Topics on Nonlinear Hyperbolic PDEs

Mathematical Institutes
Hilary Term 2022
February: 9th, 16th, 23rd
March: 2nd
Wednesdays 14:00-16:00

By Professor Gui-Qiang G. Chen

Lecture-4: 2nd March 2022

References:

1. **R. Courant and D. Hilbert: Methods of Mathematical Physics,** Vol. II. Reprint of the 1962 original. John Wiley&Sons, Inc.: New York, 1989.
2. **C. M. Dafermos: Hyperbolic Conservation Laws in Continuum Physics,** Fourth edition. Springer-Verlag: Berlin, 2016.
3. **L. C. Evans: Partial Differential Equations,** Second edition. AMS: Providence, RI, 2010.
4. **L. Hormander: Lectures on Nonlinear Hyperbolic Differential Equations** Springer-Verlag: Berlin-Heidelberg, 1997.
5. **P. D. Lax: Hyperbolic Differential Equations,** AMS: Providence, 2000.
6. **G.-Q. Chen and M. Feldman: The Mathematics of Shock Reflection-Diffraction and von Neumann's Conjectures.** Annals of Mathematics Studies, 197. *Princeton University Press*: *Princeton, NJ,* 2018.
7. **D. Serre, Systems of Conservation Laws, Vols. I, II,** Cambridge University Press: Cambridge, 1999, 2000.
8. **C. D. Sogge, Lectures on Nonlinear Wave Equations,** Second edition. International Press, Boston, MA, 2008.

Hyperbolic Conservation Laws

$$\partial_t \mathbf{u} + \nabla \cdot \mathbf{f}(\mathbf{u}) = 0$$

$$\mathbf{u} = (u_1, \cdots, u_m)^\top, \quad \mathbf{x} = (x_1, \cdots, x_d), \quad \nabla = (\partial_{x_1}, \cdots, \partial_{x_d})$$

$$\mathbf{f} = (\mathbf{f}_1, \cdots, \mathbf{f}_d): \ \mathbb{R}^m \to (\mathbb{R}^m)^d \quad \text{is a nonlinear mapping}$$

$$\mathbf{f}_i : \mathbb{R}^m \to \mathbb{R}^m \quad \text{for} \quad i = 1, \cdots, d$$

$$\partial_t \mathbf{A}(\mathbf{u}, \mathbf{u}_t, \nabla \mathbf{u}) + \nabla \cdot \mathbf{B}(\mathbf{u}, \mathbf{u}_t, \nabla \mathbf{u}) = 0$$

$$\mathbf{A}, \mathbf{B}: \ \mathbb{R}^m \times \mathbb{R}^m \times (\mathbb{R}^m)^d \to \mathbb{R}^m \ \text{are nonlinear mappings}$$

Connections and Applications:

- Fluid Mechanics and Related: Euler Equations and Related Equations
  Gas, shallow water, elastic body, reacting gas, plasma, ...
- Special Relativity: Relativistic Euler Equations and Related Equations
  General Relativity: Einstein Equations and Related Equations
- Differential Geometry: Isometric Embeddings, Nonsmooth Manifolds..
- ......

Hyperbolic Systems

$$\mathbf{u}_t + \mathbf{f}(\mathbf{u})_x = 0 \qquad \mathbf{u} = \mathbf{u}(t,x) \in \mathbb{R}^m$$

$$\mathbf{u}_t + \mathbf{A}(\mathbf{u})\mathbf{u}_x = 0 \qquad \mathbf{A}(\mathbf{u}) = \nabla \mathbf{f}(\mathbf{u})$$

The system is **strictly hyperbolic** if each $m \times m$ matrix $\mathbf{A}(\mathbf{u})$ has real distinct eigenvalues

$$\lambda_1(\mathbf{u}) < \lambda_2(\mathbf{u}) < \cdots < \lambda_m(\mathbf{u})$$

