Topics on Nonlinear Hyperbolic PDEs

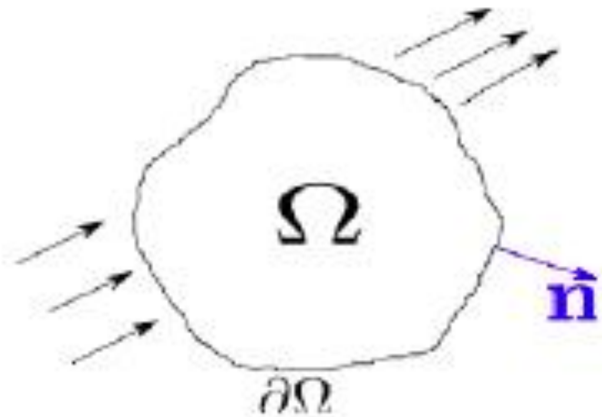
Mathematical Institutes
Hilary Term 2022
February: 9th, 16th, 23rd
March: 2nd
Wednesdays 14:00-16:00

By Professor Gui-Qiang G. Chen

Lecture-2: 16th February 2022

http://people.maths.ox.ac.uk/chengq/teach/PDECDT2022-NHPDE/CDT-NHPDE.html

Conservation Laws



Rate of Change of the Total Amount of Certain Quantity in a Fixed Region Ω
= Flux of the Quantity across the Boundary $\partial\Omega$.

Conservation Law via Calculus

$$\Longleftrightarrow \qquad \frac{d}{dt}\int_{\Omega} u d\mathbf{x} = -\int_{\partial\Omega} \mathbf{f}\cdot\mathbf{n}\, dS$$

u – Density of the Quantity
$\mathbf{n}$ – Outward Normal to Ω
$\mathbf{f}$ – Flux of the Quantity
dS – Surface Element on $\partial\Omega$

Calculus Manipulations $\Longrightarrow$

$$\partial_t u + \nabla_{\mathbf{x}}\cdot\mathbf{f}(u) = 0$$

Physical Systems with $m \geq 2$ Quantities – Density Functions

$\Longrightarrow$ **Systems of Conservation Laws**:

$$\partial_t \mathbf{u} + \nabla\cdot\mathbf{f}(\mathbf{u}) = 0$$

$$\mathbf{u} = (u_1, \cdots, u_m)^\top \qquad \mathbf{f}(\mathbf{u}) = (\mathbf{f}_1(\mathbf{u}), \cdots, \mathbf{f}_d(\mathbf{u}))$$

Euler Equations for Compressible Fluids

$$\begin{cases} \partial_t \rho + \nabla_{\mathbf{x}} \cdot (\rho \mathbf{v}) = 0 & \text{(conservation of mass)} \\ \partial_t (\rho \mathbf{v}) + \nabla_{\mathbf{x}} \cdot (\rho \mathbf{v} \otimes \mathbf{v}) + \nabla_{\mathbf{x}} p = 0 & \text{(conservation of momentum)} \\ \partial_t (\frac{1}{2}\rho |\mathbf{v}|^2 + \rho e) + \nabla_{\mathbf{x}} \cdot ((\frac{1}{2}\rho |\mathbf{v}|^2 + \rho e + p)\mathbf{v}) = 0 & \text{(conservation of energy)} \end{cases}$$

Leonhard Euler

Constitutive Relations: $p = p(\rho, e)$

- ρ – density, $\mathbf{v} = (v_1, v_2, v_3)^\top$ – fluid velocity
- p – pressure, e – internal energy

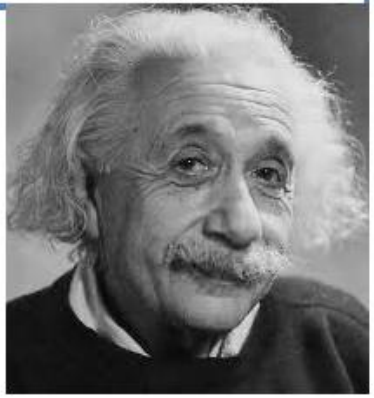
***Govern** the Flows when Convective Motions Dominate Diffusion/Dispersion, …*

e.g. , shock waves in Gases, Elastic Fluids, Shallow Water, ……

Poisson, Challis, Stokes, Kelvin, Rayleigh, Airy, Earnshaw, Riemann, Rankine, Christoffel, Mach, Clausius, Kirchhoff, Gibbs, Hugoniot, Duhem, Hadamard, Jouguet, Zamplen, Weber, Taylor, Becker, Bethe, Weyl, von Neumann, Courant, Friedrichs, ……

George Stokes

Conservation Laws and Einstein Equations



$$\mathbf{G}_{\mu\nu} = 8\pi \mathbf{T}_{\mu\nu}$$

$T_{\mu\nu}$ – Stress-energy tensor (Energy-momentum tensor)
$G_{\mu\nu}$ – Einstein tensor (Function of the metric)

These equations, with the geodesic equation, form the core of the mathematical formulation of General Relativity

Structure of the Einstein Equations

$\Longrightarrow$ **Conservation Laws of Energy and Momentum**:

$$\nabla_{\mathbf{b}} \mathbf{T}^{\mathbf{ab}} = \mathbf{T}^{\mathbf{ab}}{}_{;\mathbf{b}} = 0$$

CALCULUS OF VARIATIONS

A Field of Mathematics that deals with extremizing functionals, as opposed to ordinary calculus which deals with functions:

$$\boxed{I[w] = \int_{\Omega} L(\nabla_x w(x), w(x), x)dx}$$

- Energy or Action Functionals in Physics/Engineering/industry….
- Distance/Metric Functional in Optics (light),
  Geometry (geodesics, minimal surfaces, …), ….
- Cost Functionals in Optimization (controls, games, image processing, design,
  finance, transportation, …), …

POINT: Seek a Minimizer or Critical Point u of $I[\cdot]$:

$$\boxed{I'[u]=0}$$

Great Progress has been made in the recent four decades…

Conservation Laws and Calculus of Variations

- ❑ Systems of Euler-Lagrange Equations

- ❑ Noether's Theorem:

Any Differentiable Symmetry of the Action of a Physical System Has a Corresponding Conservation Law.

