Topics on Nonlinear Hyperbolic PDEs

Mathematical Institutes
Hilary Term 2022
February: 9th, 16th, 23rd
March: 2nd
Wednesdays 14:00-16:00

By Professor Gui-Qiang G. Chen

Lecture-3: 23rd February 2022

http://people.maths.ox.ac.uk/chengq/teach/PDECDT2022-NHPDE/CDT-NHPDE.html

References:

1. **R. Courant and D. Hilbert: Methods of Mathematical Physics,** Vol. II. Reprint of the 1962 original. John Wiley&Sons, Inc.: New York, 1989.
2. **C. M. Dafermos: Hyperbolic Conservation Laws in Continuum Physics,** Fourth edition. Springer-Verlag: Berlin, 2016.
3. **L. C. Evans: Partial Differential Equations,** Second edition. AMS: Providence, RI, 2010.
4. **L. Hormander: Lectures on Nonlinear Hyperbolic Differential Equations** Springer-Verlag: Berlin-Heidelberg, 1997.
5. **P. D. Lax: Hyperbolic Differential Equations,** AMS: Providence, 2000.
6. **G.-Q. Chen and M. Feldman: The Mathematics of Shock Reflection-Diffraction and von Neumann's Conjectures.** Annals of Mathematics Studies, 197. *Princeton University Press*: *Princeton, NJ,* 2018.
7. **D. Serre, Systems of Conservation Laws, Vols. I, II,** Cambridge University Press: Cambridge, 1999, 2000.
8. **C. D. Sogge, Lectures on Nonlinear Wave Equations,** Second edition. International Press, Boston, MA, 2008.

Hyperbolicity

$$\partial_t \mathbf{u} + \nabla \cdot \mathbf{f}(\mathbf{u}) = 0, \quad \mathbf{u} \in \mathbb{R}^m, \ \mathbf{x} \in \mathbb{R}^d$$

Plane Wave Solutions: $\mathbf{u}(t, \mathbf{x}) = \mathbf{w}(t, \omega \cdot \mathbf{x})$

$\mathbf{w}(t, \xi)$ is determined by: $\partial_t \mathbf{w} + (\nabla_{\mathbf{w}} \mathbf{f}(\mathbf{w}) \cdot \omega)\, \partial_\xi \mathbf{w} = 0$

?? **Existence** of stable plane wave solutions ??

Hyperbolicity in D: For any $\omega \in S^{d-1}$, $\mathbf{u} \in D$,

$$(\nabla_{\mathbf{u}} \mathbf{f}(\mathbf{u}) \cdot \omega)_{m\times m}\, \mathbf{r}_j(\mathbf{u}, \omega) = \lambda_j(\mathbf{u}, \omega)\, \mathbf{r}_j(\mathbf{u}, \omega), \ 1 \le j \le m$$

$\lambda_j(\mathbf{u}, \omega)$ **are real**

Main Features:

Finiteness of Propagation Speeds;
Discontinuities of Solutions, $\cdots\cdots$
Well-Posedness: Existence, Uniqueness, Stability, $\cdots$

The Simplest Example

Scalar Conservation Laws

$$\partial_t u + \nabla \cdot \mathbf{f}(u) = 0, \quad u \in \mathbb{R}, \ \mathbf{x} \in \mathbb{R}^d$$

$$\mathbf{f} : \mathbb{R} \to \mathbb{R}^d$$

Then

$$\lambda(u, \omega) = \mathbf{f}'(u) \cdot \omega, \qquad r(u, \omega) \equiv 1$$

$\Longrightarrow$ **Scalar conservation laws are always hyperbolic**

Hyperbolic Systems

$$\mathbf{u}_t + \mathbf{f}(\mathbf{u})_x = 0 \qquad\qquad \mathbf{u} = \mathbf{u}(t,x) \in \mathbb{R}^m$$

$$\mathbf{u}_t + \mathbf{A}(\mathbf{u})\mathbf{u}_x = 0 \qquad\qquad \mathbf{A}(\mathbf{u}) = \nabla \mathbf{f}(\mathbf{u})$$

The system is **strictly hyperbolic** if each $m \times m$ matrix $\mathbf{A}(\mathbf{u})$ has real distinct eigenvalues

$$\lambda_1(\mathbf{u}) < \lambda_2(\mathbf{u}) < \cdots < \lambda_m(\mathbf{u})$$

Right eigenvectors $\mathbf{r}_1(\mathbf{u}), \cdots, \mathbf{r}_m(\mathbf{u})$ (column vectors)

Left eigenvectors $\mathbf{l}_1(\mathbf{u}), \cdots, \mathbf{l}_m(\mathbf{u})$ (row vectors)

$$\mathbf{A}\mathbf{r}_i = \lambda_i \mathbf{r}_i \qquad\qquad \mathbf{l}_i \mathbf{A} = \lambda_i \mathbf{l}_i$$

Choose the bases so that

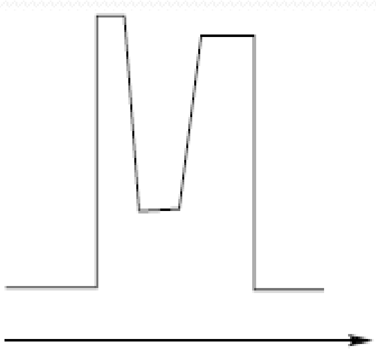
$$\mathbf{l}_i \cdot \mathbf{r}_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Global in Time Solutions to the Cauchy Problem

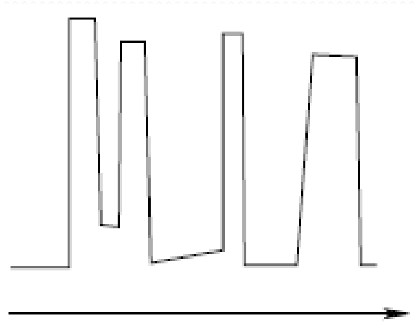
$$\mathbf{u}_t + \mathbf{f}(\mathbf{u})_x = 0, \qquad \mathbf{u}(0,x) = \bar{\mathbf{u}}(x)$$

- Construct a sequence of approximate solutions $\{\mathbf{u}^\nu\}_{\nu \geq 1}$
- Show that (a subsequence) converges: $\mathbf{u}^\nu \to \mathbf{u}$ in L^1_{loc}
- Show that the limit $\mathbf{u}$ is an entropy solution.

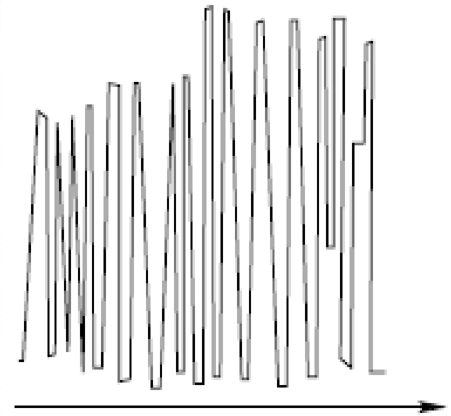
$\mathbf{u}^1$



$\mathbf{u}^2$



$\mathbf{u}^\nu$



Need: a-priori bound on the total variation (J. Glimm, 1965)

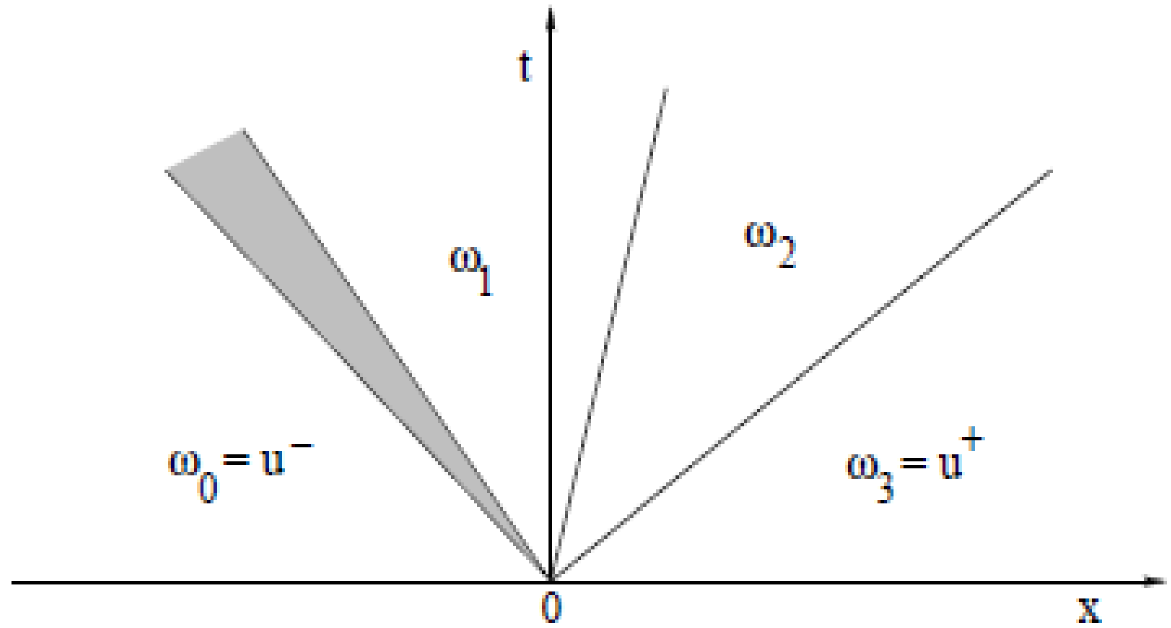
Building Block: The Riemann Problem

$$\mathbf{u}_t + \mathbf{f}(\mathbf{u})_x = 0, \qquad \mathbf{u}(0,x) = \begin{cases} \mathbf{u}^- & x < 0 \\ \mathbf{u}^+ & x > 0 \end{cases}$$

- **B. Riemann 1860:** 2×2 **Isentropic Euler equations**
- **P. Lax 1957:** $m \times m$ **systems (+ special assumptions)**
- **T.-P. Liu 1975:** $m \times m$ **systems (generic case)**

***The Riemann solutions are also the vanishing viscosity limits for general hyperbolic systems, possibly non-conservative**

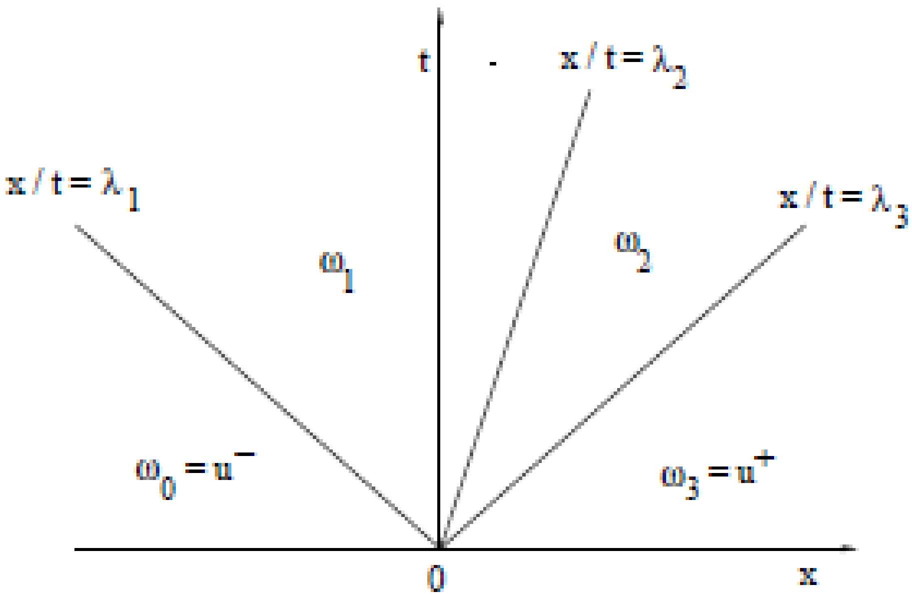
Solution to the Riemann problem



- is invariant w.r.t. rescaling symmetry: $u^\theta(t,x) \doteq u(\theta t, \theta x) \qquad \theta > 0$
- describes local behavior of BV solutions near each point (t_0, x_0)
- describes large-time asymptotics as $t \to +\infty$ (for small total variation)

Riemann Problem for Linear Systems

$$u_t + Au_x = 0 \qquad u(0,x) = \begin{cases} u^- & \text{if} \quad x < 0 \\ u^+ & \text{if} \quad x > 0 \end{cases}$$



$$u^+ - u^- = \sum_{j=1}^{n} c_j r_j \qquad \text{(sum of eigenvectors of } A\text{)}$$

intermediate states : $\omega_i \doteq u^- + \sum_{j \le i} c_j r_j$

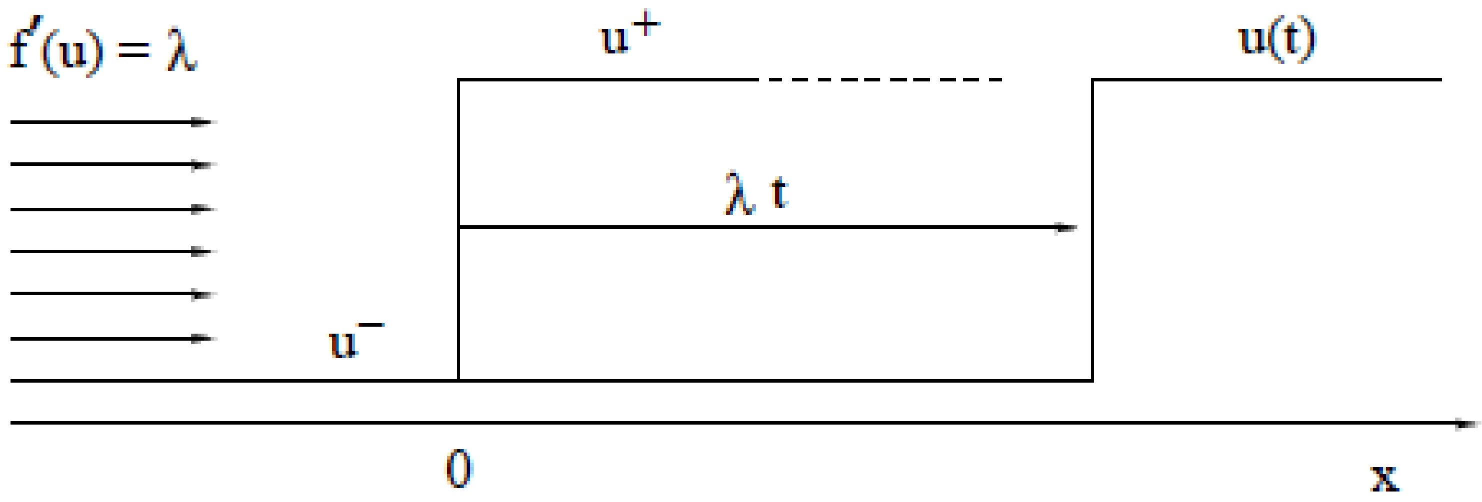
i-th jump: $\omega_i - \omega_{i-1} = c_i r_i$ travels with speed λ_i

Scalar Conservation Law

$$u_t + f(u)_x = 0 \qquad u \in \mathbb{R}$$

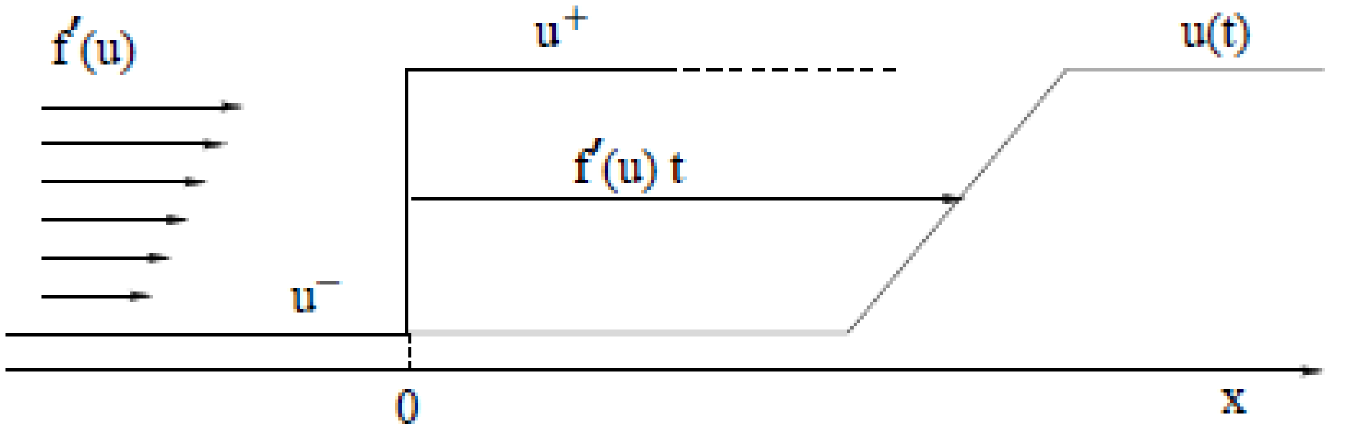
CASE 1: Linear flux: $f(u) = \lambda u$.