Right eigenvectors $\mathbf{r}_1(\mathbf{u}), \cdots, \mathbf{r}_m(\mathbf{u})$ (column vectors)
Left eigenvectors $\mathbf{l}_1(\mathbf{u}), \cdots, \mathbf{l}_m(\mathbf{u})$ (row vectors)

$$\mathbf{A}\mathbf{r}_i = \lambda_i \mathbf{r}_i \qquad \mathbf{l}_i \mathbf{A} = \lambda_i \mathbf{l}_i$$

Choose the bases so that

$$\mathbf{l}_i \cdot \mathbf{r}_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Functional Analytic Approaches for the Existence Theory:

- Compensated Compactness
- Weak Convergence Methods
- Geometric Measure Arguments
- ……

1. **C. M. Dafermos: Hyperbolic Conservation Laws in Continuum Physics,** Third edition. Springer-Verlag: Berlin, 2010.

2. **B. Dacorogna**: **Weak Continuity and Weak Lower Semicontinuity of Nonlinear Functionals**, Lecture Notes in Mathematics, Vol. 922, 1-120, Springer-Verlag, 1982.

3. **L. C. Evans**:. **Weak Convergence Methods for Nonlinear Partial Differential Equations**. CBMS-RCSM, 74. AMS: Providence, RI, 1990

4. **D. Serre:** La compacité par compensation pour les systèmes hyperboliques non linéaires de deux équations à une dimension d'espace. *J. Math. Pures Appl. (9)* 65 (1986), 423–468.

5. **G.-Q. Chen and M. Perepelitsa**: Vanishing viscosity limit of the Navier-Stokes equations to the Euler equations for compressible fluid flow. *Comm. Pure Appl. Math.* 63 (2010), 1469–1504.

5. **The references cited therein, especially more recent references.**

2x2 Hyperbolic Systems of Conservation Laws

$$\begin{cases} u_t + f(u)_x = 0 & u \in \mathbb{R}^2 \\ u|_{t=0} = u_0(x) \end{cases}$$

Assume

- $\exists$ a strictly convex entropy $\eta_*(u)$.

  $\nabla^2 \eta_*(u) > 0$

- $\exists$ globally defined Riemann Invariants

  $W = (W_1, W_2): \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ s.t.

  $\nabla W_j(u) \,/\!/\, \ell_j(u)$

↳ If $u \in C^1$:

$$\boxed{\partial_t W_j + \lambda_j(u(W))\,\partial_x W_j = 0}$$

Entropy Equation

Entropy $\eta(u)$. Entropy Flux $q(u)$

$$\nabla q(u) = \nabla \eta(u) \nabla f(u)$$

$$(\lambda_j \nabla \eta - \nabla q) \cdot r_j = 0$$

↳

$$q_{w_j} = \lambda_j \eta_{w_j}$$

$$\eta_{w_1 w_2} - \frac{\lambda_{1 w_2}}{\lambda_2 - \lambda_1} \eta_{w_1} + \frac{\lambda_{2 w_1}}{\lambda_2 - \lambda_1} \eta_{w_2} = 0$$

Genuine Nonlinearity

$$\nabla \lambda_j(u) \cdot r_j(u) \neq 0, \quad j = 1, 2.$$

$$\Longleftrightarrow \quad \frac{\partial \lambda_j}{\partial w_j} \neq 0 \quad j = 1, 2,$$

Compactness: Young Measure and Commutation Identity

Div-Curl Lemma [Murat (1978), Tartar (1979)]
Young Measure Rep. [Tartar (1979), Ball (1989), Alberti-Müller (2001)]