Any Invariance of the Variational Integral I[w] leads to a corresponding Conservation Law for the critical point of I[·]

Hyperbolicity

$$\partial_t \mathbf{u} + \nabla \cdot \mathbf{f}(\mathbf{u}) = 0, \quad \mathbf{u} \in \mathbb{R}^m, \mathbf{x} \in \mathbb{R}^d$$

Plane Wave Solutions: $\mathbf{u}(t, \mathbf{x}) = \mathbf{w}(t, \omega \cdot \mathbf{x})$

$\mathbf{w}(t, \xi)$ is determined by: $\partial_t \mathbf{w} + (\nabla_{\mathbf{w}} \mathbf{f}(\mathbf{w}) \cdot \omega)\, \partial_\xi \mathbf{w} = 0$

?? **Existence** of stable plane wave solutions ??

Hyperbolicity in D: For any $\omega \in S^{d-1}$, $\mathbf{u} \in D$,

$$(\nabla_{\mathbf{u}} \mathbf{f}(\mathbf{u}) \cdot \omega)_{m \times m} \mathbf{r}_j(\mathbf{u}, \omega) = \lambda_j(\mathbf{u}, \omega)\, \mathbf{r}_j(\mathbf{u}, \omega), \ 1 \le j \le m$$

$\lambda_j(\mathbf{u}, \omega)$ **are real**

Main Features:

Finiteness of Propagation Speeds;

Discontinuities of Solutions, $\cdots\cdots$

Well-Posedness: Existence, Uniqueness, Stability, $\cdots$

The Simplest Example

Scalar Conservation Laws

$$\partial_t u + \nabla \cdot \mathbf{f}(u) = 0, \quad u \in \mathbb{R}, \, \mathbf{x} \in \mathbb{R}^d$$

$$\mathbf{f} : \mathbb{R} \to \mathbb{R}^d$$

Then

$$\lambda(u, \omega) = \mathbf{f}'(u) \cdot \omega, \qquad r(u, \omega) \equiv 1$$

$\Longrightarrow$ **Scalar conservation laws are always hyperbolic**

Scalar Conservation Laws

$$(*)\quad \begin{cases} u_t + f(u)_x = 0 \\ u|_{t=0} = u_0(x) \in L^\infty \end{cases}$$

$f(0)=0$

$f''(u) \geq c_0 > 0$

Hamilton-Jacobi Equations

$$(**)\quad \begin{cases} W_t + f(W_x) = 0 \\ W|_{t=0} = h(x) = \int_0^x u_0(y)\,dy \quad \text{Lipschitz} \end{cases}$$

$(**) \Longrightarrow (*)$ $\qquad u(t,x) = W_x(t,x)$

$(*) \Longrightarrow (**)$ $\qquad W(t,x) = \int_0^x u(t,y)\,dy - \int_0^t f(u(\tau,0))\,d\tau$

Hamilton-Jacobi Equations

$$(**)\begin{cases} W_t + f(W_x) = 0 & f'' \geq c_0 > 0 \\ W|_{t=0} = h(x) & \text{Lipschitz} \end{cases}$$

<u>Motivation</u>: Dynamic Programming in Control Theory cf. Evans, <u>Ch. 10.3</u>

<u>The Value Function</u>

$$(A) \quad \boxed{W(t,x) = \inf\left\{ \int_0^t f^*(\dot{v}(s))\,ds + h(y) \;\middle|\; \begin{array}{l} v \in C^1 \\ v(0) = y \\ v(t) = x \end{array} \right\}}$$

is a solution of $(**)$

$\Rightarrow$ $\boxed{\text{Hopf-Lax Formula}}$. For $x \in \mathbb{R}$, $t > 0$

$$(B) \quad W(t,x) = \min_{y \in \mathbb{R}} \left\{ t f^*\left(\frac{x-y}{t}\right) + h(y) \right\}$$

Lax-Oleinik Formula

$$\begin{cases} f'' \geq c_0 > 0 \\ u_0 \in L^\infty(\mathbb{R}) \end{cases}$$

(i) For each $t>0$, for all but at most countably many values of $x \in \mathbb{R}$, $\boxed{\exists \text{ a unique } y(t,x)}$ such that

$$\min_{y\in\mathbb{R}} \left\{ t f^*\left(\frac{x-y}{t}\right) + h(y) \right\} = t f^*\left(\frac{x-y(t,x)}{t}\right) + h(y(t,x))$$

(ii) The mapping $x \longmapsto y(t,x)$ is nondecreasing

(iii) For each $t>0$

$$\boxed{u(t,x) = (f')^{-1}\left(\frac{x-y(t,x)}{t}\right) \quad \text{for a.e. } x}$$

(iv) $u(t,x)$ is a weak solution satisfying

$$\boxed{u(t,x+z) - u(t,x) \leq \frac{C}{t} z} \qquad \forall\, t>0,\ x, z \in \mathbb{R},\ z \geq 0$$

for some $C>0$.

(v) $u(t,x)$ is $\boxed{\text{unique}}$ in the class of weak solutions that satisfy (iv).

Oleinik E-Condition

$$(E) \quad \boxed{u(t, x+z) - u(t,x) \le \frac{C}{t} z. \qquad \forall t>0,\ x, z \in \mathbb{R},\ z \ge 0}$$

Lax Entropy Condition

$$\boxed{u(t, x-0) \ge u(t, x+0)}$$

$(E) \Rightarrow \quad x \longrightarrow u(t,x) - \frac{C}{t} x$ is nonincreasing