Jump travels with speed λ (contact discontinuity)

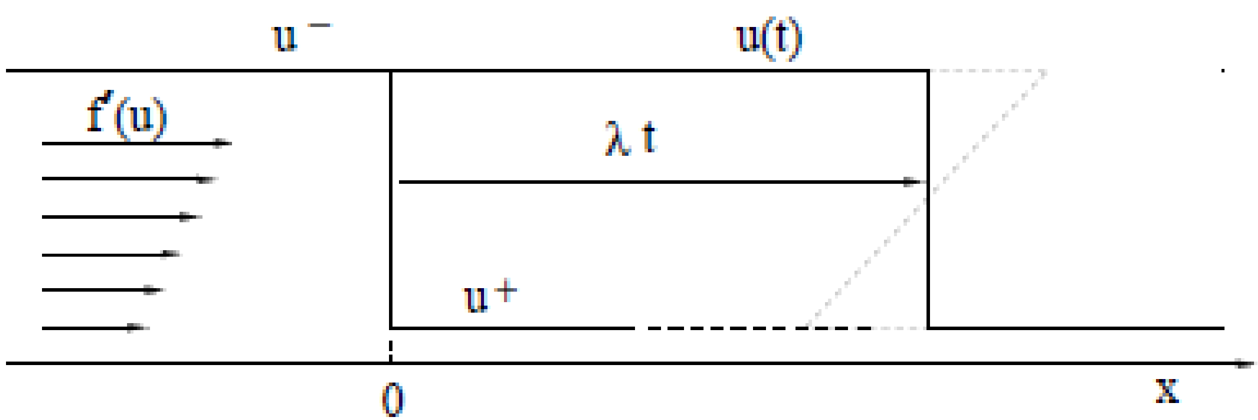


CASE 2: the flux f is convex, so that $u \mapsto f'(u)$ is increasing.

$u^+ > u^- \implies$ centered rarefaction wave



$u^+ < u^- \implies$ stable shock



$$\lambda = \frac{f(u^+) - f(u^-)}{u^+ - u^-}$$

A class of nonlinear hyperbolic systems

$$u_t + f(u)_x = 0$$

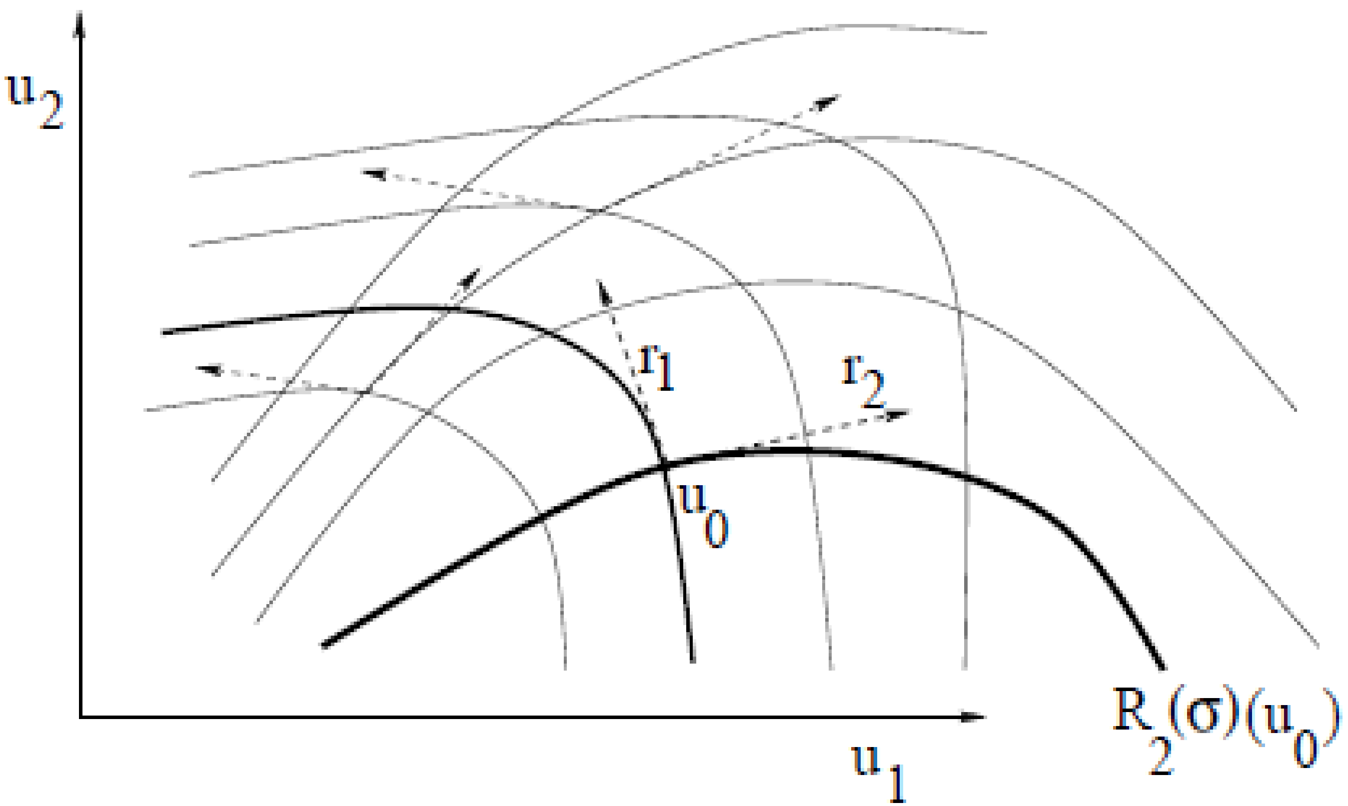
$$A(u) = Df(u) \qquad\qquad A(u)r_i(u) = \lambda_i(u)r_i(u)$$

Assumption (H) (P.Lax, 1957): Each i-th characteristic field is

- either **genuinely nonlinear**, so that $\nabla\lambda_i \cdot r_i > 0$ for all u
- or **linearly degenerate**, so that $\nabla\lambda_i \cdot r_i = 0$ for all u

genuinely nonlinear $\implies$ characteristic speed $\lambda_i(u)$ is strictly increasing along integral curves of the eigenvectors r_i

linearly degenerate $\implies$ characteristic speed $\lambda_i(u)$ is constant along integral curves of the eigenvectors r_i



Shock and Rarefaction curves

$$u_t + f(u)_x = 0 \qquad\qquad A(u) = Df(u)$$

i-rarefaction curve through u_0: $\quad \sigma \mapsto R_i(\sigma)(u_0)$

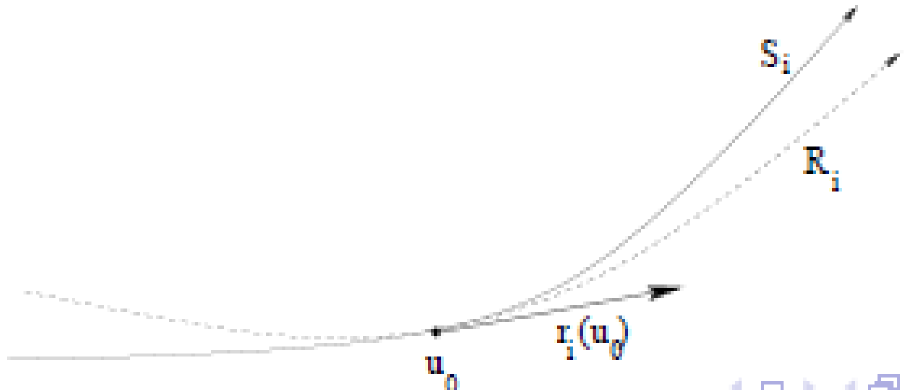
$=$ integral curve of the field of eigenvectors r_i through u_0

$$\frac{du}{d\sigma} = r_i(u), \qquad\qquad u(0) = u_0$$

i-shock curve through u_0: $\quad \sigma \mapsto S_i(\sigma)(u_0)$

$=$ set of points u connected to u_0 by an i-shock, so that

$u - u_0$ is an i-eigenvector of the averaged matrix $A(u, u_0)$



Elementary waves

$$u_t + f(u)_x = 0 \qquad\qquad u(0,x) = \begin{cases} u^- & \text{if} \quad x < 0 \\ u^+ & \text{if} \quad x > 0 \end{cases}$$

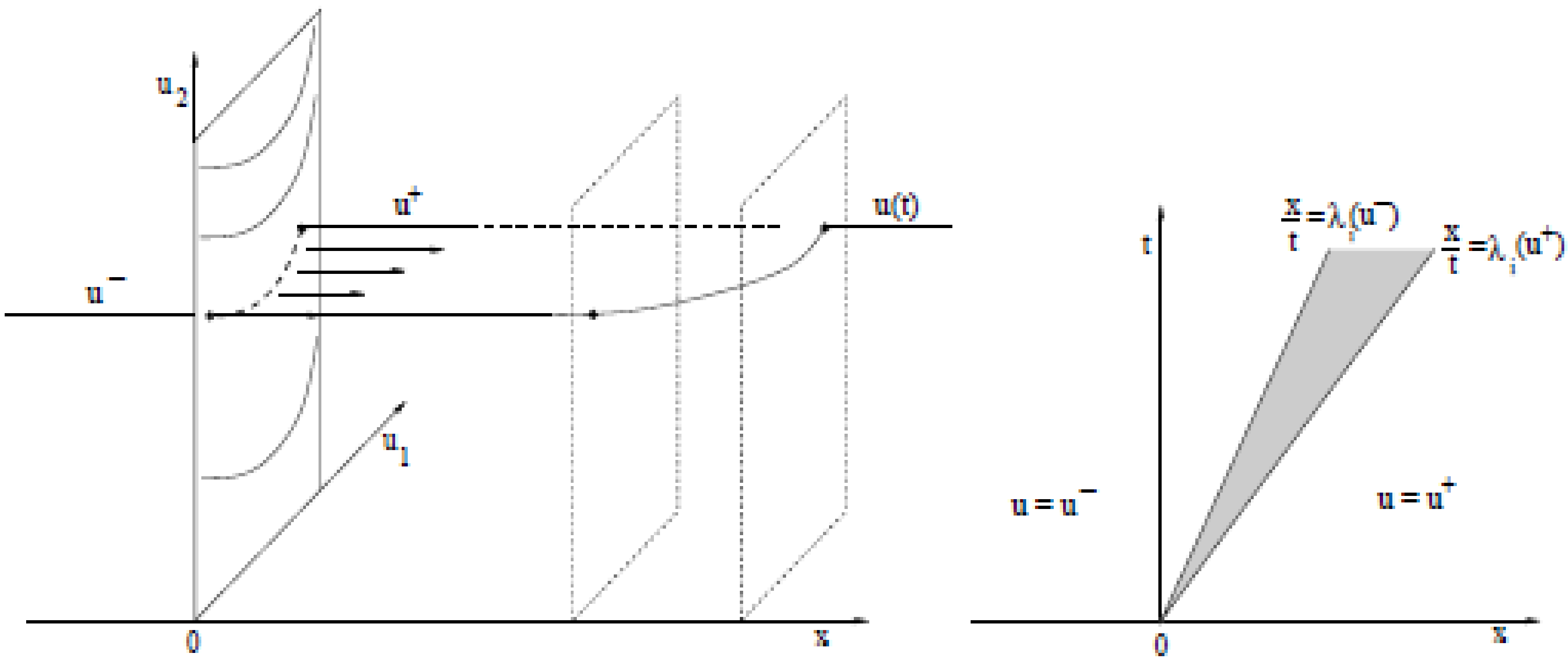
CASE 1 **(Centered rarefaction wave).** Let the i-th field be genuinely nonlinear.

If $u^+ = R_i(\sigma)(u^-)$ for some $\sigma > 0$, then

$$u(t,x) = \begin{cases} u^- & \text{if} \quad x < t\lambda_i(u^-), \\ R_i(s)(u^-) & \text{if} \quad x = t\lambda_i(s) \quad s \in [0,\sigma] \\ u^+ & \text{if} \quad x > t\lambda_i(u^+) \end{cases}$$

is a weak solution of the Riemann problem

A centered rarefaction wave



CASE 2 (**Shock or contact discontinuity**). Assume that

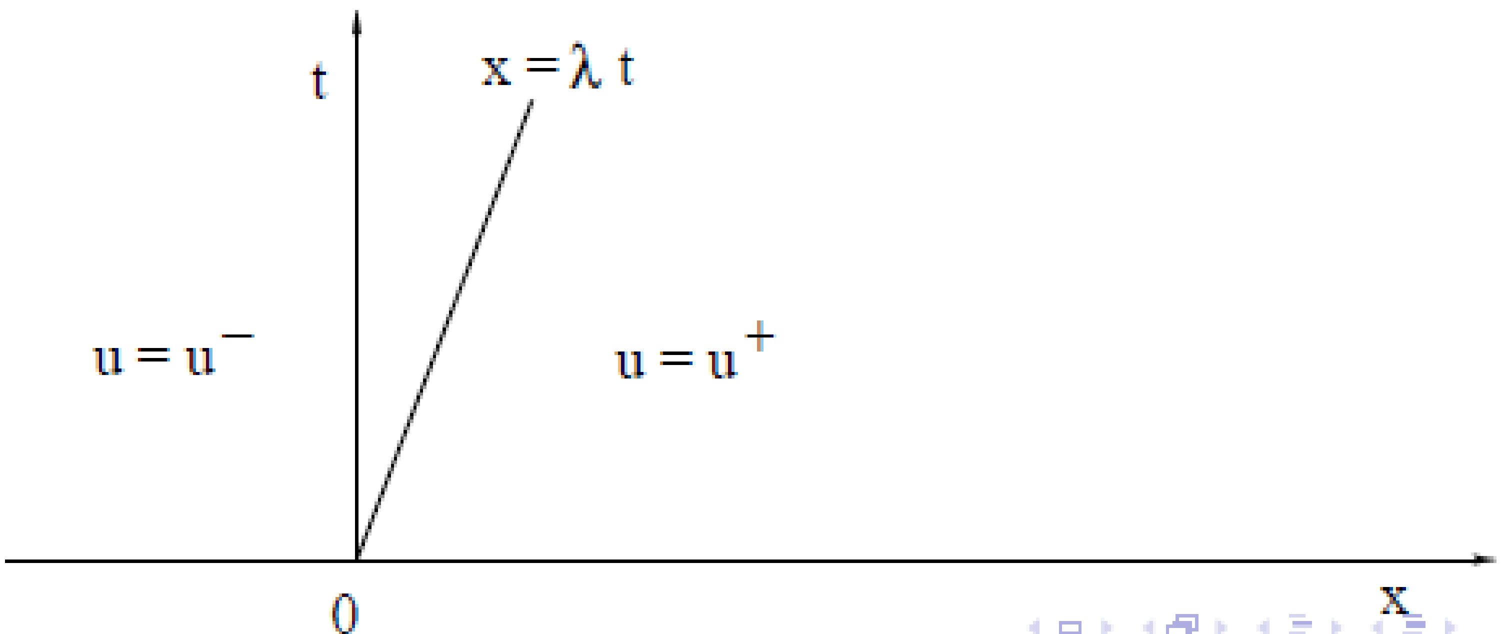
$u^+ = S_i(\sigma)(u^-)$ for some i, σ. Let $\lambda = \lambda_i(u^-, u^+)$ be the shock speed.

