0$$

* A BV-function is "measure theoretically piecewise continuous" with "jumps along a measure theoretically C^1-surface"

(ii) $-\infty < \lambda(x) \le \mu(x) < \infty$ $\mathcal{H}^{n-1}$-a.e. $x \in \mathbb{R}^n$

Notation

$$\begin{cases} F(x) := \dfrac{\lambda(x)+\mu(x)}{2} \\ \nu \in \mathbb{R}^n \ \text{unit vector} \\ H_\nu := \{ y \in \mathbb{R}^n \mid \nu \cdot (y-x) = 0 \} \ \text{hyperplane} \\ H_\nu^{\pm} := \{ y \in \mathbb{R}^n \mid \pm \nu \cdot (y-x) \geq 0 \} \ \text{half-spaces} \end{cases}$$

Fine Properties of BV functions $f \in BV(\mathbb{R}^n)$

(i) $\displaystyle \lim_{r\to 0} \fint_{B(x,r)} |f - F(x)|^{\frac{n}{n-1}} dy = 0 \quad \mathcal{H}^{n-1}\text{-a.e. } x \in \mathbb{R}^n - J$

(ii) For $\mathcal{H}^{n-1}$-a.e. $x \in J$, $\exists$ a unit vector $\nu = \nu(x)$

s.t.

$$\begin{cases} \displaystyle \lim_{r\to 0} \fint_{B(x,r)\cap H_\nu^-} |f-\mu(x)|^{\frac{n}{n-1}} dy = 0 \\ \displaystyle \lim_{r\to 0} \fint_{B(x,r)\cap H_\nu^+} |f-\lambda(x)|^{\frac{n}{n-1}} dy = 0 \end{cases}$$

↳ $\displaystyle \mu(x) = \operatorname*{ap\,lim}_{\substack{y\to x \\ y\in H_\nu^+}} f(y), \quad \lambda(x) = \operatorname*{ap\,lim}_{\substack{y\to x \\ y \in H_\nu^-}} f(y).$

↳ For $\mathcal{H}^{n-1}$-a.e. $x \in J$, f has a "measure-theoretic jump" across the hyperplane $H_{\nu(x)}$.

↳

(i) $\displaystyle f^*(x) = \lim_{r\to 0} (f)_{x,r} = F(x)$ exists $\mathcal{H}^{n-1}$-a.e. $x \in \mathbb{R}^n$

(ii) If η_ε is the standard mollifier and $f^\varepsilon = \eta_\varepsilon * f$.

$$f^*(x) = \lim_{\varepsilon\to 0} (\eta_\varepsilon * f)(x) \qquad \mathcal{H}^{n-1}\text{-a.e. } x \in \mathbb{R}^n$$

Criterion for Finite Perimeter

$E \subset \mathbb{R}^n$ is $\mathcal{L}^n$-measurable

E has locally finite perimeter

$$\Updownarrow$$

$$\mathcal{H}^{n-1}(\partial_* E \cap K) < \infty$$

for each $K \subset\subset \mathbb{R}^n$