↳ $\exists$ left and right hand limits at each point

↳ $u(t, x-0) - \frac{C}{t} x \ge u(t, x+0) - \frac{C}{t} x$

↳ $u(t, x-0) \ge u(t, x+0)$

Applications

I. Riemann Problem

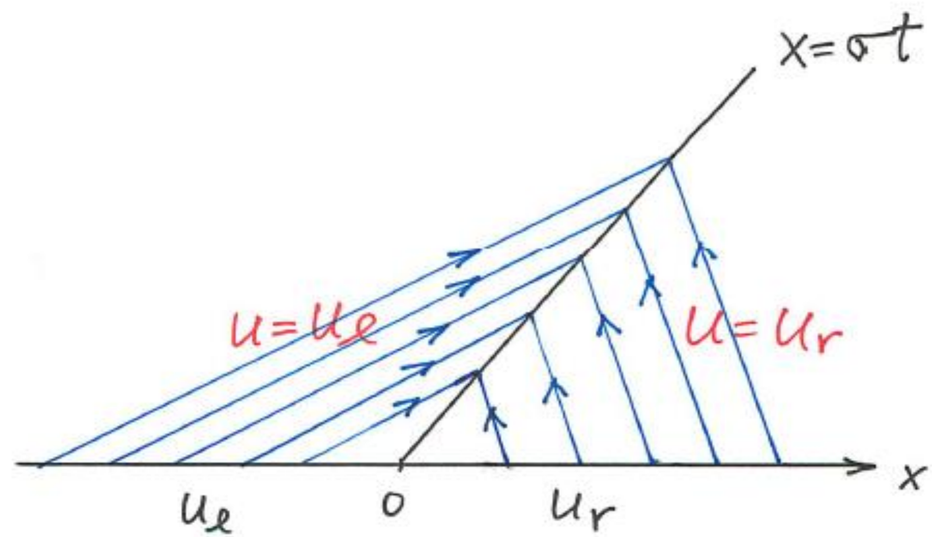
$$u_0(x) = \begin{cases} u_l & x<0 \\ u_r & x>0 \end{cases}$$

(i) $u_l > u_r \Rightarrow$ Shock Wave Solution

$$u(t,x) = \begin{cases} u_l & \frac{x}{t} < \sigma \\ u_r & \frac{x}{t} > \sigma \end{cases}$$

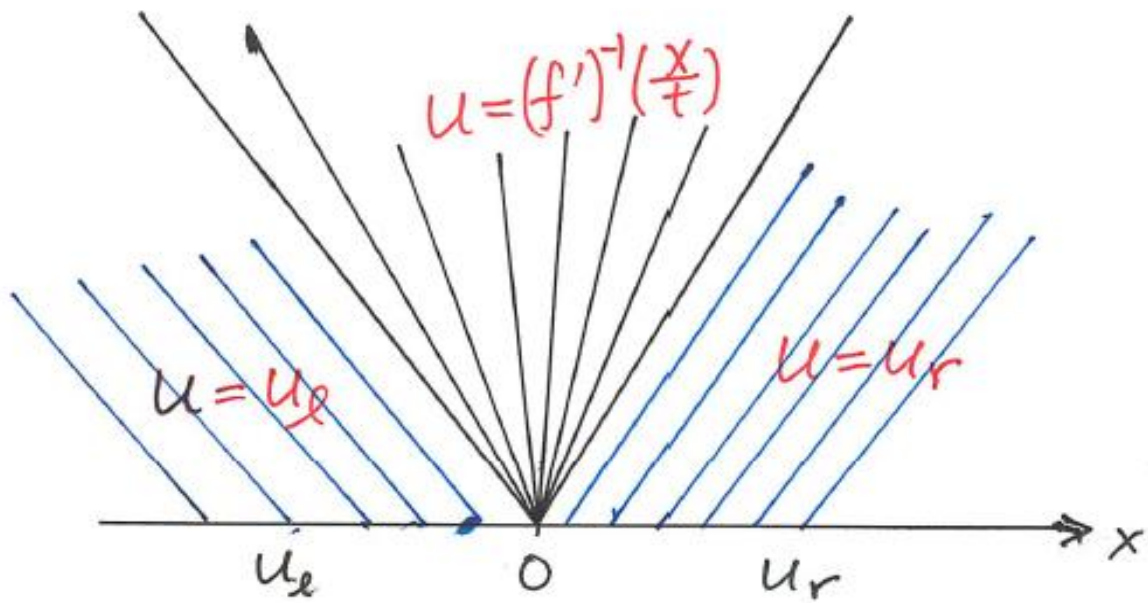
where

$$\sigma = \frac{[f]}{[u]} = \frac{f(u_r)-f(u_l)}{u_r-u_l} \quad \underline{\text{shock speed}}$$



(ii) $u_l < u_r \Longrightarrow$ Rarefaction Wave Solution

$$u(t,x) = \begin{cases} u_l & \frac{x}{t} \le f'(u_l) \\ (f')^{-1}\left(\frac{x}{t}\right) & f'(u_l) < \frac{x}{t} < f'(u_r) \\ u_r & \frac{x}{t} \ge f'(u_r) \end{cases}$$



II. Decay in L^∞

If $\begin{cases} f(0)=0 \\ u_0 \in L^1 \cap L^\infty(\mathbb{R}) \end{cases}$

$\Rightarrow$ $\exists\, C>0$ such that

$$|u(t,x)| \le \frac{C}{t^{1/2}} \qquad \forall \begin{matrix} x\in\mathbb{R} \\ t>0 \end{matrix}$$

* Different from the linear case
* The decay rate $t^{-1/2}$ is optimal.

III. Decay to the N-Wave in L^1

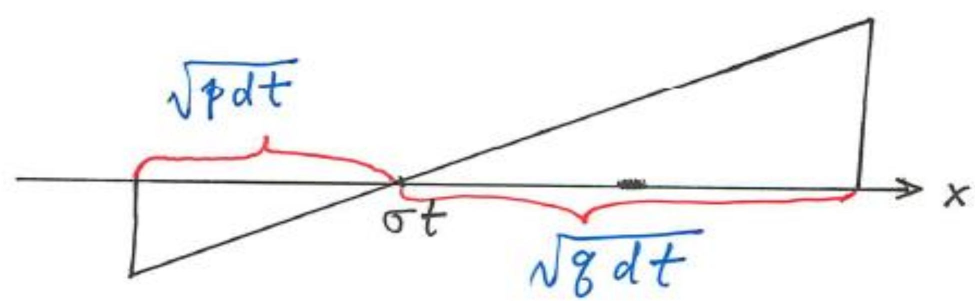
If $\begin{cases} f(0) = 0 \\ u_0 \text{ has compact support} \end{cases}$

$\Longrightarrow$

$$\| u(t,\cdot) - \underbrace{N(t,\cdot)}_{\parallel} \|_{L^1(\mathbb{R})} \leq \frac{C}{t^{1/2}} \qquad \forall t > 0$$

$$\begin{cases} \frac{1}{d}\left(\frac{x}{t} - \sigma\right) & -\sqrt{pdt} < x - \ t < \sqrt{qdt} \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{cases} p = -2 \min\limits_{y \in \mathbb{R}} \left(\int_{-\infty}^{y} u_0(x)\,dx \right) \geq 0 \\ q = 2 \max\limits_{y \in \mathbb{R}} \left(\int_{y}^{\infty} u_0(x)\,dx \right) \geq 0 \\ d = f''(0) > 0 \\ \sigma = f'(0) \end{cases}$$