Then the function

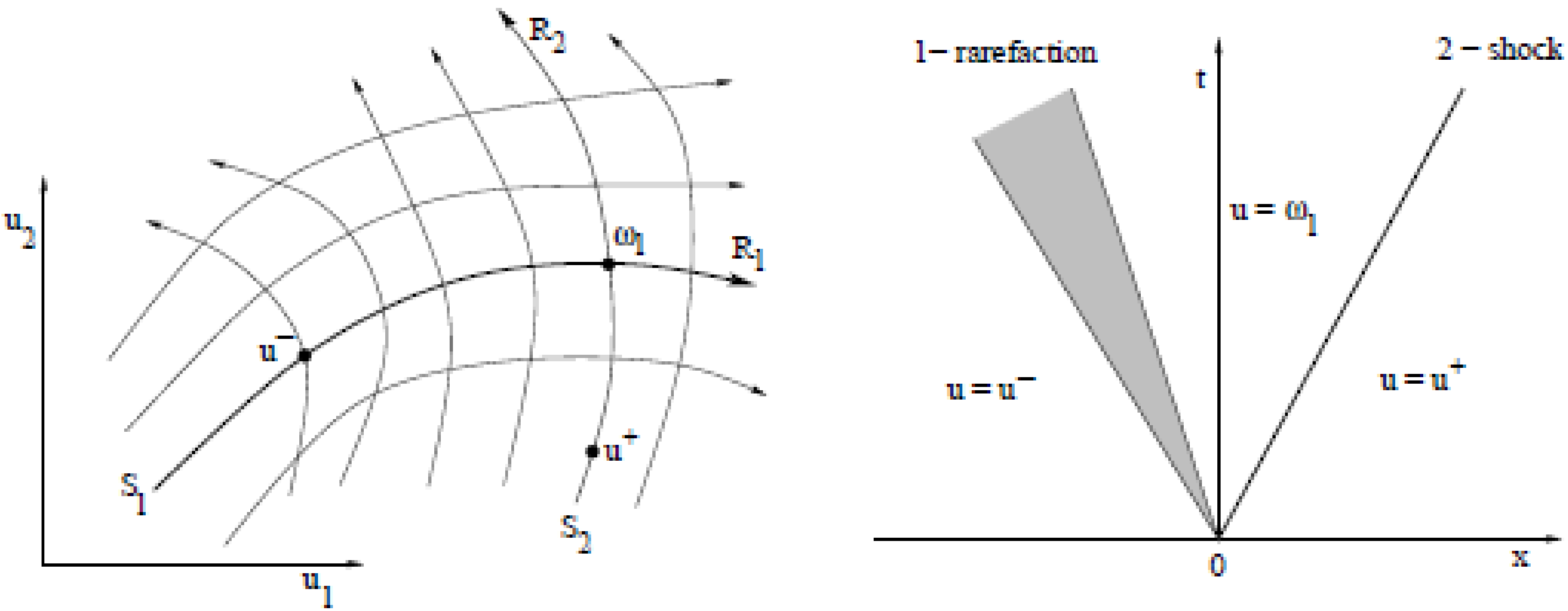
$$u(t,x) = \begin{cases} u^- & \text{if} \quad x < \lambda t, \\ u^+ & \text{if} \quad x > \lambda t, \end{cases}$$

is a weak solution to the Riemann problem.

In the genuinely nonlinear case, this shock is admissible (i.e., it satisfies the Lax condition and the Liu condition) iff $\sigma < 0$.



Solution to a 2 x 2 Riemann problem



Solution of the general Riemann problem (P. Lax, 1957)

$$u_t + f(u)_x \;=\; 0 \qquad\qquad u(0,x) \;=\; \begin{cases} u^- & \text{if} \quad x < 0 \\ u^+ & \text{if} \quad x > 0 \end{cases}$$

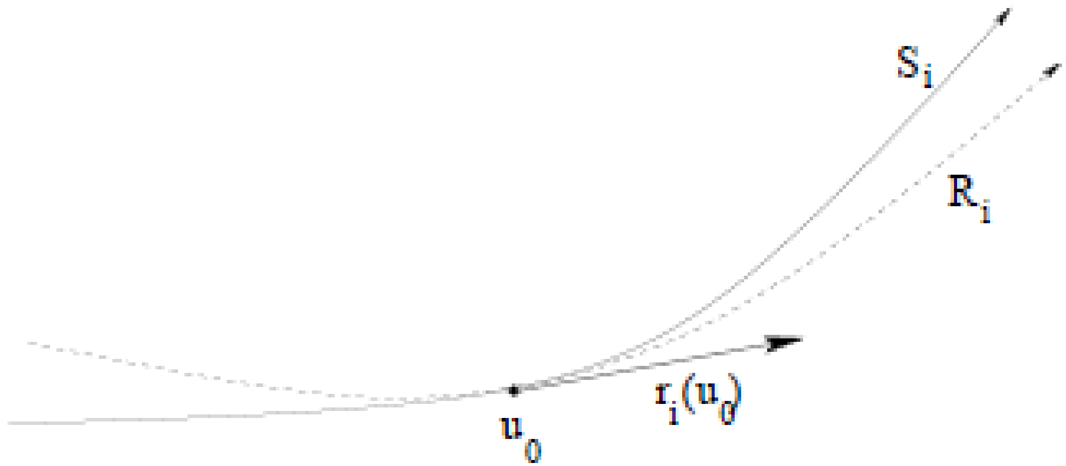
Problem: Find states $\omega_0, \omega_1, \cdots, \omega_m$ such that

$$\omega_0 = \mathbf{u}^- \qquad \omega_m = \mathbf{u}^+$$

and every couple ω_{i-1}, ω_i are connected by an elementary wave (shock or rarefaction)

$$\begin{cases} \text{either } \; \omega_i \;=\; R_i(\sigma_i)(\omega_{i-1}) & \sigma_i \geq 0 \\ \\ \text{or } \; \omega_i \;=\; S_i(\sigma_i)(\omega_{i-1}) & \sigma_i < 0 \end{cases}$$

define: $\Psi_i(\sigma)(u) = \begin{cases} R_i(\sigma)(u) & \text{if } \sigma \geq 0 \\ S_i(\sigma)(u) & \text{if } \sigma < 0 \end{cases}$



$$(\sigma_1, \sigma_2, \dots, \sigma_n) \mapsto \Psi_n(\sigma_n) \circ \cdots \circ \Psi_2(\sigma_2) \circ \Psi_1(\sigma_1)(u^-)$$

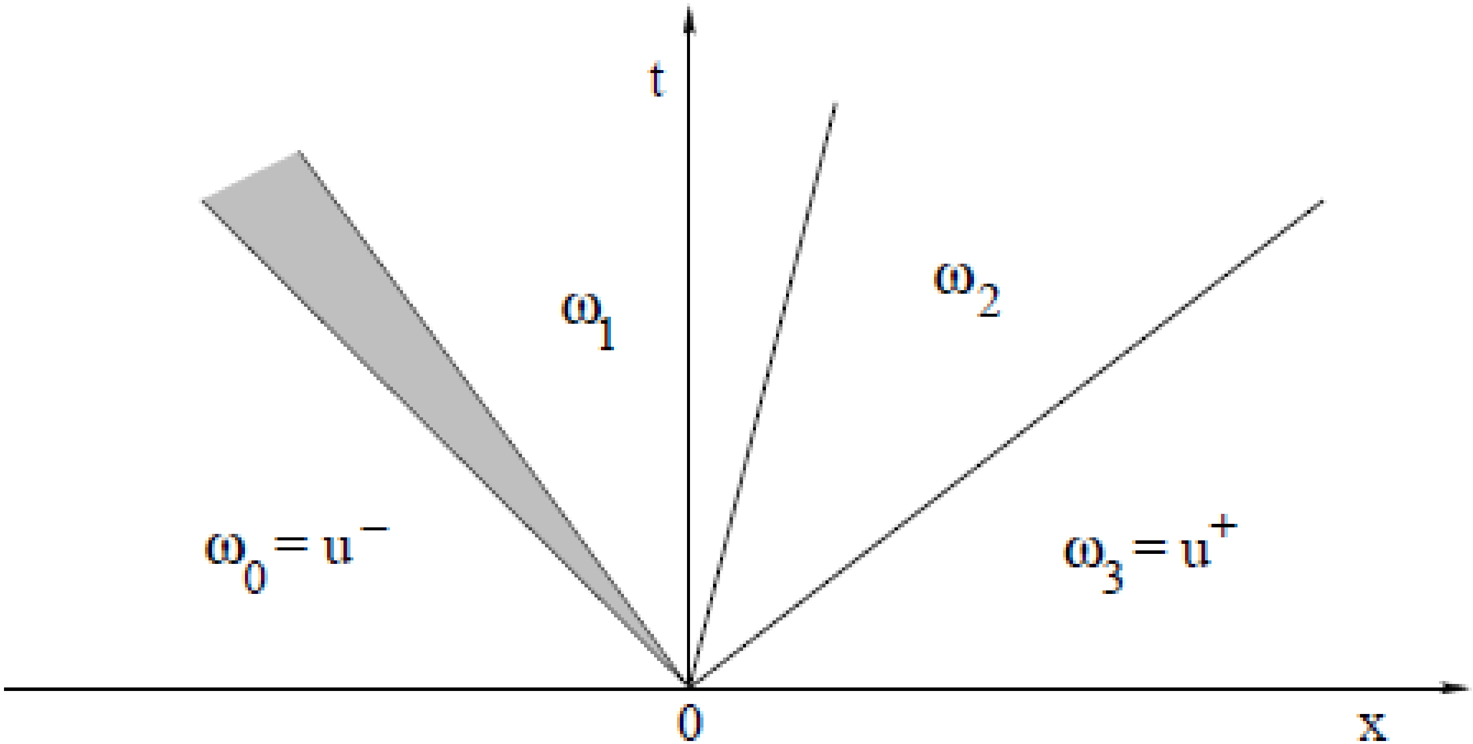
Jacobian matrix at the origin: $J \doteq \left(r_1(u^-) \middle| r_2(u^-) \middle| \cdots \middle| r_n(u^-) \right)$

always has full rank

If $|u^+ - u^-|$ is small, then the implicit function theorem yields existence and uniqueness of the intermediate states $\omega_0, \omega_1, \dots, \omega_n$

General solution of the Riemann problem

Concatenation of elementary waves (shocks, rarefactions, or contact discontinuities)



Global solution to the Cauchy problem

$$u_t + f(u)_x = 0, \qquad u(0,x) = \bar{u}(x)$$

Theorem (Glimm, 1965).

Assume:

- *system is strictly hyperbolic*
- *each characteristic field is either linearly degenerate or genuinely nonlinear*

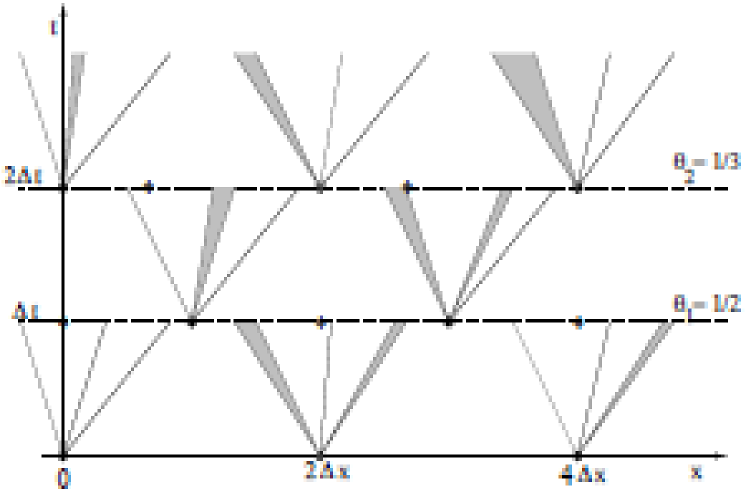
Then there exists a constant $\delta > 0$ such that, for every initial condition $\bar{u} \in \mathbf{L}^1(\mathbb{R};\ \mathbb{R}^n)$ with

$$Tot.Var.(\bar{u}) \leq \delta,$$

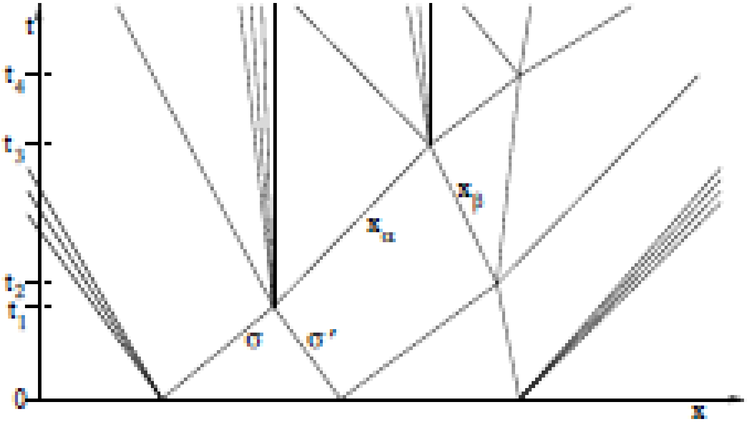
the Cauchy problem has an entropy admissible weak solution $u = u(t,x)$ defined for all $t \geq 0$.