$\Longrightarrow$

$$\langle \nu_{t,r}(\boldsymbol{\lambda}), \eta_1(\boldsymbol{\lambda}) q_2(\boldsymbol{\lambda}) - q_1(\boldsymbol{\lambda}) \eta_2(\boldsymbol{\lambda}) \rangle$$
$$= \langle \nu_{t,r}(\boldsymbol{\lambda}), \eta_1(\boldsymbol{\lambda}) \rangle \langle \nu_{t,r}(\boldsymbol{\lambda}), q_2(\boldsymbol{\lambda}) \rangle - \langle \nu_{t,r}(\boldsymbol{\lambda}), q_1(\boldsymbol{\lambda}) \rangle \langle \nu_{t,r}(\boldsymbol{\lambda}), \eta_2(\boldsymbol{\lambda}) \rangle$$

for entropy-entropy flux pairs $(\eta_j, q_j) = (\eta^{\psi_j}, q^{\psi_j})$ **with** *compactly supported* C^2**–functions** $\psi_j(s), j = 1, 2$, **where** $\nu_{t,r}(\lambda)$ **is the associated Young measure (probability measure) for the sequence** $U^\varepsilon(t, r)$**.**

Issue: **Is $\nu_{t,r}$ a Dirac measure?** $\Longrightarrow$ **Compactness of $U^\varepsilon(t, x)$ in L^1**

Reduction of the Young Measure:

Scalar Conservation Laws: $u^\varepsilon \overset{*}{\rightharpoonup} u$, a.e.

$\hookrightarrow u(t,x) = \langle \nu_{t,x}, \lambda \rangle$

$$\left\langle \nu_{t,x}, \begin{vmatrix} \eta_1(\lambda) & q_1(\lambda) \\ \eta_2(\lambda), & q_2(\lambda) \end{vmatrix} \right\rangle = \begin{vmatrix} \langle \nu_{t,x}, \eta_1(\lambda)\rangle & \langle \nu_{t,x}, q_1(\lambda)\rangle \\ \langle \nu_{t,x}, \eta_2(\lambda)\rangle & \langle \nu_{t,x}\, q_2(\lambda)\rangle \end{vmatrix}$$

Choose. $(\eta_1(\lambda), q_1(\lambda)) = (\lambda - u(t,x), f(\lambda) - f(u(t,x)))$

$(\eta_2(\lambda), q_2(\lambda)) = (f(\lambda) - f(u(t,x)), \int_{u(t,x)}^{\lambda} (f'(s))^2 ds)$

$$\left\langle \nu_{t,x} \begin{vmatrix} \lambda - u & f(\lambda) - f(u) \\ f(\lambda) - f(u) & \int_u^\lambda (f'(s))^2 ds \end{vmatrix} \right\rangle$$

$$= \begin{vmatrix} \boxed{\langle \nu_{t,x}\, \lambda - u\rangle}^{=0} & \langle \nu_{t,x}\, f(\lambda) - f(u)\rangle \\ \langle \nu_{t,x}, f(\lambda) - f(u)\rangle & \langle \nu_{t,x}, \int_u^\lambda (f'(s))^2 ds\rangle \end{vmatrix}$$

$$\langle \nu_{t,x}, (\lambda - u) \int_u^\lambda (f'(s))^2 ds - (f(\lambda) - f(u))^2 \rangle$$
$$+ \langle \nu_{t,x}, f(\lambda) - f(u) \rangle^2 = 0$$

* G.-Q. Chen and Y.-G. Lu, *Chinese Sci. Bull.* 34 (1989), no. 1, 15–19.

$$(\lambda-u)\int_u^\lambda (f'(s))^2 ds - (f(\lambda)-f(u))^2$$

$$=(\lambda-u)\int_u^\lambda \left(f'(s)-\frac{1}{\lambda-u}\int_u^\lambda f'(\tau)d\tau\right)^2 ds \geq 0.$$

$\Rightarrow$

$$\begin{cases} \langle \nu_{t,x},\ f(\lambda)-f(u)\rangle = 0 \\ \langle \nu_{t,x},\ \underbrace{(\lambda-u)\int_u^\lambda \left(f'(s)-\frac{1}{\lambda-u}\int_u^\lambda f'(\tau)d\tau\right)^2 ds}\rangle = 0 \end{cases}$$

$$\| \\ (\lambda-u)\int_u^\lambda \left(\int_u^\lambda f''(\lambda+\theta(s-\tau))(s-\tau)d\tau\right)^2 ds$$