N- Wave

Ex

$$\begin{cases} u_t + \left(\frac{u^2}{2}\right)_x = 0 \\ u|_{t=0} = u_0(x) = \begin{cases} 0 & x<0 \\ 1 & 0\le x\le 1 \\ 0 & x>1 \end{cases} \end{cases}$$

Solution

$0\le t\le 2$

$$u(t,x) = \begin{cases} 0 & x<0 \\ \frac{x}{t} & 0<x<t \\ 1 & t<x<1+\frac{t}{2} \\ 0 & x>1+\frac{t}{2} \end{cases}$$

$t\ge 2$

$$u(t,x) = \begin{cases} 0 & x<0 \\ \frac{x}{t} & 0<x<\sqrt{2t} \\ 0 & x>\sqrt{2t} \end{cases}$$

L^∞:

$$|u(t,x)| \le \left|\frac{x}{t}\right| \le \frac{\sqrt{2t}}{t} = \frac{\sqrt{2}}{\sqrt{t}}$$

L^1:

$$N(t,x) = \begin{cases} \frac{x}{t} & 0<x<\sqrt{2t} \\ 0 & \text{otherwise}. \end{cases}$$

$$[\sigma=0,\ d=1,\ p=0,\ q=2]$$

L^1-Theory for Scalar Conservation Laws

$$(*)\quad \begin{cases} \partial_t u + \nabla_x \cdot f(u) = 0 \\ u|_{t=0} = u_0(x) \end{cases}$$

$f: \mathbb{R} \longrightarrow \mathbb{R}^d$ Given Smooth function on $\mathbb{R}$

$f(u) = (f_1(u), \cdots, f_d(u))$

Admissible Solutions

$\boxed{u \in L^\infty(\mathbb{R}_+ \times \mathbb{R}^d),}$ $\mathbb{R}_+ = [0, \infty)$

$$(**)\quad \boxed{\int_0^\infty \int_{\mathbb{R}^d} \Big(\eta(u)\partial_t \psi + \sum_{j=1}^{d} q_j(u)\, \partial_{x_j} \psi \Big)\, dx\, dt + \int_{\mathbb{R}^d} \psi(0,x)\, \eta(u_0(x))\, dx \geq 0,}$$

for any entropy $\eta(u)$, $\underline{\eta''(u) \geq 0}$, and Corresponding entropy flux $q(u) = \int^u \eta'(v) f'(v)\, dv$, and any test function $\psi \in C_0^\infty(\mathbb{R}_+ \times \mathbb{R}^d)$, $\underline{\psi \geq 0}$

$$(**)\quad \int_0^\infty\int_{\mathbb{R}^d}\left(\eta(u)\partial_t\psi + q(u)\cdot\nabla_x\psi\right)dx\,dt + \int_{\mathbb{R}^d}\psi(0,x)\,\eta(u_0(x))\,dx \geq 0$$

1. $\partial_t \eta(u(t,x)) + \nabla_x\cdot q(u(t,x)) \leq 0 \quad \mathcal{D}'$

$\hookrightarrow$ $\boxed{\partial_t \eta(u) + \nabla_x q(u) =: \mu \quad \text{Radon Measure}}$

2. $(\eta(u), q(u)) = (\pm u, \pm f(u)) \Rightarrow$ $u(t,x)$ Weak Solution

3. Lipschitz Convex functions can be approximated by

$$\left\{c_0 u + \sum_{j=1}^m c_j (u-u_j)^+\right\}$$

$$\boxed{(**)} \Leftarrow \text{Sufficient for } (\eta, q) = \begin{cases} (\pm u, \pm f(u)) \\ ((u-\bar{u})^+, \operatorname{sign}(u-\bar{u})^+(f(u)-f(\bar{u}))) \end{cases} \quad \forall\, \bar{u}\in\mathbb{R}.$$

$\Updownarrow$

$$\boxed{(|u-\bar{u}|,\ \operatorname{sign}(u-\bar{u})(f(u)-f(\bar{u})))}$$

Kruzkov's family of Entropy-Entropy Flux Pairs

L^1-Theory

Theorem 1 (Existence & Uniqueness) $u_0 \in L^\infty(\mathbb{R}^d)$

$\Longrightarrow$ $\exists\, 1$ admissible solution u of $(*)$ and

$$u(t,\cdot) \in C^0([0,\infty); L^1_{loc}(\mathbb{R}^d)).$$

Theorem 2 (Stability in L^1 & Monotonicity in L^∞).

$$\begin{cases} u_0 \in L^\infty(\mathbb{R}^d) \longrightarrow u(t,\cdot) \in C^0([0,\infty); L^1_{loc}(\mathbb{R}^d)) \\ v_0 \in L^\infty(\mathbb{R}^d) \longrightarrow v(t,\cdot) \in C^0([0,\infty); L^1_{loc}(\mathbb{R}^d)) \end{cases}$$

$\Longrightarrow$ $\exists\; s = s(\|u_0\|_{L^\infty}, \|v_0\|_{L^\infty}) > 0$ such that, $\forall\, t > 0,\ R > 0$.

$$\begin{cases} \int_{|x|<r} [u(t,x) - v(t,x)]^+ dx \le \int_{|x|<r+st} [u_0(x) - v_0(x)]^+ dx \\ \|u(t,\cdot) - v(t,\cdot)\|_{L^1(B_r)} \le \|u_0(\cdot) - v_0(\cdot)\|_{L^1(B_{r+st})} \end{cases}$$

$\hookrightarrow$ $\begin{cases} \text{If } u_0(x) \le v_0(x),\ a\text{-}e \Longrightarrow u(t,x) \le v(t,x)\ .\ a\text{-}e \text{ on } \mathbb{R}_+ \times \mathbb{R}^d \\ \inf u_0(x) \le u(t,x) \le \sup u_0(x);\quad \inf v_0(x) \le v(t,x) \le \sup v_0(x) \end{cases}$

Ideas of Proof: Theorem 2

1. $(\eta(u;v), q(u;v)) = \left((u-v)^+, \operatorname{sign}(u-v)^+ (f(u)-f(v))\right)$

   Entropy-Entropy Flux Pair in the Variable u, for fixed v

   – – Variable v, for fixed u

   $\phi(t,x;\tau,y) \geq 0$ Lipschitz, $\operatorname{supp}\phi \subset\subset \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{R}^d$