Construction of a sequence of approximate solutions by piecing together solutions of Riemann problems

- on a fixed grid in t-x plane (Glimm scheme)

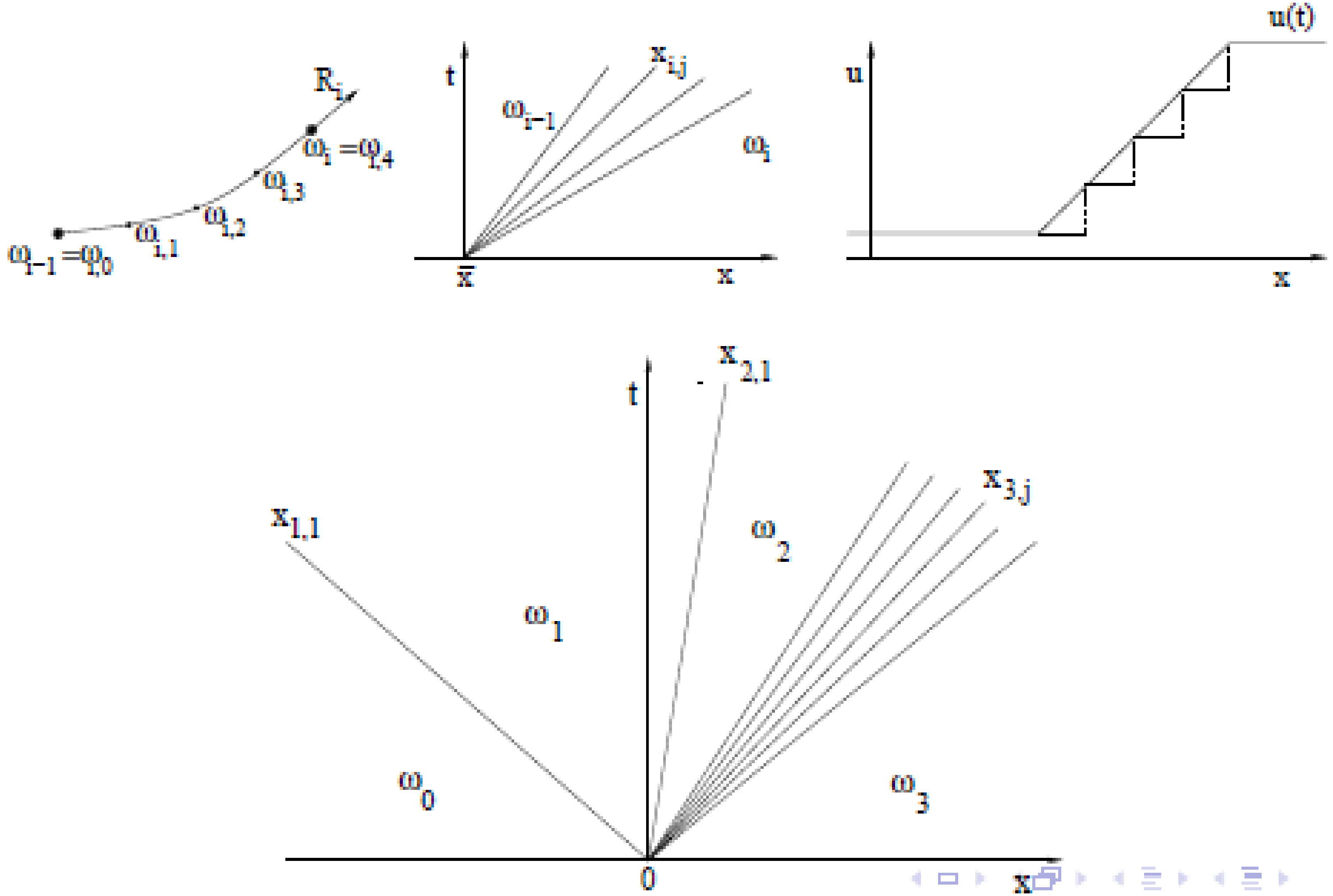


- at points where fronts interact (front tracking)

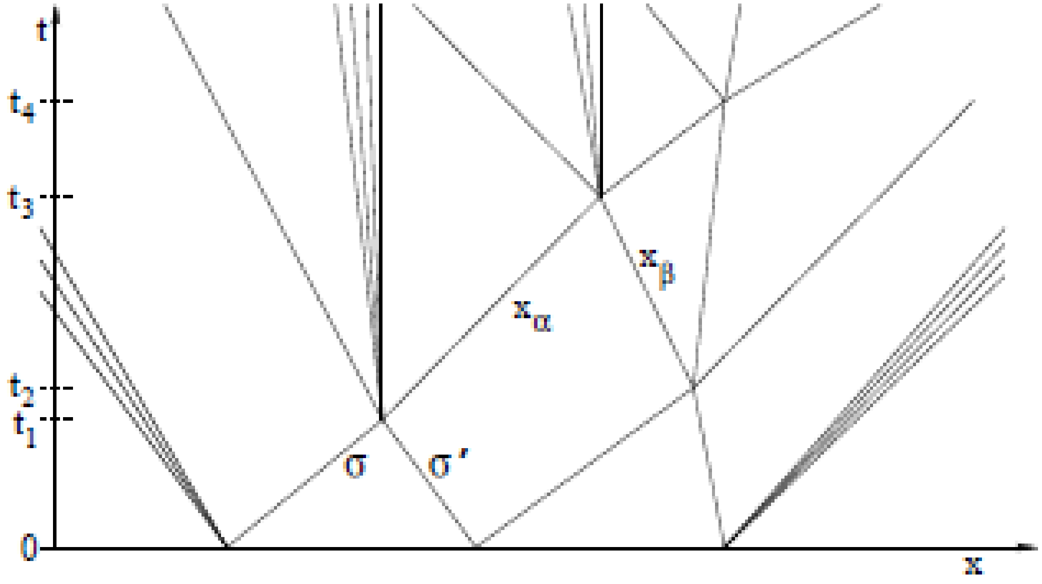


Piecewise constant approximate solution to a Riemann problem

replace centered rarefaction waves with piecewise constant rarefaction fans



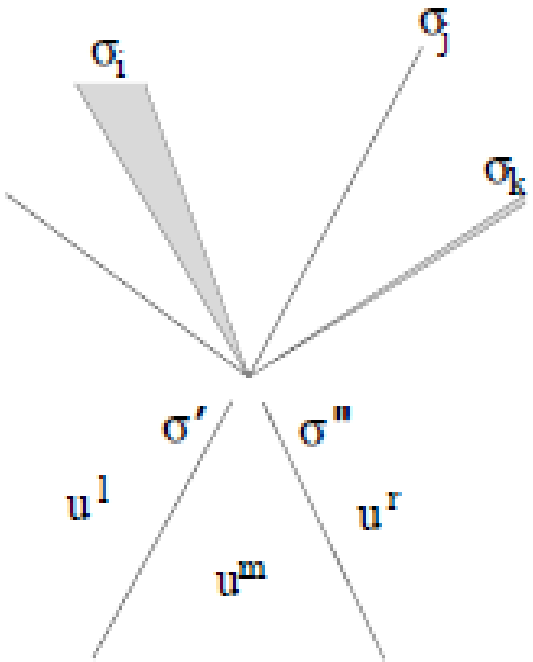
Front Tracking Approximations



- Approximate the initial data $\bar{u}$ with a piecewise constant function
- Construct a piecewise constant approximate solution to each Riemann problem at $t = 0$
- at each time t_j where two fronts interact, construct a piecewise constant approximate solution to the new Riemann problem ...
- NEED TO CHECK: $\begin{cases} \text{- total variation remains small} \\ \text{- number of wave fronts remains finite} \end{cases}$

Interaction estimates

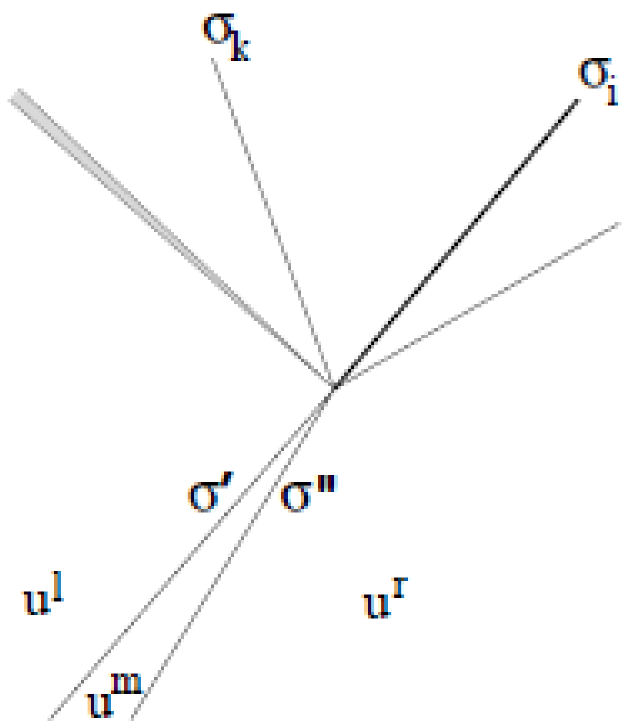
GOAL: estimate the strengths of the waves in the solution of a Riemann problem, depending on the strengths of the two interacting waves σ', σ''



Incoming: a j-wave of strength σ' and an i-wave of strength σ''

Outgoing: waves of strengths $\sigma_1, \cdots, \sigma_m$. Then

$$|\sigma_i - \sigma''| + |\sigma_j - \sigma'| + \sum_{k \neq i,j} |\sigma_k| = O(1) \cdot |\sigma' \sigma''|$$



Incoming: two i-waves of strengths σ' and σ''

Outgoing: waves of strengths $\sigma_1, \cdots, \sigma_m$. Then

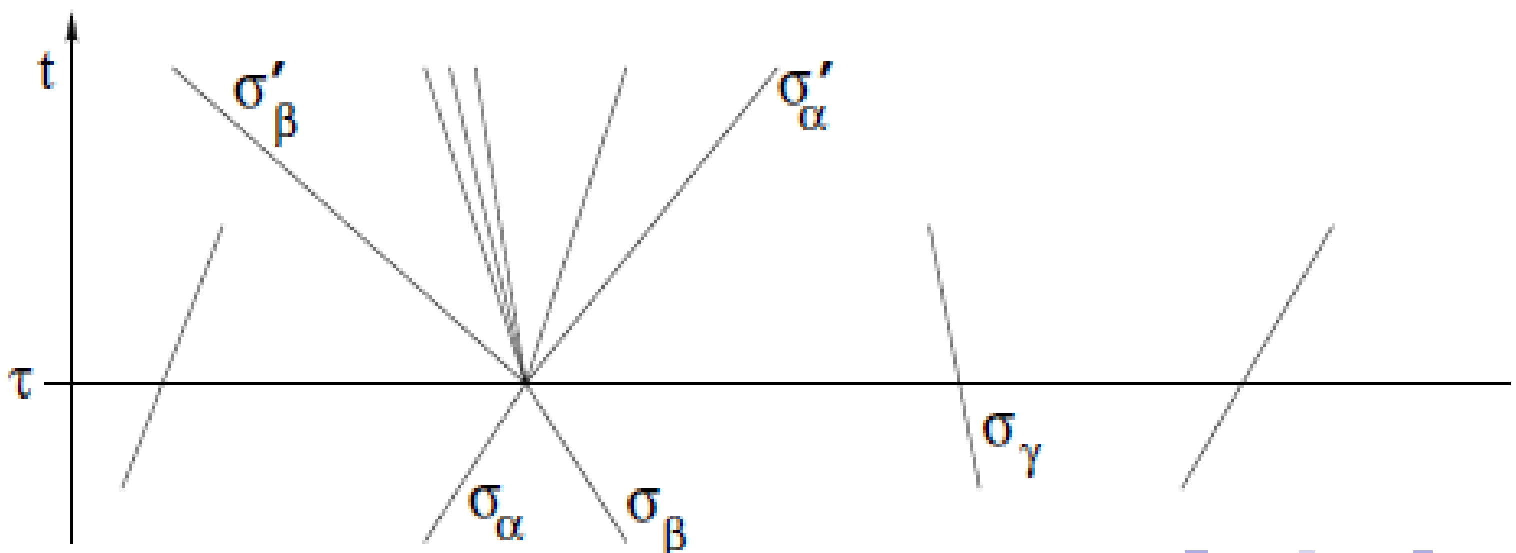
$$|\sigma_i - \sigma' - \sigma''| + \sum_{k \neq i} |\sigma_k| \ = \ \mathcal{O}(1) \cdot |\sigma' \sigma''| \Big(|\sigma'| + |\sigma''| \Big)$$

Glimm functionals

Total strength of waves: $V(t) \doteq \sum_{\alpha} |\sigma_\alpha|$

Wave interaction potential: $Q(t) \doteq \sum_{(\alpha,\beta)\in\mathcal{A}} |\sigma_\alpha \sigma_\beta|$

$\mathcal{A} \doteq$ couples of *approaching* wave fronts



Changes in V, Q at time τ when the fronts $\sigma_\alpha, \sigma_\beta$ interact:

$$\Delta V(\tau) \;=\; \mathcal{O}(1)|\sigma_\alpha \sigma_\beta|$$

$$\Delta Q(\tau) \;=\; -|\sigma_\alpha \sigma_\beta| + \mathcal{O}(1) \cdot V(\tau-)|\sigma_\alpha \sigma_\beta|$$

Choosing a constant C_0 large enough, the map

$$t \mapsto V(t) + C_0 Q(t)$$

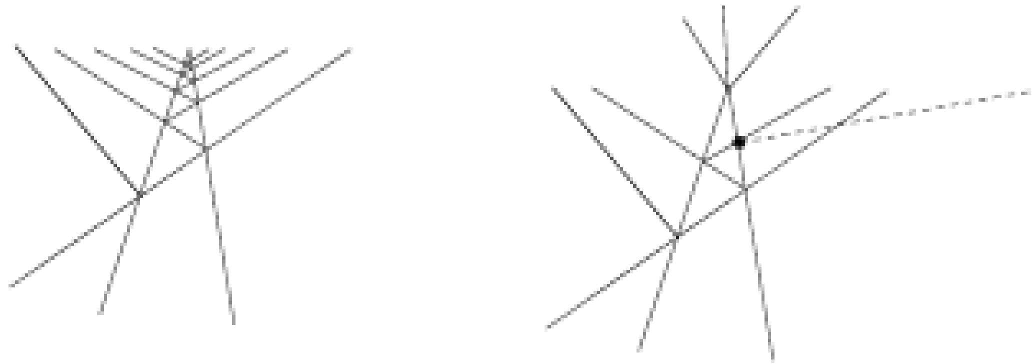
is nonincreasing, as long as V remains small

Total variation initially small $\quad\Longrightarrow\quad$ global BV bounds

$$\text{Tot.Var.}\{u(t,\cdot)\} \;\leq\; V(t) \;\leq\; V(0) + C_0 Q(0)$$

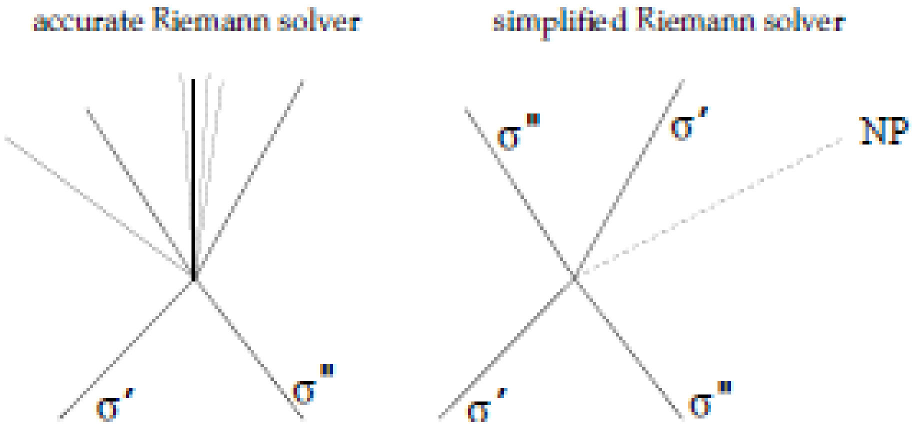
Front tracking approximations can be constructed for all $t \geq 0$

Keeping finite the number of wave fronts



At each interaction point, the **Accurate Riemann Solver** yields a solution, possibly introducing several new fronts