$\Rightarrow$

- $\langle \nu_{t,x}, f(\lambda)\rangle = f(u(t,x))$
- If $f''(u) \geq 0$

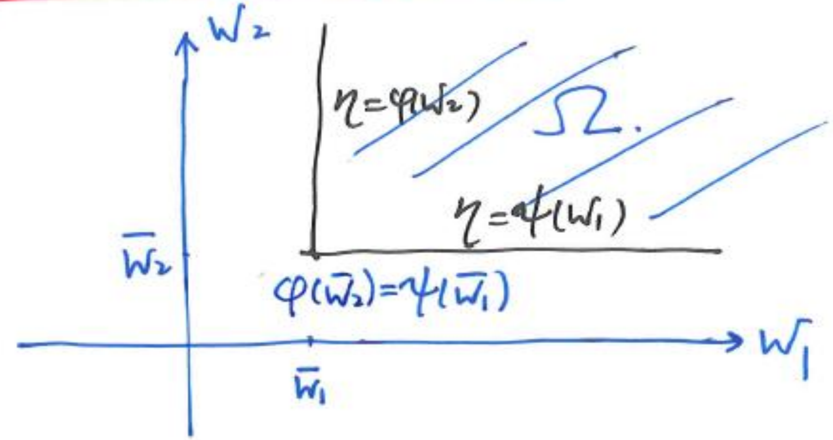
$\hookrightarrow$ $\boxed{\nu_{t,x} = \delta_{u(t,x)}}$

The Goursat Entropy Pairs for 2x2 Hyperbolic Systems of Conservation Laws

$$\eta_{w_1 w_2} - \frac{\lambda_{1w_2}}{\lambda_2 - \lambda_1}\eta_{w_1} + \frac{\lambda_{2w_1}}{\lambda_2 - \lambda_1}\eta_{w_2} = 0 \qquad (*)$$

$$q_{w_j} = \lambda_j \eta_{w_j} \qquad (**)$$

Goursat Problem for (*)



Well-posed !

Goursat Entropy Pairs

$\exists$ two families of entropy pairs:

$$\begin{cases} \eta_a(w) = I_1(w)\,a(w_1) + \int_{\bar{w}_1}^{w_1} J_1(\xi; w)\,a(\xi)\,d\xi \\ q_a(w) = K_1(w)\,a(w_1) + \int_{\bar{w}_1}^{w_1} L_1(\xi; w)\,a(\xi)\,d\xi \end{cases}$$

$$\begin{cases} \eta_b(w) = I_2(w)\,b(w_2) + \int_{\bar{w}_2}^{w_2} J_2(\xi; w)\,b(\xi)\,d\xi \\ q_b(w) = K_2(w)\,b(w_2) + \int_{\bar{w}_2}^{w_2} L_2(\xi; w)\,b(\xi)\,d\xi \end{cases}$$

where (I_i, J_i, K_i, L_i), $i = 1, 2$, are unique smooth functions and independent of $\bar{w}_1$ and $\bar{w}_2$:

$$\begin{cases} I_1(w_1, \bar{w}_2) = 1 \\ J_1(\xi; w_1, \bar{w}_2) = 0 \end{cases} \qquad \begin{cases} I_2(\bar{w}_1, w_2) = 1 \\ J_2(\xi; \bar{w}_1, w_2) = 0 \end{cases}$$

$$\begin{cases} I_i(w) > 0, \\ K_i = \lambda_i I_i \\ \dfrac{\partial K_i(w)}{\partial w_i} + L_i(w_i; w) = \lambda_i(w)\left(\dfrac{\partial I_i(w)}{\partial w_i} + J(w_i; w)\right) \\ \dfrac{\partial L_i(\xi; w)}{\partial w_i} = \lambda_i(w)\,\dfrac{\partial J_i(\xi; w)}{\partial w_i} \\ \dfrac{\partial K_i(w)}{\partial w_j} = \lambda_j(w)\,\dfrac{\partial I_i(w)}{\partial w_j}, \qquad i \neq j \\ \dfrac{\partial L_i(\xi; w)}{\partial w_j} = \lambda_j(w)\,\dfrac{\partial J_i(\xi; w)}{\partial w_j} \end{cases}$$