2. Fix (τ, y), in $(**)$, choose

$$\begin{cases} (\eta, q) = (\eta(u; v(\tau,y)), q(u; v(\tau,y))) & \text{w.r.t } (t,x) \\ \psi(t,x) := \phi(t,x;\tau,y) \end{cases}$$

$\Rightarrow$

$$\boxed{\begin{aligned} &\int_0^\infty \int_{\mathbb{R}^d} \left\{ \partial_t \phi(t,x;\tau,y)\, \eta(u(t,x); v(\tau,y)) + \nabla_x \phi(t,x;\tau,y)\, q(u(t,x); v(\tau,y)) \right\} dx\, dt \\ &+ \int_{\mathbb{R}^d} \phi(0,x;\tau,y)\, \eta(u_0(x); v(\tau,y))\, dx \geq 0 \end{aligned}} \quad (A)$$

Fix (t,x), in $(**)$, choose

$$\begin{cases} (\eta, q) = \big(\eta(U(t,x); v),\ q(U(t,x); v)\big) & \text{w.r.t. } (\tau, y) \\ \psi(\tau, y) := \phi(t, x; \tau, y) \end{cases}$$

$$\Rightarrow \boxed{\begin{aligned} &\int_0^\infty \int_{\mathbb{R}^d} \Big\{ \partial_\tau \phi(t,x;\tau,y)\, \eta\big(U(t,x); v(\tau,y)\big) + \nabla_y \phi(t,x;\tau,y)\, q\big(U(t,x); v(\tau,y)\big) \Big\} dy\, d\tau \\ &\qquad + \int_{\mathbb{R}^d} \phi(t,x;0,y)\, \eta\big(U(t,x); v_0(y)\big)\, dy \geq 0 \end{aligned}} \quad (B)$$

$$\boxed{\int_0^\infty \int_{\mathbb{R}^d} (A)\, dt\, dy + \int_0^\infty \int_{\mathbb{R}^d} (B)\, dt\, dx}$$

$$\Rightarrow \int_0^\infty \int_{\mathbb{R}^d} \int_0^\infty \int_{\mathbb{R}^d} \Big\{ \underline{(\partial_t + \partial_\tau)\phi(t,x;\tau,y)}\, \eta(u(t,x);\, v(\tau,y))$$

$$+ \sum_{j=1}^{d} \underline{(\partial_{x_j} + \partial_{y_j})\phi(t,x;\tau,y)}\, q_j(u(t,x);\, v(\tau,y)) \Big\}\, dx\, dt\, dy\, d\tau$$

$$+ \int_0^\infty \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \phi(0,x;\tau,y)\, \eta(u_0(x);\, v(\tau,y))\, dx\, dy\, d\tau$$

$$+ \int_0^\infty \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \phi(t,x;0,y)\, \eta(u(t,x);\, v_0(y))\, dx\, dy\, d\tau \geq 0$$

$$\forall \begin{cases} \phi(t,x;\tau,y) \geq 0 \quad \text{Lipschitz} \\ \operatorname{Supp}\phi \subset\subset \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{R}^d \end{cases}$$

3. Choose

$$\phi(t,x;\tau,y)=\frac{1}{\varepsilon^{d+1}}\psi\left(\frac{x+y}{2},\frac{t+\tau}{2}\right)\rho\left(\frac{t-\tau}{2\varepsilon}\right)\prod_{j=1}^{d}\rho\left(\frac{x_j-y_j}{2\varepsilon}\right),\quad \varepsilon>0$$

$$\rho\in C_0^\infty(\mathbb{R}),\quad \rho\geq 0\quad \int_{-\infty}^{\infty}\rho(z)\,dz=1$$

$$\underline{\psi\in Lip(\mathbb{R}_+\times\mathbb{R}^d)},\quad \psi\geq 0,\quad \operatorname{supp}\psi\subset\subset\mathbb{R}_+\times\mathbb{R}^d$$

$$\left\{\begin{aligned}
&(\partial_t+\partial_\tau)\phi(t,x;\tau,y)=\frac{1}{\varepsilon^{d+1}}\underline{\partial_t\psi\left(\frac{x+y}{2},\frac{t+\tau}{2}\right)}\rho\left(\frac{t-\tau}{\varepsilon}\right)\prod_{j=1}^{d}\rho\left(\frac{x_j-y_j}{\varepsilon}\right),\\
&(\partial_{x_j}+\partial_{y_j})\phi(t,x;\tau,y)=\frac{1}{\varepsilon^{d+1}}\underline{\partial_{x_j}\psi\left(\frac{x+y}{2},\frac{t+\tau}{2}\right)}\rho\left(\frac{t-\tau}{\varepsilon}\right)\prod_{j=1}^{d}\rho\left(\frac{x_j-y_j}{\varepsilon}\right),
\end{aligned}\right.$$

$$\left\{\begin{aligned}
&|\eta(u(t,x);v_0(y))-\eta(u_0(x);v_0(y))|\leq|u(t,x)-u_0(x)|,\\
&|\eta(u_0(x);v(t,x))-\eta(u_0(x);v_0(y))|\leq|v(\tau,y)-v_0(y)|.
\end{aligned}\right.$$

$$\frac{1}{\varepsilon^{d+1}}\rho\left(\frac{t-\tau}{\varepsilon}\right)\prod_{j=1}^{d}\rho\left(\frac{x_j-y_j}{\varepsilon}\right)\longrightarrow\delta_{t=\tau}\prod_{j=1}^{d}\delta_{x_j=y_j}\qquad(\mathcal{M})$$

4. $\begin{cases} \varepsilon \longrightarrow 0 \\ \text{Convergence Theorem} \end{cases}$

$\Longrightarrow$

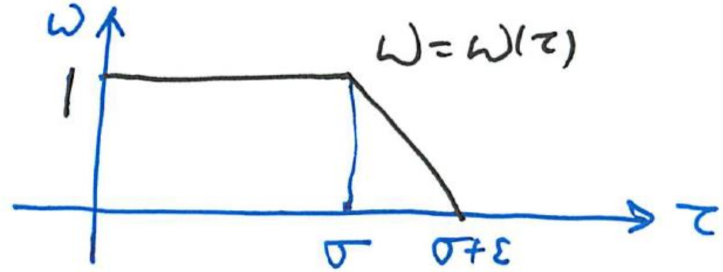
(c) $$\boxed{\begin{aligned} &\int_0^\infty \int_{\mathbb{R}^d} \Big\{ \partial_t \psi(t,x)\, \eta\big(u(t,x);\, v(t,x)\big) \\ &\qquad + \sum_{j=1}^d \partial_{x_j} \psi(t,x)\, q_j\big(u(t,x);\, v(t,x)\big) \Big\}\, dx\,dt \\ &+ \int_{\mathbb{R}^d} \psi(0,x)\, \eta\big(u_0(x);\, v_0(x)\big)\, dx \geq 0 \end{aligned}}$$