The total number of fronts can become infinite in finite time



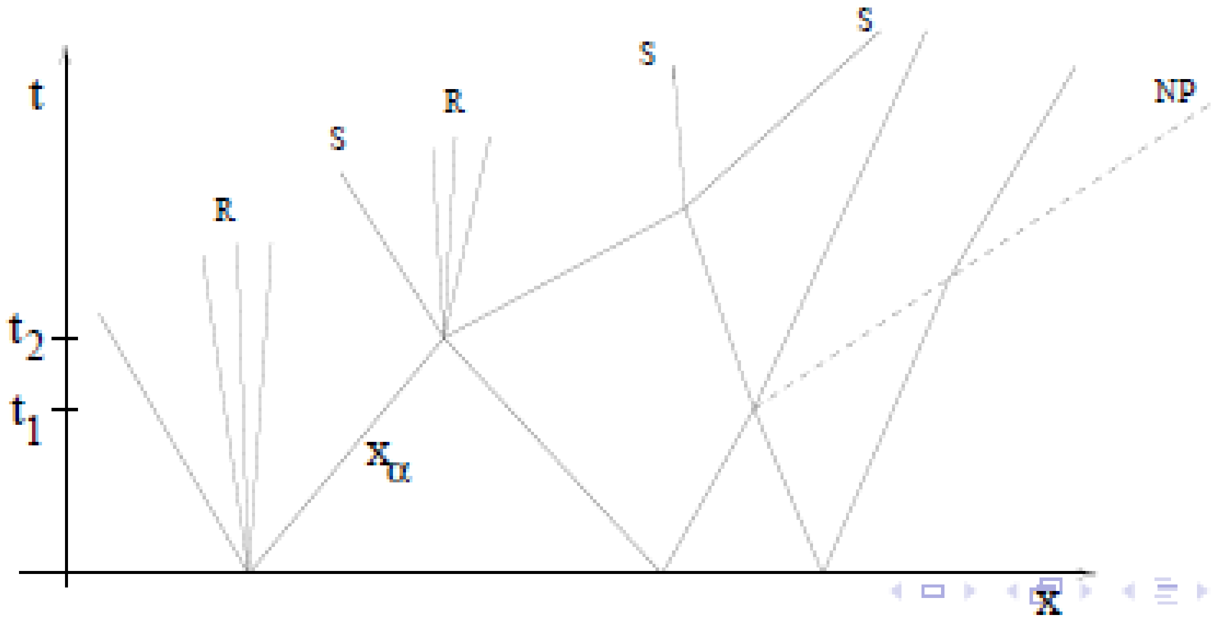
Need: a **Simplified Riemann Solver**, producing only one *"non-physical"* front

A sequence of approximate solutions

$$u_t + f(u)_x = 0 \qquad u(0,x) = \bar{u}(x)$$

$(u_\nu)_{\nu \geq 1}$ sequence of approximate front tracking solutions

- initial data satisfy $\|u_\nu(0,\cdot) - \bar{u}\|_{\mathbf{L}^1} \leq \varepsilon_\nu \to 0$
- all shock fronts in u_ν are entropy-admissible
- each rarefaction front in u_ν has strength $\leq \varepsilon_\nu$
- at each time $t \geq 0$, the total strength of all non-physical fronts in $u_\nu(t,\cdot)$ is $\leq \varepsilon_\nu$



Existence of a convergent subsequence

$$\text{Tot.Var.}\{u_\nu(t,\cdot)\} \leq C$$

$$\begin{aligned} \big\|u_\nu(t) - u_\nu(s)\big\|_{\mathbf{L}^1} &\leq (t-s)\cdot [\text{total strength of all wave fronts}]\cdot[\text{maximum speed}] \\ &\leq L\cdot(t-s) \end{aligned}$$

Helly's compactness theorem $\Longrightarrow$ a subsequence converges

$$u_\nu \to u \qquad\qquad \text{in } \mathbf{L}^1_{loc}$$

Claim: $u = \lim_{\nu \to \infty} u_\nu$ is a weak solution

$$\iint \left\{ \phi_t u + \phi_x f(u) \right\} dxdt \; = \; 0 \qquad \phi \in \mathcal{C}^1_c\Big( ]0, \infty[\times \mathbb{R} \Big)$$

Need to show:

$$\lim_{\nu \to \infty} \iint \left\{ \phi_t u_\nu + \phi_x f(u_\nu) \right\} dxdt \; = \; 0$$

$$
\begin{aligned}
&\int_0^{\infty}\int_{-\infty}^{\infty}\left\{\phi_t(t,x)u_\nu(t,x)+\phi_x(t,x)f\left(u_\nu(t,x)\right)\right\}\,dxdt \\
&\quad = \sum_j \int_{\partial\Gamma_j} \Phi_\nu \cdot \mathbf{n}\, d\sigma
\end{aligned}
$$

$$
\begin{aligned}
&\limsup_{\nu\to\infty}\left|\sum_j \int_{\partial\Gamma_j} \Phi_\nu \cdot \mathbf{n}\, d\sigma\right| \\
&\le \limsup_{\nu\to\infty}\left|\sum_{\alpha\in\mathcal{S}\cup\mathcal{R}\cup\mathcal{NP}}\left[\dot{x}_\alpha(t)\cdot\Delta u_\nu(t,x_\alpha)-\Delta f\left(u_\nu(t,x_\alpha)\right)\right]\phi(t,x_\alpha(t))\right| \\
&\le \left(\max_{t,x}\left|\phi(t,x)\right|\right)\cdot\limsup_{\nu\to\infty}\left\{\mathcal{O}(1)\cdot\sum_{\alpha\in\mathcal{R}}\varepsilon_\nu|\sigma_\alpha| \;+\; \mathcal{O}(1)\cdot\sum_{\alpha\in\mathcal{NP}}|\sigma_\alpha|\right\} \\
&= 0
\end{aligned}
$$

The Glimm scheme

$$u_t + f(u)_x = 0 \qquad\qquad u(0,x) = \bar{u}(x)$$

Assume: all characteristic speeds satisfy $\lambda_i(u) \in [0,1]$

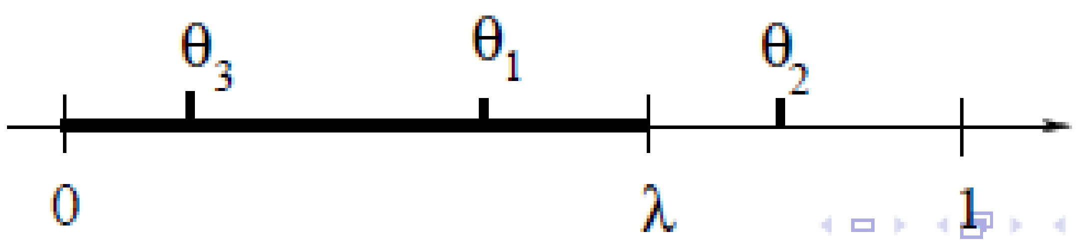
This is not restrictive. If $\lambda_i(u) \in [-M, M]$, simply change coordinates:

$$y = x + Mt, \qquad\qquad \tau = 2Mt$$

Choose:

- a grid in the t-x plane with step size $\Delta t = \Delta x$
- a sequence of numbers $\theta_1, \theta_2, \theta_3, \ldots$ *uniformly distributed* over $[0,1]$

$$\lim_{N\to\infty} \frac{\#\{j\,;\ \ 1 \le j \le N,\ \ \theta_j \in [0,\lambda]\ \}}{N} = \lambda \qquad\qquad \text{for each } \lambda \in [0,1].$$

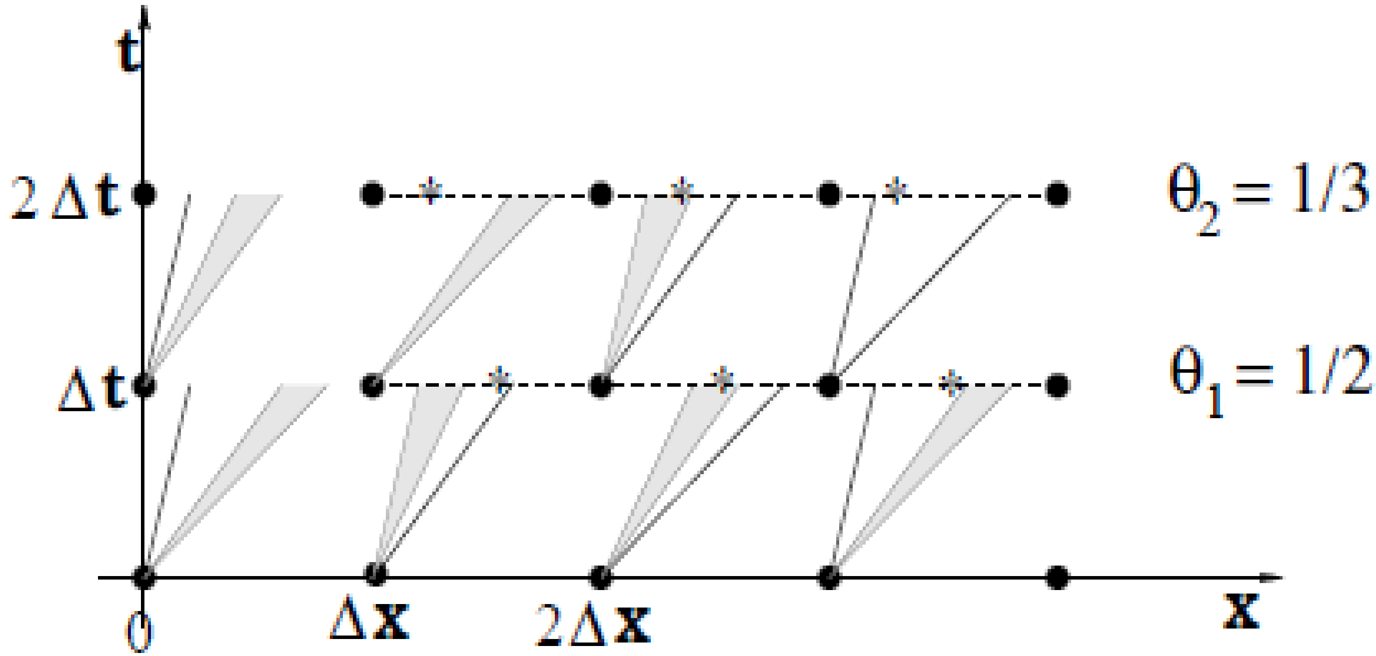


Glimm approximations

Grid points : $x_j = j \cdot \Delta x, \quad t_k = k \cdot \Delta t$

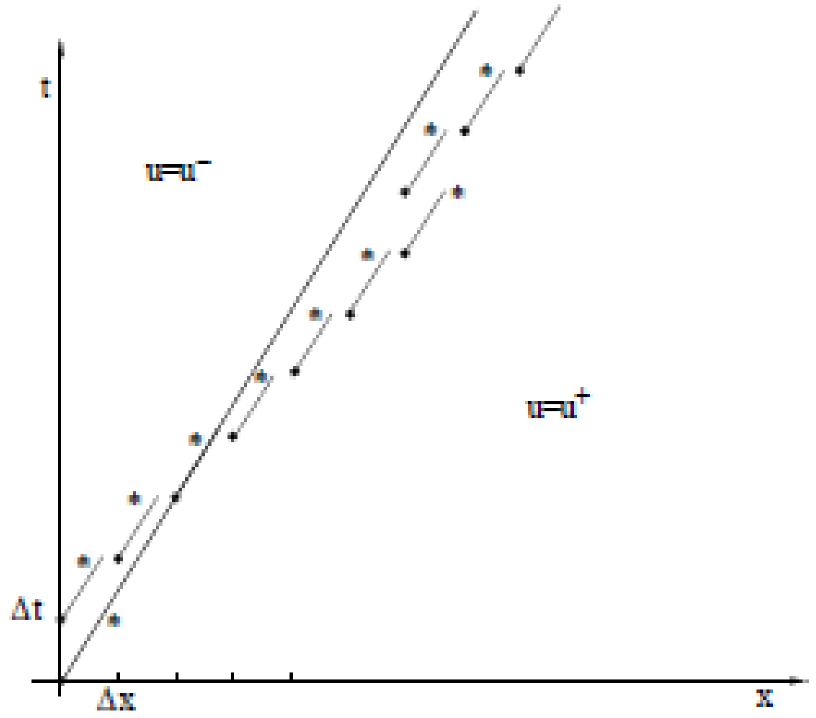
• for each $k \geq 0$, $u(t_k, \cdot)$ is piecewise constant, with jumps at the points x_j. The Riemann problems are solved exactly, for $t_k \leq t < t_{k+1}$

• at time t_{k+1} the solution is again approximated by a piecewise constant function, by a sampling technique



Example: Glimm's scheme applied to a solution containing a single shock

$$U(t,x) = \begin{cases} u^+ & \text{if} \quad x > \lambda t \\ u^- & \text{if} \quad x < \lambda t \end{cases}$$



Fix $T > 0$, take $\Delta x = \Delta t = T/N$

$$x(T) = \#\{j\ ;\ 1 \le j \le N,\ \ \theta_j \in [0,\lambda]\ \} \cdot \Delta t$$

$$= \frac{\#\{j\ ;\ 1 \le j \le N,\ \ \theta_j \in [0,\lambda]\ \}}{N} \cdot T \ \to\ \lambda T \qquad \text{as } N \to \infty$$

Rate of convergence for Glimm approximations

Random sampling at points determined by the equidistributed sequence $(\theta_k)_{k\geq 1}$

$$\lim_{N\to\infty} \frac{\#\{j\ ;\ \ 1 \leq j \leq N,\ \ \theta_j \in [0,\lambda]\ \}}{N} = \lambda \qquad \text{for each } \lambda \in [0,1].$$

Need fast convergence to uniform distribution. Achieved by choosing:

$$\theta_1 = 0.1\,, \quad \ldots \quad , \quad \theta_{759} = 0.957\,, \quad \ldots \quad , \quad \theta_{39022} = 0.22093\,, \quad \ldots$$

Convergence rate: $$\lim_{\Delta x\to 0} \frac{\left\| u^{\text{Glimm}}(T,\cdot) - u^{\text{exact}}(T,\cdot)\right\|_{\mathbf{L}^1}}{\sqrt{\Delta x}\cdot |\ln \Delta x|} = 0$$

(A.Bressan & A.Marson, 1998)

Bressan, A.: Hyperbolic Systems of Conservation Laws. The One-Dimensional Cauchy Problem. Oxford University Press: Oxford, 2000.

Dafermos, C: Hyperbolic Conservation Laws in Continuum Physics, 4rd Edition, Springer-Verlag: Berlin, 2016.

Functional Analytic Approaches for the Existence Theory:

- Compensated Compactness
- Weak Convergence Methods
- Geometric Measure Arguments
- ……

1. **C. M. Dafermos:** **Hyperbolic Conservation Laws in Continuum Physics,** Third edition. Springer-Verlag: Berlin, 2010.
2. **B. Dacorogna**: **Weak Continuity and Weak Lower Semicontinuity of Nonlinear Functionals**, Lecture Notes in Mathematics, Vol. 922, 1-120, Springer-Verlag, 1982.
3. **L. C. Evans**: **Weak Convergence Methods for Nonlinear Partial Differential Equations**. CBMS-RCSM, 74. AMS: Providence, RI, 1990
4. **D. Serre,** La compacité par compensation pour les systèmes hyperboliques non linéaires de deux équations à une dimension d'espace. *J. Math. Pures Appl. (9)* 65 (1986), 423–468.
5. **The references cited therein, especially more recent references.**

Young Measures

$K \subset \mathbb{R}^m$, $\Omega \subset \mathbb{R}^n$ bounded open

$u^k : \Omega \longrightarrow \mathbb{R}^m$ measurable

$u^k(y) \in K$, a.e.

$\Longrightarrow \exists \{\nu_y \in Prob.(\mathbb{R}^m)\}_{y \in \Omega}$ s.t.

- $\boxed{\operatorname{Supp} \nu_y \subset \overline{K}}$ $\forall y \in \Omega$
- $\forall f \in C(\mathbb{R}^m; \mathbb{R})$, $\exists \{u^{k_j}\}_{j=1}^{\infty} \subset \{u^k\}$.

$$\boxed{\begin{aligned} w^*\text{-}\lim f(u^{k_j}) &= \langle \nu_y(\lambda), f(\lambda) \rangle \\ &= \int f(\lambda)\, d\nu_y(\lambda) \end{aligned}}$$

- $u^{k_j} \longrightarrow u$ a.e. $\Longleftrightarrow$ $\nu_y(\lambda) = \delta_{u(y)}(\lambda)$

Dirac mass

* This theorem can be extended to more general cases.