Reduction of the Young Measure

Thm. If $\frac{\partial \lambda_j}{\partial w_j} \neq 0$, $j=1,2$ (Genuinely Nonlinear)

$$\Longrightarrow \quad \nu_{t,x} = \delta_{u(t,x)}$$

Proof. If $\nu_{t,x} \neq \delta_{u(t,x)}$, we denote

$[w_1^-, w_1^+] \times [w_2^-, w_2^+]$ the smallest rectangle containing $\text{supp}\, \nu_{t,x}$.

1. **Claim.** If $w_1^- < w_1^+$, then $\exists\, c_1(t,x)$ s.t.

$$\langle \nu, q_a \rangle = c_1 \langle \nu, \eta_a \rangle$$

$\forall\, a \in C$, $a(w_1) = 0$ when $\begin{cases} w_1 \geq \bar{w}_1 \\ \text{or} \\ w_1 \leq \bar{w}_1 \end{cases}$ for $\bar{w}_1 \in (w_1^-, w_1^+)$

- Choose $\begin{cases} a_0(w_1) = (w_1 - w_1^*)_+, & w_1^* \geq \bar{w}_1, \quad |w_1^+ - w_1^*| \ll 1 \\ a(w_1) = 0, & w_1 \geq \bar{w}_1 \end{cases}$

$\hookrightarrow \begin{cases} \eta_{a_0}(w) > 0 & \forall\, w \in \{\bar{w}_1 < w_1 < w_1^+\} \cap \text{supp}\, \nu \\ \eta_{a_0} q_a - \eta_a q_{a_0} \equiv 0 \end{cases}$

$\hookrightarrow \langle \nu, \eta_{a_0} \rangle \langle \nu, q_a \rangle = \langle \nu, \eta_a \rangle \langle \nu, q_{a_0} \rangle$

$\hookrightarrow$ $$\langle v, \xi_a \rangle = C_1(\bar{w}_1) \langle v, \eta_a \rangle$$

$$\forall a \in C, \quad a(w_1) = 0, \quad w_1 \geq \bar{w}_1.$$

$$\left[C_1(\bar{w}_1) = \frac{\langle v, \xi_{a_0} \rangle}{\langle v, \eta_{a_0} \rangle} \right]$$

Similarly, $\forall a \in C, \quad a(w_1) = 0$ when $w_1 \leq \bar{w}_1$,

$$\langle v, \xi_a \rangle = C_1(\bar{w}_1) \langle v, \eta_a \rangle$$

- <u>claim</u> $C_1(\bar{w}_1) \not\rightarrow \bar{w}_1$.

For any $\tilde{w}_1 \in (w_1^-, w_1^+)$, choose

$$\begin{cases} a_1(w_1) = 0 & w_1 \leq \tilde{w}_1 \\ a_2(w_1) = 0 & w_1 \geq \bar{w}_1 \end{cases} \quad \text{for } \tilde{w}_1 < \bar{w}_1$$

$$\begin{cases} a_1(w_1) = 0 & w_1 \geq \tilde{w}_1 \\ a_2(w_1) = 0 & w_1 \leq \bar{w}_1 \end{cases} \quad \text{for } \tilde{w}_1 > \bar{w}_1$$