$\forall\, \psi \in Lip(\mathbb{R}_+ \times \mathbb{R}^d)$, $\psi \geq 0$, $\operatorname{Supp}\psi \subset\subset \mathbb{R}_+ \times \mathbb{R}^d$

5. $|q(u;v)| \le s\,\eta(u;v)$ $\quad \forall u \in [\inf u_0(x), \sup u_0(x)]$
$\forall v \in [\inf v_0(x), \sup v_0(x)]$

For $r>0,\ t\ge 0,\ \varepsilon>0$ $\quad \boxed{\psi(\tau,x)=\omega(\tau)\chi(\tau,x)}$

$$\omega(\tau)=\begin{cases} 1 & 0<\tau<\sigma \\ \dfrac{\sigma-\tau}{\varepsilon}+1 & \sigma\le\tau<\sigma+\varepsilon \\ 0 & \sigma+\varepsilon\le\tau<\infty \end{cases}$$



$$\chi(\tau,x)=\begin{cases} 1 & |x|<r+s(t-\tau) \\ \dfrac{r+s(t-\tau)-|x|}{\varepsilon}+1 & r+s(t-\tau)\le|x|<r+s(t-\tau)+\varepsilon \\ 0 & |x|\ge r+s(t-\tau)+\varepsilon \end{cases}$$

$$(C)\Rightarrow \frac{1}{\varepsilon}\int_t^{t+\varepsilon}\int_{|x|<r}[u(\tau,x)-v(\tau,x)]^+\,dx\,d\tau$$
$$\le \int_{|x|<r+st}[u_0(x)-v_0(x)]^+\,dx - \frac{1}{\varepsilon}\int_0^t\int_{r+s(t-\tau)\le|x|<r+s(t-\tau)+\varepsilon}\left[s\eta(u;v)+\frac{q(u;v)\cdot x}{|x|}\right]dx\,d\tau + O(\varepsilon).$$

$$\xRightarrow{\varepsilon\to 0+} \int_{|x|<r}[u(t,x)-v(t,x)]^+\,dx \le \int_{|x|\le r+st}[u_0(x)-v_0(x)]^+\,dx$$

6. Interchanging the roles of u and v

$$\Rightarrow \int_{|x|<r} [v(t,x)-u(t,x)]^+ dx \le \int_{|x|\le r+st} [v_0(x)-u_0(x)]^+ dx$$

7. <u>Steps 5–6</u>

$$\Rightarrow \|u(t,\cdot)-v(t,\cdot)\|_{L^1(B_r)} \le \|u_0(\cdot)-v_0(\cdot)\|_{L^1(B_{r+st})}.$$

8. $u_0(x) \le v_0(x)$ a.e $\Longrightarrow$ $u(t,x) \le v(t,x)$ a.e.

$v_0(x) = \sup_x u_0(x)$ $\Longrightarrow$ $u(t,x) \le \sup_x u_0(x)$ a.e

$v_0(x) = \inf_x u_0(x)$ $\Longrightarrow$ $u(t,x) \ge \inf_x u_0(x)$ a.e.

Similarly, we have

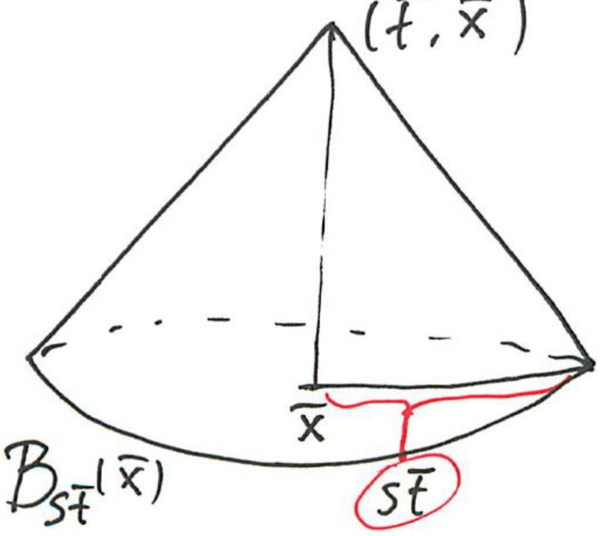
$$\inf_x v_0(x) \le v(t,x) \le \sup_x v_0(x) \quad \text{a.e}$$

Direct Applications

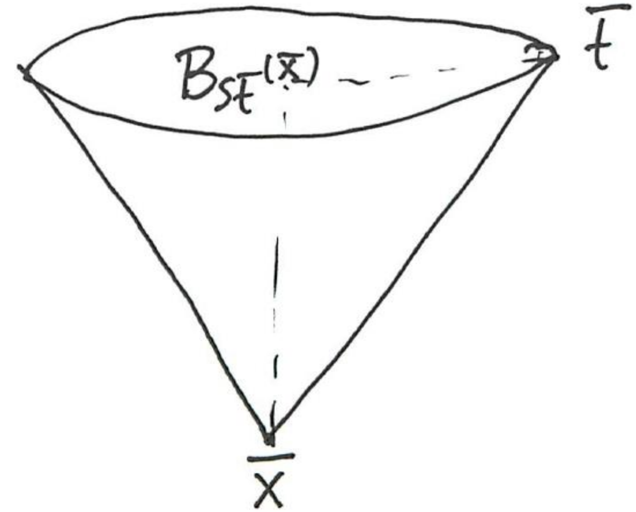
1. $\exists$ at most one admissible weak solution of $(*)$
2. The value $u(\bar{t}, \bar{x})$ depends solely on the restriction of the initial data to the ball $B_{s\bar{t}}(\bar{x})$

$\hookrightarrow$ $\boxed{\text{Finite Propagation Speed}}$

Domain of Dependence



Domain of Influence



Direct Applications

3. $u_0 \in BV_{loc}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \Longrightarrow u(t,x) \in BV_{loc}(\mathbb{R}_+ \times \mathbb{R}^d)$