Remarks

1. The deviation between the weak and strong convergence is measured by the spreading of the support of ν_y.

$$\boxed{\begin{aligned}&\|f(w^*\text{-}\lim u^k) - w^*\text{-}\lim f(u^k)\|_{L^\infty}\\ &\le C \sup_y (\operatorname{diam}(\operatorname{supp} \nu_y))\end{aligned}}$$

for $f \in Lip(\mathbb{R}^m; \mathbb{R})$

$$\|f(w^*\text{-}\lim u^k) - w^*\text{-}\lim f(u^k)\|_{L^\infty}$$

$$= \|f(\langle \nu_y, \lambda \rangle) - \langle \nu_y, f(\lambda) \rangle\|_{L^\infty}$$

$$= \|\langle \nu_y, f(\lambda) - f(\langle \nu_y, \lambda \rangle) \rangle\|_{L^\infty}$$

$$\le C \|\langle \nu_y, |\lambda - \langle \nu_y, \lambda \rangle| \rangle\|_{L^\infty}$$

$$\le C \sup_y (\operatorname{diam}(\operatorname{supp} \nu_y)).$$

2. The Young measure family $\{\nu_y\}_{y\in\Omega}$ can be thought of as the limiting probability distribution of the values of $\{u^k(y)\}$ near the point y as $k\to\infty$.

$\Omega\subset\mathbb{R}^n$, $y\in\Omega$.

$\exists\ \delta_0>0$ s.t. $B(y,\delta)\subset\Omega$, $0<\delta\le\delta_0$

Define

$$\langle \nu^k_{y,\delta}, \phi\rangle = \frac{1}{|B(y,\delta)|}\phi(u^k(x))\,dx$$

$$\forall\,\phi\in C_c(\mathbb{R}^m;\mathbb{R})$$

$$\nu^k_{y,\delta}(\lambda) \triangleq \frac{1}{|B(y,\delta)|}\delta_{u^k(x)}\,dx$$

$$\boxed{\nu_y(\lambda) = \lim_{\delta\to0}\lim_{k\to\infty}\nu^k_{y,\delta}}$$

Weak Continuity of

2x2 Determinants

$\Omega \subset \mathbb{R}_+ \times \mathbb{R}$ bounded open

$U^k : \Omega \to \mathbb{R}^4$ measurable

$$\begin{cases} \text{w-}\lim\limits_{k\to\infty} U^k = U \quad \text{in } L^2(\Omega;\mathbb{R}^4) \\ \begin{cases} \dfrac{\partial u_1^k}{\partial t} + \dfrac{\partial u_2^k}{\partial x} \\ \dfrac{\partial u_3^k}{\partial t} + \dfrac{\partial u_4^k}{\partial x} \end{cases} \quad \text{compact in } H^{-1}_{loc}(\Omega) \end{cases}$$

$$\Longrightarrow \quad \begin{vmatrix} u_1^k & u_2^k \\ u_3^k & u_4^k \end{vmatrix} \longrightarrow \begin{vmatrix} u_1 & u_2 \\ u_3 & u_4 \end{vmatrix} \quad \mathcal{D}'.$$

Subsequentially

Another Form $\quad V^k = (V_1^k, V_2^k) \in L^2(\Omega;\mathbb{R}^2)$

$W^k = (W_1^k, W_2^k) \in L^2(\Omega;\mathbb{R}^2)$

$$\begin{cases} \text{w-}\lim\limits_{k\to\infty} (V^k, W^k) = (V, W), \quad L^2(\Omega) \\ \begin{cases} \operatorname{div} V^k \\ \operatorname{curl} W^k \end{cases} \quad \text{compact in } H^{-1}_{loc}(\Omega) \end{cases}$$

$$\Longrightarrow \quad V^k \cdot W^k \longrightarrow V \cdot W \quad \mathcal{D}'.$$

Div-Curl Lemma

$\Omega \subset \mathbb{R}^n$ open, bounded

$p, q > 1, \quad \frac{1}{p} + \frac{1}{q} = 1$

$v^k \in L^p(\Omega; \mathbb{R}^n)$

$w^k \in L^q(\Omega; \mathbb{R}^n)$

$$\begin{cases} v^k \rightharpoonup v & \text{weakly in } L^p(\Omega; \mathbb{R}^n) \\ w^k \rightharpoonup w & \text{weakly in } L^q(\Omega; \mathbb{R}^n). \end{cases}$$

$$\begin{cases} \operatorname{div} v^k & \text{compact in } W^{-1,p}_{loc}(\Omega; \mathbb{R}) \\ \operatorname{curl} w^k & \text{compact in } W^{-1,q}_{loc}(\Omega; \mathbb{R}). \end{cases}$$

$\Longrightarrow$

$$v^k \cdot w^k \longrightarrow v \cdot w \quad \mathcal{D}'$$

Compensated Compact Embedding Lemma

$\Omega \subset \mathbb{R}^n$ bounded open

$\hookrightarrow$

(Compact set of $W^{-1,q}_{loc}(\Omega)$)

$\cap$ (Bounded set of $W^{-1,r}_{loc}(\Omega)$)

$\subset$ (Compact set of $W^{-1,p}_{loc}(\Omega)$)

for any $1 < q \leq p < r < \infty$

2x2 Hyperbolic Systems of Conservation Laws

$$\begin{cases} u_t + f(u)_x = 0 & u \in \mathbb{R}^2 \\ u|_{t=0} = u_0(x) \end{cases}$$

Assume

- $\exists$ a strictly convex entropy $\eta_*(u)$.

  $\nabla^2 \eta_*(u) > 0$

- $\exists$ globally defined Riemann Invariants

  $W = (W_1, W_2) : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ s.t.

  $\nabla W_j(u) \,//\, \ell_j(u)$

If $u \in C^1$:

$$\boxed{\partial_t W_j + \lambda_j(u(W))\, \partial_x W_j = 0}$$

Entropy Equation

Entropy $\eta(u)$. Entropy Flux $q(u)$

$$\nabla q(u) = \nabla \eta(u) \nabla f(u)$$

$$(\lambda_j \nabla \eta - \nabla q) \cdot r_j = 0$$

$$\boxed{\begin{aligned} & q_{w_j} = \lambda_j \eta_{w_j} \\ & \eta_{w_1 w_2} - \frac{\lambda_{1 w_2}}{\lambda_2 - \lambda_1} \eta_{w_1} + \frac{\lambda_{2 w_1}}{\lambda_2 - \lambda_1} \eta_{w_2} = 0 \end{aligned}}$$

Genuine Nonlinearity

$$\nabla \lambda_j(u) \cdot r_j(u) \neq 0. \quad j = 1, 2.$$

$$\Longleftrightarrow \quad \frac{\partial \lambda_j}{\partial w_j} \neq 0. \quad j = 1, 2,$$

Method of Vanishing Viscosity

$$\begin{cases} u_t + f(u)_x = 0 & u \in \mathbb{R}^2 \\ u|_{t=0} = u_0(x) \in L^\infty \cap L^1(\mathbb{R}) \end{cases}$$

Viscosity Approximation

$$\begin{cases} u_t + f(u)_x = \varepsilon u_{xx} \\ u|_{t=0} = u_0^\varepsilon(x) \longrightarrow u_0(x) \text{ a.e.} \end{cases}$$

$\hookrightarrow$ $u^\varepsilon = u^\varepsilon(t, x)$

- Invariant Regions or L^p Estimates

$$\|u^\varepsilon\|_{L^\infty} \le C \quad \text{or} \quad \|u^\varepsilon\|_{L^p} \le C$$

- Dissipation Estimate

$$\|\sqrt{\varepsilon}\, u_x^\varepsilon\|_{L^2} \le C \quad \forall \varepsilon.$$

Dissipation Estimate

$$\nabla\bar{\eta}_*(u^\varepsilon)\times\left[\, u^\varepsilon_t + f(u^\varepsilon)_x = \varepsilon u^\varepsilon_{xx} \right]$$

$$\varepsilon(u^\varepsilon_x)^T\nabla^2\bar{\eta}_*(u^\varepsilon)u^\varepsilon_x \geq c_0\varepsilon|u^\varepsilon_x|^2$$

$$= -\bar{\eta}_*(u^\varepsilon)_t - \bar{q}_*(u^\varepsilon)_x + \varepsilon\bar{\eta}_*(u^\varepsilon)_{xx}$$

$$c_0\int_0^T\!\!\int \varepsilon|u^\varepsilon_x|^2\,dx\,dt$$

$$\leq \int\bar{\eta}_*(u^\varepsilon_0)\,dx - \int\bar{\eta}_*(u^\varepsilon(T,x))\,dx.$$

$$\leq \int\bar{\eta}_*(u^\varepsilon_0)\,dx \leq C \quad \cancel{\varepsilon}$$

- $\bar{\eta}_*(u) = \eta_*(u) - \eta_*(0) - \nabla\eta(0)\,u \geq c_0 > 0$

  $\bar{q}_*(u) = q_*(u) - q_*(0) - \nabla\eta(0)(f(u) - f(0)).$

H^{-1}- Compactness

$$\eta(u^\varepsilon)_t + q(u^\varepsilon)_x$$
$$= \varepsilon(\nabla\eta(u^\varepsilon)\, u_x^\varepsilon)_x - \varepsilon\,(u_x^\varepsilon)^T \nabla^2\eta(u^\varepsilon)\, u_x^\varepsilon$$
$$= I_1^\varepsilon + I_2^\varepsilon$$

- $\|I_1^\varepsilon\|_{H^{-1}(\Omega)} \le \sqrt{\varepsilon}\, \|\sqrt{\varepsilon}\, u_x^\varepsilon\|_{L^2}\, \|\nabla\eta(u^\varepsilon)\|_{L^\infty} \le \sqrt{\varepsilon}\, C \to 0$
- $\|I_2^\varepsilon\|_{L^1(\Omega)} \le \|\nabla^2\eta(u^\varepsilon)\|_{L^\infty} \|\sqrt{\varepsilon}\, u_x^\varepsilon\|_{L^2}^2 \le C$

$\hookrightarrow$ I_2^ε compact in $W^{-1,q}(\Omega)$, $1<q<2$

$\hookrightarrow$ $I_1^\varepsilon + I_2^\varepsilon$ compact in $W^{-1,q}(\Omega)$, $1<q<2$

But $\eta(u^\varepsilon)_t + q(u^\varepsilon)_x$ bounded in $W^{-1,\infty}(\Omega)$

$\hookrightarrow$ **Lemma**

$$\boxed{\eta(u^\varepsilon)_t + q(u^\varepsilon)_x \text{ is compact in } H^{-1}_{loc}}$$

$$\forall\, (\eta, q) \in C^2$$

Commutation Identity for Young Measure.

$\{\nu_{t,x}\}_{(t,x)\in\mathbb{R}_+^2} \Updownarrow \{u^\varepsilon\}_{\varepsilon>0}.$

- $\operatorname{Supp} \nu_{t,x} \subset\subset \mathbb{R}^2$
- For any entropy pairs (η, q),

$$(*) \quad \left\langle \nu_{t,x}, \begin{vmatrix} \eta_1 & q_1 \\ \eta_2 & q_2 \end{vmatrix} \right\rangle = \begin{vmatrix} \langle \nu_{t,x}, \eta_1 \rangle & \langle \nu_{t,x}, q_1 \rangle \\ \langle \nu_{t,x}, \eta_2 \rangle & \langle \nu_{t,x}, q_2 \rangle \end{vmatrix} \quad \text{a.e. } (t,x)$$

$$\overset{?}{\Longrightarrow} \quad \nu_{t,x} = \delta_{u(t,x)}(\lambda) \quad ??$$

* If $f(u) = Au$ (linear)

$\hookrightarrow$ $(*)$ is trivial.

The imbalance of $(*)$ is enforced by the nonlinearity of $f(u)$.

Proof of (*)

$\forall\ (\eta_i, q_i) \in C$, $i = 1, 2$.

$U^\varepsilon = (\eta_1(u^\varepsilon), q_1(u^\varepsilon), \eta_2(u^\varepsilon), q_2(u^\varepsilon))$ uniformly bdd

$\hookrightarrow \exists\ \{\varepsilon_k\}_{k=1}^\infty$, s.t $\varepsilon_k \to 0$ as $k \to \infty$.

- $U^{\varepsilon_k} \xrightarrow{*} (\langle \nu_{t,x}, \eta_1(\lambda)\rangle, \langle \nu_{t,x}, q_1(\lambda)\rangle, \langle \nu_{t,x}\ \eta_2(\lambda)\rangle, \langle \nu_{t,x}\ q_2(\lambda)\rangle) = U(t,x)$

- $\begin{vmatrix} U_1^{\varepsilon_k} & U_2^{\varepsilon_k} \\ U_3^{\varepsilon_k} & U_4^{\varepsilon_k} \end{vmatrix} \xrightarrow{*} \boxed{\left\langle \nu_{t,x}, \begin{vmatrix} \eta_1(\lambda) & q_1(\lambda) \\ \eta_2(\lambda) & q_2(\lambda) \end{vmatrix} \right\rangle}$

Div-Curl

$$\begin{vmatrix} U_1 & U_2 \\ U_3 & U_4 \end{vmatrix} = \boxed{\begin{vmatrix} \langle \nu_{t,x}\ \eta_1(\lambda)\rangle & \langle \nu_{t,x}, q_1(\lambda)\rangle \\ \langle \nu_{t,x}\ \eta_2(\lambda)\rangle & \langle \nu_{t,x}, q_2(\lambda)\rangle \end{vmatrix}}.$$

Compactness: Young Measure and Commutation Identity

Div-Curl Lemma [Murat (1978), Tartar (1979)]
Young Measure Rep. [Tartar (1979), Ball (1989), Alberti-Müller (2001)]

$\Longrightarrow$

$$\langle \nu_{t,r}(\boldsymbol{\lambda}), \eta_1(\boldsymbol{\lambda}) q_2(\boldsymbol{\lambda}) - q_1(\boldsymbol{\lambda}) \eta_2(\boldsymbol{\lambda}) \rangle$$
$$= \langle \nu_{t,r}(\boldsymbol{\lambda}), \eta_1(\boldsymbol{\lambda}) \rangle \langle \nu_{t,r}(\boldsymbol{\lambda}), q_2(\boldsymbol{\lambda}) \rangle - \langle \nu_{t,r}(\boldsymbol{\lambda}), q_1(\boldsymbol{\lambda}) \rangle \langle \nu_{t,r}(\boldsymbol{\lambda}), \eta_2(\boldsymbol{\lambda}) \rangle$$

for entropy-entropy flux pairs $(\eta_j, q_j) = (\eta^{\psi_j}, q^{\psi_j})$ **with** *compactly supported* C^2**–functions** $\psi_j(s), j = 1, 2$, **where** $\nu_{t,r}(\lambda)$ **is the associated Young measure (probability measure) for the sequence** $U^\varepsilon(t, r)$.

Issue: **Is** $\nu_{t,r}$ **a Dirac measure?** $\Longrightarrow$ **Compactness of** $U^\varepsilon(t, x)$ **in** L^1

Reduction of the Young Measure:

Scalar Conservation Laws: $u^\varepsilon \xrightarrow{*} u$, a.e.