$\Rightarrow$ $$\eta_{a_1} \xi_{a_2} - \eta_{a_2} \xi_{a_1} \equiv 0$$

$\hookrightarrow$ $$C_1(\bar{w}_1) = C_1(\tilde{w}_1)$$

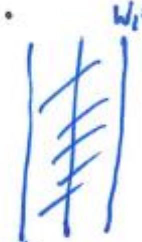
2. <u>claim</u> If $W_2^- < W_2^+$, then $\exists\ C_2(t,x)$ s.t.

$$\langle \nu, \xi_b \rangle = C_2 \langle \nu, \eta_b \rangle$$

$\forall\, b \in C$, $b(W_2) = 0$ when $\begin{cases} W_2 \geq \bar{W}_2 \\ \text{or } W_2 \leq \bar{W}_2 \end{cases}$ $\bar{W}_2 \in (W_2^-, W_2^+)$.

3. $\forall\, \alpha \in (W_1^-, W_1^+)$, choose W_1^*, $\bar{W}_1$ s.t.

$$W_1^* < \alpha < \bar{W}_1, \quad \bar{W}_1 - W_1^* << 1$$



Choose (η_a, ξ_a): $a(W_1) = (W_1 - W_1^*)_+$

choose $(\eta_{\bar{a}}, \xi_{\bar{a}})$: $\bar{a}(W_1) = (W_1 - \bar{W}_1)_-$

$$\langle \nu, \eta_a \xi_{\bar{a}} - \eta_{\bar{a}} \xi_a \rangle = \langle \nu, \eta_a \rangle \langle \nu, \xi_{\bar{a}} \rangle - \langle \nu, \eta_{\bar{a}} \rangle \langle \nu, \xi_a \rangle = 0$$

<u>We know</u> that, On $\{W_1 < W_1^*\} \cup \{W_1 > \bar{W}_1\}$,

$$\eta_a \xi_{\bar{a}} - \eta_{\bar{a}} \xi_a \equiv 0$$

On $\{w_1^* \le w_1 \le \bar{w}_1\}$.

$$\begin{cases} \eta_a(w) = I_1(w)(w_1 - w_1^*) + \frac{1}{2} J_1(\alpha; w)(w_1 - w_1^*)^2 + O(|w_1 - w_1^*|^3) \\ \xi_a(w) = K_1(w)(w_1 - w_1^*) + \frac{1}{2} L_1(\alpha; w)(w_1 - w_1^*)^2 + O(|w_1 - w_1^*|^3) \end{cases}$$

$$\begin{cases} \eta_{\bar{a}}(w) = I_1(w)(w_1 - \bar{w}_1) - \frac{1}{2} J_1(\alpha; w)(w_1 - \bar{w}_1)^2 + O(|w_1 - w_1^*|^3) \\ \xi_{\bar{a}}(w) = K_1(w)(w_1 - \bar{w}_1) - \frac{1}{2} L_1(\alpha; w)(w_1 - \bar{w}_1)^2 + O(|w_1 - w_1^*|^3) \end{cases}$$

$$(\eta_a \xi_{\bar{a}} - \eta_{\bar{a}} \xi_a)(w)$$
$$= \frac{1}{2}(\bar{w}_1 - w_1^*)(\bar{w}_1 - w_1)(w_1 - w_1^*)\left(I^2 \frac{\partial \lambda_1}{\partial w_1}\right)(\alpha, w_2)$$
$$+ O\left((\bar{w}_1 - w_1^*)^2(\bar{w}_1 - w_1)(w_1 - w_1^*)\right)$$

$$\Rightarrow \left\langle \nu, (\bar{w}_1 - w_1)_+ (w_1 - w_1^*)_+ \left(\underline{\left(I^2 \frac{\partial \lambda_1}{\partial w_1}\right)(\alpha, w_2)} + O(|\bar{w}_1 - w_1^*|) \right) \right\rangle = 0$$

$$\ne 0$$

$$\Rightarrow \quad \operatorname{Supp} \nu \cap \{w_1^* \le w_1 \le \bar{w}_1\} = \phi \quad \forall \; \begin{matrix} w_1^* < \bar{w}_1 \\ |\bar{w}_1 - w_1^*| << 1. \end{matrix}$$