$\Uparrow$

For fixed $t>0$
$$\begin{cases} u(t,\cdot) \in BV_{loc}(\mathbb{R}^d) \\ TV_{B_r}(u(t,\cdot)) \le TV_{B_{r+st}}(u_0(\cdot)), \quad \forall r>0 \end{cases}$$

$s \sim \|u_0\|_{L^\infty}$

Proof ① $\{e_j\}_{j=1}^d$ — standard orthonormal basis of $\mathbb{R}^d$

For $j=1,\cdots,d$, $\quad v(t,x) := u(t, x+he_i) \quad h>0$

[admissible solution of $(*)$. with initial data $v_0(x) = u_0(x+he_i)$

$$\xRightarrow{\text{Theorem 2}} \int_{|x|<r} |u(t, x+he_i) - u(t,x)|\, dx$$
$$\le \int_{|x|<r+st} |u_0(x+he_i) - u_0(x)|\, dx \underset{\substack{\uparrow \\ u_0 \in BV}}{\le} C|h|.$$

$\Longrightarrow u(t,\cdot) \in BV_{loc}(\mathbb{R}^d)$

Proof (Conti.)

(2) (1) $\Rightarrow$ $\partial_{x_j} u(t,\cdot) \in \mathcal{M}_{loc}(\mathbb{R}_+ \times \mathbb{R}^d)$

$\hookrightarrow$ $\nabla_x \cdot f(u(t,x)) \in \mathcal{M}_{loc}(\mathbb{R}_+ \times \mathbb{R}^d)$

$\hookrightarrow$ $u_t = -\nabla_x \cdot f(u(t,x)) \in \mathcal{M}_{loc}(\mathbb{R}_+ \times \mathbb{R}^d)$

$\hookrightarrow$ $u \in BV_{loc}(\mathbb{R}_+ \times \mathbb{R}^d)$

Direct Applications

4. $u_0 \in L^\infty(\mathbb{R}^d)$

$\Longrightarrow \exists\ S = S(\|u_0\|_{L^\infty(\mathbb{R}^d)})$ s.t. $\forall\ p \in [1,\infty),\ t>0,\ r>0$

$$\boxed{\|u(t,\cdot)\|_{L^p(B_r)} \le \|u_0(\cdot)\|_{L^p(B_{r+St})}}$$

Similar Proof

Existence Proof Via the Method of Vanishing Viscosity

Consider

$$(\maltese)\quad \begin{cases} \partial_t u + \nabla_x \cdot f(u) = \varepsilon \Delta u \\ u|_{t=0} = u_0(x) \in L^\infty \cap L^1(\mathbb{R}^d). \end{cases} \qquad \varepsilon > 0$$

$$\Delta = \sum_{j=1}^{d} \partial_{x_j}^2 \quad \text{Laplace's operator}$$

The Parabolic Theory

$\hookrightarrow$ $\exists$ 1 Global Smooth Solution $u^\varepsilon(t,x)$

Question

$$\underbrace{u^\varepsilon(t,x) \longrightarrow u(t,x) \quad \text{a.e.}}_{\text{Admissible Solution ??}} \quad ??$$

Fact I.
$$\begin{cases} u_0 \in L^\infty \cap L^1 \longrightarrow u^\varepsilon(t,x) \\ v_0 \in L^\infty \cap L^1 \longrightarrow v^\varepsilon(t,x) \end{cases}$$

$\Longrightarrow \quad \forall\, t > 0$

- $$\begin{cases} \int_{\mathbb{R}^d} \left[u^\varepsilon(t,x) - v^\varepsilon(t,x)\right]^+ dx \le \int_{\mathbb{R}^d} \left[u_0(x) - v_0(x)\right]^+ dx \\ \int_{\mathbb{R}^d} \left| u^\varepsilon(t,x) - v^\varepsilon(x,t)\right| dx \le \int_{\mathbb{R}^d} \left| u_0(x) - v_0(x)\right| dx. \end{cases}$$

- If $u_0(x) \le v_0(x)$ a.e $\Longrightarrow$ $u^\varepsilon(t,x) \le v^\varepsilon(t,x)$ a.e.

- $\inf u_0(\cdot) \le u^\varepsilon(t,x) \le \sup u_0(\cdot)$

  $\inf v_0(\cdot) \le v^\varepsilon(t,x) \le \sup v_0(\cdot)$

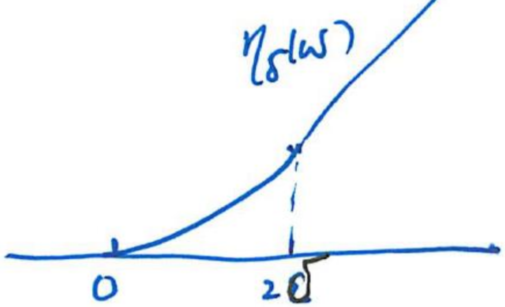
Proof of Fact I To simplify the notation, we drop the subscript ε & denote $u^\varepsilon, v^\varepsilon$ by u, v

① **The Parabolic Theory**: $u_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$

$\hookrightarrow$ $u(t,x),\ D_x^\alpha u(t,x) \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, $t>0$

② For $\delta > 0$

$$\eta_\delta(w) = \begin{cases} 0 & -\infty < w \le 0 \\ \dfrac{w^2}{4\delta} & 0 < w \le 2\delta \\ w - \delta & 2\delta < w < \infty \end{cases}$$



$\hookrightarrow$

$$(\mathrm{V}) \begin{cases} \partial_t \eta_\delta(u-v) + \nabla\cdot\big(\eta_\delta'(u-v)(f(u)-f(v))\big) \\ \qquad - \eta_\delta''(u-v)(f(u)-f(v))\cdot\nabla(u-v) \\ = \varepsilon \Delta \eta_\delta(u-v) - \varepsilon \underbrace{\eta_\delta''(u-v)}_{\ge 0} |\nabla_x(u-v)|^2 \end{cases}$$

③ Fix $0<s<t<\infty$.