$\hookrightarrow u(t,x) = \langle \nu_{t,x}, \lambda \rangle$

$$\boxed{\left\langle \nu_{t,x}, \begin{vmatrix} \eta_1(\lambda) & q_1(\lambda) \\ \eta_2(\lambda), & q_2(\lambda) \end{vmatrix} \right\rangle = \begin{vmatrix} \langle \nu_{t,x}, \eta_1(\lambda)\rangle & \langle \nu_{t,x}, q_1(\lambda)\rangle \\ \langle \nu_{t,x}, \eta_2(\lambda)\rangle & \langle \nu_{t,x}, q_2(\lambda)\rangle \end{vmatrix}}$$

Choose.
$$(\eta_1(\lambda), q_1(\lambda)) = (\lambda - u(t,x),\ f(\lambda) - f(u(t,x)))$$
$$(\eta_2(\lambda), q_2(\lambda)) = \left(f(\lambda) - f(u(t,x)),\ \int_{u(t,x)}^{\lambda} (f'(s))^2\, ds\right)$$

$$\left\langle \nu_{t,x} \begin{vmatrix} \lambda - u & f(\lambda) - f(u) \\ f(\lambda) - f(u) & \int_u^\lambda (f'(s))^2 ds \end{vmatrix} \right\rangle$$

$$= \begin{vmatrix} \underbrace{\boxed{\langle \nu_{t,x}\ \lambda - u\rangle}}_{=0} & \langle \nu_{t,x}\ f(\lambda) - f(u)\rangle \\ \langle \nu_{t,x}, f(\lambda) - f(u)\rangle & \langle \nu_{t,x}, \int_u^\lambda (f'(s))^2 ds\rangle \end{vmatrix}$$

$$\boxed{\begin{aligned} &\left\langle \nu_{t,x}, (\lambda - u)\int_u^\lambda (f'(s))^2\, ds - (f(\lambda) - f(u))^2 \right\rangle \\ &\quad + \langle \nu_{t,x}, f(\lambda) - f(u)\rangle^2 = 0 \end{aligned}}$$

* G.-Q. Chen and Y.-G. Lu, *Chinese Sci. Bull.* 34 (1989), no. 1, 15–19.

$$(\lambda-u)\int_u^\lambda (f'(s))^2 ds - (f(\lambda)-f(u))^2$$

$$=(\lambda-u)\int_u^\lambda \left(f'(s)-\frac{1}{\lambda-u}\int_u^\lambda f'(\tau)d\tau\right)^2 ds \geq 0.$$

$\Rightarrow$

$$\begin{cases} \langle \nu_{t,x},\ f(\lambda)-f(u)\rangle = 0 \\ \langle \nu_{t,x},\ \underbrace{(\lambda-u)\int_u^\lambda \left(f'(s)-\frac{1}{\lambda-u}\int_u^\lambda f'(\tau)d\tau\right)^2 ds}_{} \rangle = 0 \end{cases}$$

$$\| $$

$$(\lambda-u)\int_u^\lambda \left(\int_u^\lambda f''(\lambda+\theta(s-\tau))(s-\tau)d\tau\right)^2 ds$$

$\Rightarrow$

- $\langle \nu_{t,x}, f(\lambda)\rangle = f(u(t,x))$
- If $f''(u) > 0$

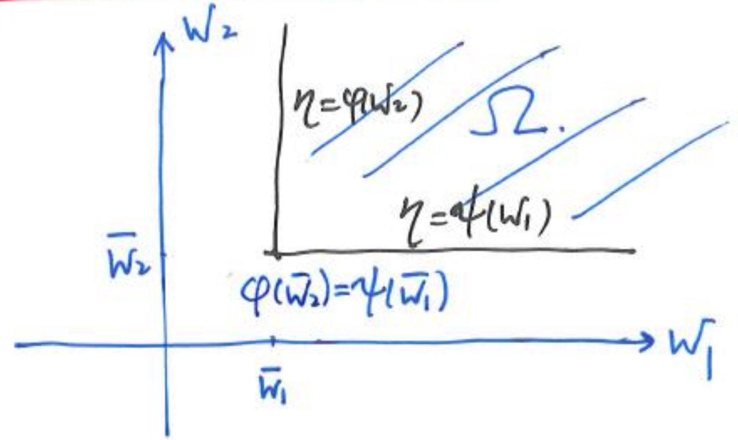
$\hookrightarrow$ $\boxed{\nu_{t,x} = \delta_{u(t,x)}}$

The Goursat Entropy Pairs for 2x2 Hyperbolic Systems of Conservation Laws

$$\boxed{\eta_{w_1 w_2} - \frac{\lambda_{1 w_2}}{\lambda_2 - \lambda_1}\eta_{w_1} + \frac{\lambda_{2 w_1}}{\lambda_2 - \lambda_1}\eta_{w_2} = 0} \qquad (*)$$

$$\boxed{q_{w_j} = \lambda_j \eta_{w_j}} \qquad (**)$$

Goursat Problem for $(*)$



Well-posed!

Goursat Entropy Pairs

$\exists$ two families of entropy pairs:

$$\begin{cases} \eta_a(w) = I_1(w)\, a(w_1) + \int_{\bar{w}_1}^{w_1} J_1(\xi; w)\, a(\xi)\, d\xi \\ q_a(w) = K_1(w)\, a(w_1) + \int_{\bar{w}_1}^{w_1} L_1(\xi; w)\, a(\xi)\, d\xi \end{cases}$$

$$\begin{cases} \eta_b(w) = I_2(w)\, b(w_2) + \int_{\bar{w}_2}^{w_2} J_2(\xi; w)\, b(\xi)\, d\xi \\ q_b(w) = K_2(w)\, b(w_2) + \int_{\bar{w}_2}^{w_2} L_2(\xi; w)\, b(\xi)\, d\xi \end{cases}$$

where (I_i, J_i, K_i, L_i), $i = 1, 2$, are unique smooth functions and independent of $\bar{w}_1$ and $\bar{w}_2$:

$$\begin{cases} I_2(\bar{w}_1, w_2) = 1 \\ J_2(\xi; \bar{w}_1, w_2) = 0 \end{cases}$$

$$\begin{cases} I_i(w) > 0, \quad K_i = \lambda_i I_i \qquad \begin{cases} I_1(w_1, \bar{w}_2) = 1 \\ J_1(\xi; w_1, \bar{w}_2) = 0 \end{cases} \\ \dfrac{\partial K_i(w)}{\partial w_1} + L_i(w_i; w) = \lambda_i(w) \left(\dfrac{\partial I_i(w)}{\partial w_i} + J(w_i; w) \right) \\ \dfrac{\partial L_i(\xi; w)}{\partial w_i} = \lambda_i(w) \dfrac{\partial J_i(\xi; w)}{\partial w_i} \\ \dfrac{\partial K_i(w)}{\partial w_j} = \lambda_j(w) \dfrac{\partial I_i(w)}{\partial w_j}, \qquad i \neq j \\ \dfrac{\partial L_i(\xi; w)}{\partial w_j} = \lambda_j(w) \dfrac{\partial J_i(\xi; w)}{\partial w_j} \end{cases}$$

Reduction of the Young Measure

<u>Thm</u>. If $\frac{\partial \lambda_j}{\partial w_j} \neq 0$, $j=1,2$ (Genuinely Nonlinear)

$$\Longrightarrow \quad \nu_{t,x} = \delta_{u(t,x)}$$

<u>Proof</u>. If $\nu_{t,x} \neq \delta_{u(t,x)}$, we denote $[w_1^-, w_1^+] \times [w_2^-, w_2^+]$ the smallest rectangle containing $\operatorname{supp} \nu_{t,x}$.

1. <u>Claim</u>: If $w_1^- < w_1^+$, then $\exists\ C_1(t,x)$ s.t.

$$\langle \nu, q_a \rangle = C_1 \langle \nu, \eta_a \rangle$$

$$\forall\ a \in C, \quad a(w_1) = 0 \text{ when } \begin{cases} w_1 \geq \bar{w}_1 \\ \text{or} \\ w_1 \leq \bar{w}_1 \end{cases} \text{ for } \bar{w}_1 \in (w_1^-, w_1^+)$$

- Choose $\begin{cases} a_0(w_1) = (w_1 - w_1^*)_+, & w_1^* \geq \bar{w}_1, \quad |w_1^+ - w_1^*| \ll 1 \\ a(w_1) = 0, & w_1 \geq \bar{w}_1 \end{cases}$

$$\hookrightarrow \begin{cases} \eta_{a_0}(w) > 0 & \forall\ w \in \{\bar{w}_1 < w_1 < w_1^+\} \cap \operatorname{supp} \nu \\ \eta_{a_0} q_a - \eta_a q_{a_0} \equiv 0 \end{cases}$$

$$\hookrightarrow \langle \nu, \eta_{a_0} \rangle \langle \nu, q_a \rangle = \langle \nu, \eta_a \rangle \langle \nu, q_{a_0} \rangle$$

$\hookrightarrow \quad \langle v, \mathfrak{z}_a \rangle = C_1(\bar{w}_1) \langle v, \eta_a \rangle$

$\forall a \in C, \quad a(w_1) = 0, \ w_1 \geq \bar{w}_1.$

$$\left[C_1(\bar{w}_1) = \frac{\langle v, \mathfrak{z}_{a_0} \rangle}{\langle v, \eta_{a_0} \rangle} \right]$$

Similarly, $\forall a \in C, \quad a(w_1) = 0$ when $w_1 \leq \bar{w}_1,$

$$\langle v, \mathfrak{z}_a \rangle = C_1(\bar{w}_1) \langle v, \eta_a \rangle$$

- <u>claim</u> $C_1(\bar{w}_1) \not\to \bar{w}_1.$

For any $\tilde{w}_1 \in (w_1^-, w_1^+)$, choose

$$\begin{cases} a_1(w_1) = 0 & w_1 \leq \tilde{w}_1 \\ a_2(w_1) = 0 & w_1 \geq \bar{w}_1 \end{cases} \quad \text{for } \tilde{w}_1 < \bar{w}_1$$

$$\begin{cases} a_1(w_1) = 0 & w_1 \geq \tilde{w}_1 \\ a_2(w_1) = 0 & w_1 \leq \bar{w}_1 \end{cases} \quad \text{for } \tilde{w}_1 > \bar{w}_1$$

$\Longrightarrow \quad \eta_{a_1} \mathfrak{z}_{a_2} - \eta_{a_2} \mathfrak{z}_{a_1} \equiv 0$

$\hookrightarrow \quad C_1(\bar{w}_1) = C_1(\tilde{w}_1)$

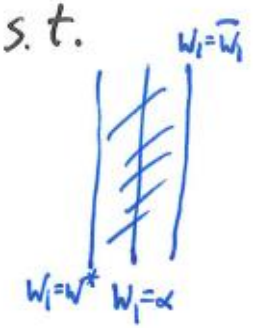
2. <u>claim</u> If $W_2^- < W_2^+$, then $\exists\ C_2(t,x)$ s.t.

$$\langle \nu, \xi_b \rangle = C_2 \langle \nu, \eta_b \rangle$$

$\forall b \in C$, $b(W_2)=0$ when $\begin{cases} W_2 \ge \bar{W}_2 \\ \text{or } W_2 \le \bar{W}_2 \end{cases}$ $\bar{W}_2 \in (W_2^-, W_2^+)$.

3. $\forall \alpha \in (W_1^-, W_1^+)$, choose $W_1^*, \bar{W}_1$ s.t.

$$W_1^* < \alpha < \bar{W}_1, \quad \bar{W}_1 - W_1^* \ll 1$$



Choose (η_a, ξ_a): $a(W_1) = (W_1 - W_1^*)_+$

choose $(\eta_{\bar{a}}, \xi_{\bar{a}})$: $\bar{a}(W_1) = (W_1 - \bar{W}_1)_-$

$$\langle \nu, \eta_a \xi_{\bar{a}} - \eta_{\bar{a}} \xi_a \rangle = \langle \nu, \eta_a \rangle \langle \nu, \xi_{\bar{a}} \rangle - \langle \nu, \eta_{\bar{a}} \rangle \langle \nu, \xi_a \rangle$$
$$= 0$$

<u>We know</u> that, On $\{W_1 < W_1^*\} \cup \{W_1 > \bar{W}_1\}$,

$$\eta_a \xi_{\bar{a}} - \eta_{\bar{a}} \xi_a \equiv 0$$

On $\{w_1^* \le w_1 \le \bar{w}_1\}$.

$$\begin{cases} \eta_a(w) = I_1(w)(w_1-w_1^*) + \frac{1}{2} J_1(\alpha; w)(w_1-w_1^*)^2 + O(|w_1-w_1^*|^3) \\ g_a(w) = K_1(w)(w_1-w_1^*) + \frac{1}{2} L_1(\alpha; w)(w_1-w_1^*)^2 + O(|w_1-w_1^*|^3) \end{cases}$$

$$\begin{cases} \eta_{\bar{a}}(w) = I_1(w)(w_1-\bar{w}_1) - \frac{1}{2} J_1(\alpha; w)(w_1-\bar{w}_1)^2 + O(|w_1-w_1^*|^3) \\ g_{\bar{a}}(w) = K_1(w)(w_1-\bar{w}_1) - \frac{1}{2} L_1(\alpha; w)(w_1-\bar{w}_1)^2 + O(|w_1-w_1^*|^3) \end{cases}$$

$$(\eta_a g_{\bar{a}} - \eta_{\bar{a}} g_a)(w)$$
$$= \frac{1}{2}(\bar{w}_1-w_1^*)(\bar{w}_1-w_1)(w_1-w_1^*)\left(I^2 \frac{\partial \lambda_1}{\partial w_1}\right)(\alpha, w_2)$$
$$+ O\left((\bar{w}_1-w_1^*)^2(\bar{w}_1-w_1)(w_1-w_1^*)\right)$$

$$\Rightarrow \left\langle \nu,\ (\bar{w}_1-w_1)_+(w_1-w_1^*)_+\left(\underbrace{\left(I^2 \frac{\partial \lambda_1}{\partial w_1}\right)(\alpha, w_2)}_{\neq 0} + O(|\bar{w}_1-w_1^*|)\right)\right\rangle = 0$$

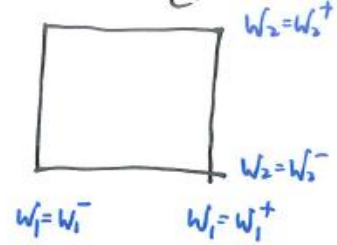
$$\Rightarrow \quad \operatorname{supp} \nu \cap \{w_1^* \le w_1 \le \bar{w}_1\} = \phi \quad \forall \begin{matrix} w_1^* < \bar{w}_1 \\ |\bar{w}_1 - w_1^*| \ll 1. \end{matrix}$$

$$\hookrightarrow \quad \operatorname{supp} \nu \cap \{w_1^- < w_1 < w_1^+\} = \phi$$

Similarly.

$$\hookrightarrow \quad \operatorname{supp} \nu \cap \{w_2^- < w_2 < w_2^+\} = \phi$$

4. If



$$\mathrm{Supp}\,\nu \cap \left(\{W_1 = W_1^{\pm}\} \cup \{W_2 = W_2^{\pm}\}\right) \neq \phi$$

for example

$$\mathrm{Supp}\,\nu \cap \{W_1 = W_1^-\} \neq \phi$$

$\Rightarrow$ $\nu(\{W_1 = W_1^-\}) \neq 0$

then we follow Step 3 to choose

$$\alpha = W_1^-, \quad \bar{W}_1 = W_1^- + \varepsilon, \quad W_1^* = W_1^- - \varepsilon.$$

to conclude

$$\mathrm{Supp}\,\nu \cap \{W_1^- - \varepsilon \le W_1 \le W_1^- + \varepsilon\} = \phi$$

for sufficiently small $\varepsilon \geq 0$

$\Rightarrow$ Contradiction

5. Conclusion

$$\mathrm{Supp}\,\nu \cap \left([W_1^-, W_1^+] \times [W_2^-, W_2^+]\right) = \phi$$

$\Rightarrow$ Contradiction

$\Longrightarrow$ $\nu_{t,x} = \delta_{u(t,x)}$ Single point support.