$\int_s^t \int_{\mathbb{R}^d} (\mathrm{V})\, dx\, d\tau$

$\hookrightarrow \int_{\mathbb{R}^d} \eta_\delta\big(u(t,x)-v(t,x)\big)dx - \int_{\mathbb{R}^d} \eta_\delta\big(u(s,x)-v(s,x)\big)dx$

$$\leq \int_s^t \int_{\mathbb{R}^d} \underbrace{\eta_\delta''(u-v)\big(f(u)-f(v)\big)\cdot\nabla_x(u-v)}_{\text{Uniformly bdd for fixed } \varepsilon>0}\, dx\, d\tau$$

$\delta \to 0$ pointwise 0

$\longrightarrow 0$

Dominated Convergence Theorem

③ (Conti).

$\hookrightarrow$

$$(\hat{A}) \quad \int_{\mathbb{R}^d} [u(t,x)-v(t,x)]^+ dx \le \int_{\mathbb{R}^d} [u(s,x)-v(s,x)]^+ dx \xrightarrow{s\to 0} \int_{\mathbb{R}^d} [u_0(x)-v_0(x)]^+ dx$$

④ <u>Interchange the role of u and v</u>

$$(\hat{B}) \quad \int_{\mathbb{R}^d} [v(t,x)-u(t,x)]^+ dx \le \int_{\mathbb{R}^d} [v_0(x)-u_0(x)]^+ dx$$

$$\xRightarrow{(\hat{A})+(\hat{B})} \int_{\mathbb{R}^d} |u(t,x)-v(t,x)|\, dx \le \int_{\mathbb{R}^d} |u_0(x)-v_0(x)|\, dx$$

⑤ $u_0(x) \le v_0(x) \xRightarrow{(\hat{A})} u(t,x) \le v(t,x)$ a.e.

Choose $v(t,x) = \sup u_0(x) \xRightarrow{(\hat{A})} u(t,x) \le \sup u_0(x)$. a.e.

Choose $v(t,x) = \inf u_0(x) \xRightarrow{(\hat{B})} u(t,x) \ge \inf u_0(x)$. a.e.

Similarly

$$\inf v_0(x) \le v(t,x) \le \sup v_0(x)$$

$\square$

Fact II

$$\begin{cases} u_0 \in L^\infty \cap L^1(\mathbb{R}^d) \\ \int_{\mathbb{R}^d} |u_0(x+y) - u_0(x)|\, dx \le \omega(|y|), \quad \forall y \in \mathbb{R}^d \end{cases}$$

$\omega(r) \nearrow r \nearrow^{+\infty}_{0}$, $\quad \omega(r) \downarrow 0$ as $r \downarrow 0$

$\Rightarrow$ $\exists\ C = C(\|u_0\|_{L^\infty})$ s.t. $\forall\ t > 0$

$$\begin{cases} \int_{\mathbb{R}^d} |u^\varepsilon(t, x+y) - u^\varepsilon(t,x)|\, dx \le \omega(|y|), \quad y \in \mathbb{R}^d \\ \int_{\mathbb{R}^d} |u^\varepsilon(t+h, x) - u^\varepsilon(t,x)|\, dx \le C\left(h^{\frac{2}{3}} + \varepsilon h^{\frac{1}{3}}\right)\|u_0\|_{L^1(\mathbb{R}^d)} \\ \qquad + 2\omega(h^{\frac{1}{3}}) \qquad h > 0 \end{cases}$$

Anisotropic Mixed Hyperbolic-Parabolic Equations

$$\partial_t u + \nabla_{\mathbf{x}} \cdot \mathbf{f}(u) = \nabla_{\mathbf{x}} \cdot (\mathbf{A}(u)\nabla_{\mathbf{x}} u) + h(u;t,\mathbf{x}), \qquad \mathbf{x} \in \mathbb{R}^d$$

for the unknown function $u\colon \mathbb{R}_+ \times \mathbb{R}^d \longrightarrow \mathbb{R}$, where

- $\mathbf{f} \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^d)$ is the convection flux function
- $\mathbf{A}(u) = (a_{ij}(u)) \geq 0$ is the symmetric diffusion matrix so that

$$\mathbf{A}(u) = \boldsymbol{\alpha}(u)\boldsymbol{\alpha}(u)^\top, \qquad \boldsymbol{\alpha}(u) = (\boldsymbol{\alpha}_1(u), \cdots, \boldsymbol{\alpha}_d(u))$$

  where $\boldsymbol{\alpha}_k(u) = (\alpha_{1k}(u), \ldots, \alpha_{dk}(u))^\top \in L^\infty_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^d),\ k = 1, \ldots, d$
- $h(u;t,\mathbf{x})$ is the source term – forcing, reaction, stochastic noise, $\cdots$

  *Multiplicative white noise: $h(u;t,\mathbf{x}) = \sigma(u)\partial_t W(t)$

Applications: Convection, Diffusion, $\cdots\cdots$

- Viscous-inviscid two phase flows, $\cdots\cdots$
- Sedimentation-consolidation processes, fluids in porous media $\cdots\cdots$

Example: $\partial_t u + \nabla_{\mathbf{x}} \cdot \mathbf{f}(u) = \Delta_{\mathbf{x}} u_+, \qquad u_+ = \max\{u, 0\}$

- $\{u > 0\}$: Viscous phase
- $\{u < 0\}$: Inviscid phase
- $\{u = 0\}$: Free boundary interface separating the two phases

Kinetic Formulation:

$$\mathrm{d}\chi(\xi;u) + \big(\mathbf{f}'(\xi)\cdot\nabla_{\mathbf{x}}\chi(\xi;u) - \nabla\cdot(\mathbf{A}(\xi)\nabla\chi(\xi;u))\big)\mathrm{d}t$$
$$= \delta(\xi-u)\sigma(\xi)\mathrm{d}W + \partial_\xi(m_u+n_u-p_u)(t,\mathbf{x};\xi)\mathrm{d}t$$

in the sense of distributions *a.s.*

Dissipation measure: $m_u(t,\mathbf{x};\xi)\ge 0$

Parabolic defect measure:

$$n_u(t,\mathbf{x};\xi) = \delta(\xi-u(t,\mathbf{x}))|\nabla\cdot\boldsymbol{\beta}(u(t,\mathbf{x}))|^2$$

Ito correction measure: $p_u(t,\mathbf{x};\xi) = \frac{1}{2}\sigma(\xi)\delta(\xi-u)$

Kinetic function: Quasi-Maxwellian χ is the Heavside function on $\mathbb{R}^2$:

$$\chi(\xi;u) = \begin{cases} +1 & \text{for } \xi < u, \\ 0 & \text{for } \xi \ge u. \end{cases}$$

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