1D Isentropic Euler Equations: Entropy-Entropy Flux Pairs

$$\begin{cases} \rho_t + m_x = 0, & (m = \rho v) \\ m_t + (\frac{m^2}{\rho} + p(\rho))_x = 0. \end{cases}$$

Strict Hyperbolicity: Fails when $\rho \to 0$ (vacuum)

Entropy Pair (η, q): $\nabla q(U) = \nabla \eta(U) \nabla F(U)$

Convex Entropy: $\nabla^2 \eta(U) > 0$ **Weak Entropy**: $\eta(\rho, \rho v)|_{\rho=0} = 0$

Weak entropy pairs are represented as

$$\eta^\psi(\rho, \rho v) = \int_{\mathbb{R}} \chi(s)\psi(s)\,\mathrm{d}s, \quad q^\psi(\rho, \rho v) = \int_{\mathbb{R}} (\theta s + (1-\theta)v)\chi(s)\psi(s)\,\mathrm{d}s$$

by C^2-functions $\psi(s)$, where $\chi(s)$ is the weak entropy kernel:

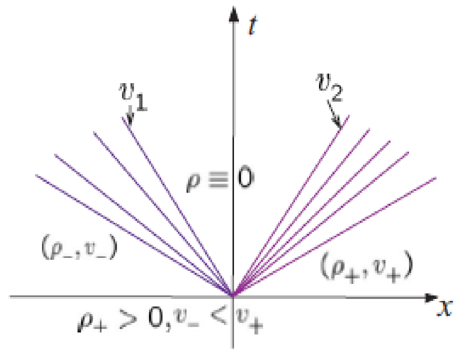
$$\chi(s) := [\rho^{2\theta} - (v-s)^2]_+^\lambda, \qquad \theta = \frac{\gamma - 1}{2}, \lambda = \frac{3-\gamma}{2(\gamma-1)}.$$

Physical Convex Entropy: Mechanical energy-energy flux pair (η_*, q_*):

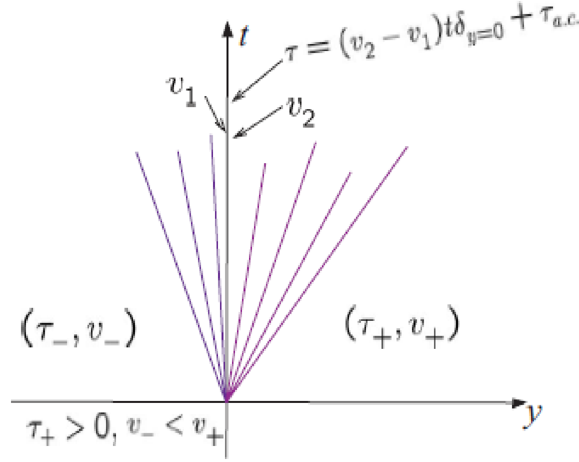
$$\eta_*(\rho, m) = \frac{1}{2}\frac{m^2}{\rho} + \rho e(\rho), \qquad q_*(\rho, m) = \frac{1}{2}\frac{m^3}{\rho^2} + m(e(\rho) + \frac{p}{\rho})$$

Cavitation and Concentration: Pressure $p(\rho) = a\rho^{\gamma}, \gamma > 1$

$$\partial_t \rho + \partial_x(\rho v) = 0, \qquad \partial_t(\rho v) + \partial_x(\rho v^2 + p(\rho)) = 0$$



$$(t,x) \to (t,y): \ y_t = \rho(t,x), y_x = -(\rho v)(t,x); \qquad \tau(t,y) = 1/\rho(t,x)$$



$$\partial_t \tau - \partial_y v = 0, \qquad \partial_t v + \partial_y p(1/\tau) = 0$$

Cavitation and Concentration: Pressure $p(\rho) = a\rho^{\gamma}, \gamma > 1$

$$\partial_t \rho + \partial_x(\rho v) = 0, \qquad \partial_t(\rho v) + \partial_x(\rho v^2 + p(\rho)) = 0$$

Theorem

Let the Cauchy initial data satisfy

$$0 \le \rho_0(x) \le C_0, \qquad |m_0(x)| \le C_0 \rho_0(x)$$

for some $C_0 > 0$. Then there exists a global entropy solution $(\rho, m)(t,x) = (\rho, \rho v)(t,x)$ of the Cauchy problem such that

$$0 \le \rho(t,x) \le C, \qquad |m(t,x)| \le C\rho(t,x),$$

where $C > 0$ is a constant depending only on $\gamma > 1$, $a > 0$, and $C_0 > 0$.

DiPerna: $\gamma = \frac{N+2}{N}, N \ge 5$ odd, Ding-Luo & Chen: $\gamma \in (1, \frac{5}{3}]$,
Lions-Perthame-Tadmor: $\gamma \ge 3$, Lions-Perthame-Souganidis: $\gamma \in (\frac{5}{3}, 3)$,
Chen-LeFloch: General pressure laws

Entropy Pairs and the Young Measure-Valued Solution $\nu_{t,r}$

Weak entropy pairs are represented as

$$\eta^\psi(\rho, \rho v) = \int_{\mathbb{R}} \chi(s)\psi(s)\,\mathrm{d}s, \quad q^\psi(\rho, \rho v) = \int_{\mathbb{R}} (\theta s + (1-\theta)v)\chi(s)\psi(s)\,\mathrm{d}s$$

by *compactly supported* C^2-test functions $\psi(s)$, for

$$\chi(s) := [\rho^{2\theta} - (u-s)^2]_+^\lambda, \qquad \theta = \frac{\gamma-1}{2}, \lambda = \frac{3-\gamma}{2(\gamma-1)}.$$

Let $\nu_{t,r}$ be the Young measure determined by the solutions of the Navier-Stokes equations. Then $\nu_{t,r}$ **is confined by the following communtative relations**:

$$\begin{aligned} &(\gamma-1)(s_2-s_1)\left(\overline{\chi(s_1)\chi(s_2)} - \overline{\chi(s_1)}\,\overline{\chi(s_2)}\right) \\ &= (3-\gamma)\left(\overline{v\chi(s_2)}\,\overline{\chi(s_1)} - \overline{v\chi(s_1)}\,\overline{\chi(s_2)}\right) \qquad \text{for a.e. } s_1, s_2 \in \mathbb{R} \end{aligned}$$

where $\overline{f(s)} := \langle \nu_{t,r}, f(s;\rho,v)\rangle$.

- **If supp $\nu_{t,r}$ is bounded**, then $\nu_{t,r}(\boldsymbol{\lambda}) = \nu_{(\rho(t,r),m(t,r))}(\boldsymbol{\lambda})$.

  DiPerna: $\gamma = \frac{N+2}{N}$, $N \geq 5$ odd, Ding-Luo & Chen: $\gamma \in (1, \frac{5}{3}]$,
  Lions-Perthame-Tadmor: $\gamma \geq 3$, Lions-Perthame-Souganidis: $\gamma \in (\frac{5}{3}, 3)$,
  Chen-LeFloch: General pressure laws

Measure-Valued Solution: Reduction for $\gamma = 3$ (cf. LPT)

When $\gamma = 3$, then the commutation relation becomes

$$\overline{\chi(s_1)\chi(s_2)} = \overline{\chi(s_1)}\ \overline{\chi(s_2)},$$

which implies

$$\overline{\chi(s)}^2 = \overline{\chi(s)^2},$$

by taking $s_1 = s_2$. That is,

$$\langle \nu_{t,r}, \left(\chi(s) - \overline{\chi(s)}\right)^2 \rangle = 0 \qquad \text{for any } s \in \mathbb{R}$$

This implies that ν must be a Dirac mass on the set $\{\rho > 0\}$ or be supported completely in the vacuum $V = \{\rho = 0\}$, that is, the measure-valued solution $\nu_{t,r}$ is a Dirac mass in the phase coordinates (ρ, m):

$$\nu_{t,r}(\rho, m) = \delta_{(\rho(t,r), m(t,r))}(\rho, m).$$

Bounded Supported Measure-Valued Solution for $\gamma > 3$ —-Arguments by Lions-Perthame-Tadmor

Let $A := \cup\{(u-\rho^\theta, \rho^\theta + u) : (\rho, u) \in \operatorname{supp}\nu\}$ be open set

Let $J = (s_-, s_+)$ be any bounded connected component of A

Note that $\operatorname{supp}\chi(s) = \{(\rho, u) : u - \rho^\theta \leq s \leq u + \rho^\theta\}$.

By definition of J, $\chi(s) > 0$ for *a.e.* $s \in J$, so that

$$\frac{1-\theta}{\theta}\frac{1}{s_2-s_1}\left(\frac{\overline{u\chi(s_2)}}{\overline{\chi(s_2)}} - \frac{\overline{u\chi(s_1)}}{\overline{\chi(s_1)}}\right) = \frac{\overline{\chi(s_1)\chi(s_2)}}{\overline{\chi(s_1)}\,\overline{\chi(s_2)}} - 1$$

- Taking $s_1, s_2 \to s$ yields $\frac{\partial}{\partial s}\left(\frac{\overline{u\chi(s)}}{\overline{\chi(s)}}\right) = \frac{1-\theta}{\theta}\left(\frac{\overline{\chi^2(s)}}{\left(\overline{\chi(s)}\right)^2} - 1\right) \leq 0.$

  Then the function $\frac{\overline{u\chi(s)}}{\overline{\chi(s)}}$ is non-increasing on J.

- Set $u_0 = \frac{s_-+s_+}{2}$. Then $\lim_{s\to s_+}\frac{\overline{u\chi(s)}}{\overline{\chi(s)}} \geq u_0 \geq \lim_{s\to s_-}\frac{\overline{u\chi(s)}}{\overline{\chi(s)}}$.

- Then $\frac{\overline{u\chi(s)}}{\overline{\chi(s)}}$ is constant, which implies $\overline{\chi(s)^2} = \overline{\chi(s)}^2$.

Measure-Valued Solution: Any Connected Component J of the Support Is Bounded for $\gamma > 3$

Strategy: On the contrary, let $\inf\{s : s \in J\} = -\infty$.
Fix $M_0 > 0$ such that $M_0 + 1 \in J$ and restrict $s_2 \in (M_0, M_0+1)$;
Choose $s_1 \le -2|M_0| \ll -1$ to reach the contradiction.

New Observation: $\int_{M_0}^{M_0+1} \frac{\overline{\chi(s_1)\chi(s_2)}}{\overline{\chi(s_1)}} ds_2 \le C(\lambda)|s_1|^{\lambda}, \qquad \lambda < 0.$

By LPT's argument: $\frac{\overline{\chi(s_1)\chi(s_2)}}{\overline{\chi(s_1)}} \ge \overline{\chi(s_2)}$ a.e. $s_1, s_2 \in J, s_1 < s_2,$

$$\Longrightarrow \int_{M_0}^{M_0+1} \frac{\overline{\chi(s_1)\chi(s_2)}}{\overline{\chi(s_1)}} ds_2 \ge \int_{M_0}^{M_0+1} \overline{\chi(s_2)} ds_2 = C(M_0, \lambda) > 0$$

Combining the TWO facts, we have

$$0 < C(M_0, \lambda) = \int_{M_0}^{M_0+1} \frac{\overline{\chi(s_1)\chi(s_2)}}{\overline{\chi(s_1)}} ds_2 \le C(\lambda)|s_1|^{\lambda},$$

which is a contradiction when $s_1 \to -\infty$.

*The case when J is unbounded from above can be treated similarly.

Measure-Valued Solution: Any Connected Component J of the Support Is Bounded for $\gamma \in (1,3)$, I: Strategy

On the contrary, suppose that J is unbounded from below.
Let $M_0 = \sup\{s : s \in J\} \in (-\infty, \infty]$.
Let $s_1, s_2, s_3 \in (-\infty, M_0)$ with $s_1 < s_2 < s_3$. The commutation relation $\Longrightarrow$

$$(s_2 - s_1)\frac{\overline{\chi(s_1)\chi(s_2)}}{\overline{\chi(s_1)}} + (s_3 - s_2)\frac{\overline{\chi(s_3)\chi(s_2)}}{\overline{\chi(s_3)}} = (s_3 - s_1)\overline{\chi(s_2)}\frac{\overline{\chi(s_1)\chi(s_3)}}{\overline{\chi(s_1)}\,\overline{\chi(s_3)}}.$$

Differentiating this equation in s_2 and dividing by $(s_3 - s_1)$, we obtain

$$\frac{s_2 - s_1}{s_3 - s_1}\frac{\overline{\chi(s_1)\chi'(s_2)}}{\overline{\chi(s_1)}} + \frac{s_3 - s_2}{s_3 - s_1}\frac{\overline{\chi(s_3)\chi'(s_2)}}{\overline{\chi(s_3)}} + \frac{1}{s_3 - s_1}\frac{\overline{\chi(s_1)\chi(s_2)}}{\overline{\chi(s_1)}}$$
$$-\frac{1}{s_3 - s_1}\frac{\overline{\chi(s_3)\chi(s_2)}}{\overline{\chi(s_3)}} = \overline{\chi'(s_2)}\frac{\overline{\chi(s_1)\chi(s_3)}}{\overline{\chi(s_1)}\,\overline{\chi(s_3)}}.$$

Strategy: Take $s_1 \to -\infty$ and show that the left-hand side has a smaller order than the right-hand side $\Longrightarrow$ Contradiction.

Measure-Valued Solution: Any Connected Component J of the Support Is Bounded for $\gamma \in (1,3)$, II: Steps

- As before: For any $s_1, s_3 \in J$, $\dfrac{\overline{\chi(s_1)\chi(s_3)}}{\overline{\chi(s_1)}\;\overline{\chi(s_3)}} \geq 1$;
- $\overline{\chi(s)} \geq 0$ is not identically zero and $\overline{\chi(s)} \to 0$ as $s \to \inf J, \sup J$,
$\Longrightarrow$ there exists s_2 such that $\overline{\chi'(s_2)} > 0, \quad \overline{\chi(s_2)} > 0$.
- Let $s_3 > s_2$ be points such that $\overline{\chi(s_3)} > 0$ and let $s_1 \to -\infty$. From the 1st identity, $\dfrac{\overline{\chi(s_1)\chi(s_2)}}{\overline{\chi(s_1)}} = \overline{\chi(s_2)}\dfrac{\overline{\chi(s_1)\chi(s_3)}}{\overline{\chi(s_1)}\;\overline{\chi(s_3)}} + o(1)$, as $s_1 \to -\infty$.
- From the 2nd equation, by throwing away the negative terms, we obtain

$$\overline{\chi'(s_2)}\frac{\overline{\chi(s_1)\chi(s_3)}}{\overline{\chi(s_1)}\;\overline{\chi(s_3)}} \leq \Big(\frac{2\lambda}{s_2 - s_1} + \frac{1}{s_3 - s_1}\Big)\frac{\overline{\chi(s_1)\chi(s_2)}}{\overline{\chi(s_1)}} + o(1).$$

$$\Longrightarrow \qquad \Big(\overline{\chi'(s_2)} - \frac{2\lambda\overline{\chi(s_2)}}{s_2 - s_1} - \frac{\overline{\chi(s_2)}}{s_3 - s_1}\Big)\frac{\overline{\chi(s_1)\chi(s_3)}}{\overline{\chi(s_1)}\;\overline{\chi(s_3)}} \leq o(1).$$

Contradiction as $s_1 \to -\infty$.

*The proof above is much simpler than that by LeFloch-Westdickenberg 2009.

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