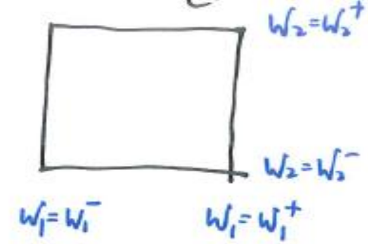
$$\hookrightarrow \quad \operatorname{Supp} \nu \cap \{w_1^- < w_1 < w_1^+\} = \phi$$

Similarly.

$$\hookrightarrow \quad \operatorname{Supp} \nu \cap \{w_2^- < w_2 < w_2^+\} = \phi$$

4. If

$$\mathrm{Supp}\,\nu \cap \left(\{W_1 = W_1^{\pm}\} \cup \{W_2 = W_2^{\pm}\} \right) \neq \phi$$



for example

$$\mathrm{Supp}\,\nu \cap \{W_1 = W_1^-\} \neq \phi$$

$$\Rightarrow \quad \nu(\{W_1 = W_1^-\}) \neq 0$$

then we follow Step 3 to choose

$$\alpha = W_1^-, \quad \overline{W_1} = W_1^- + \varepsilon, \quad W_1^* = W_1^- - \varepsilon.$$

to conclude

$$\mathrm{Supp}\,\nu \cap \{ W_1^- - \varepsilon \leq W_1 \leq W_1^- + \varepsilon \} = \phi$$

for sufficiently small $\varepsilon > 0$

$\Rightarrow$ Contradiction

5. Conclusion

$$\mathrm{Supp}\,\nu \cap \left([W_1^-, W_1^+] \times [W_2^-, W_2^+] \right) = \phi$$

$\Rightarrow$ Contradiction

$$\Longrightarrow \quad \nu_{t,x} = \delta_{u(t,x)} \quad \text{Single point support.}$$

1D Isentropic Euler Equations: Entropy-Entropy Flux Pairs

$$\begin{cases} \rho_t + m_x = 0, & (m = \rho v) \\ m_t + (\frac{m^2}{\rho} + p(\rho))_x = 0. \end{cases}$$

Strict Hyperbolicity: Fails when $\rho \to 0$ (vacuum)

Entropy Pair (η, q): $\nabla q(U) = \nabla \eta(U) \nabla F(U)$

Convex Entropy: $\nabla^2 \eta(U) > 0$ **Weak Entropy**: $\eta(\rho, \rho v)|_{\rho=0} = 0$

Weak entropy pairs are represented as

$$\eta^\psi(\rho, \rho v) = \int_{\mathbb{R}} \chi(s)\psi(s)\,\mathrm{d}s, \;\; q^\psi(\rho, \rho v) = \int_{\mathbb{R}} (\theta s + (1-\theta)v)\chi(s)\psi(s)\,\mathrm{d}s$$

by C^2-functions $\psi(s)$, where $\chi(s)$ is the weak entropy kernel:

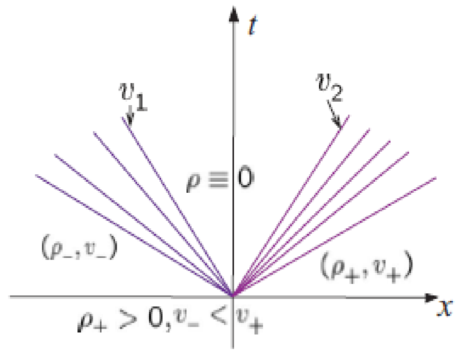
$$\chi(s) := [\rho^{2\theta} - (v-s)^2]_+^\lambda, \qquad \theta = \frac{\gamma - 1}{2}, \lambda = \frac{3-\gamma}{2(\gamma-1)}.$$

Physical Convex Entropy: Mechanical energy-energy flux pair (η_*, q_*):

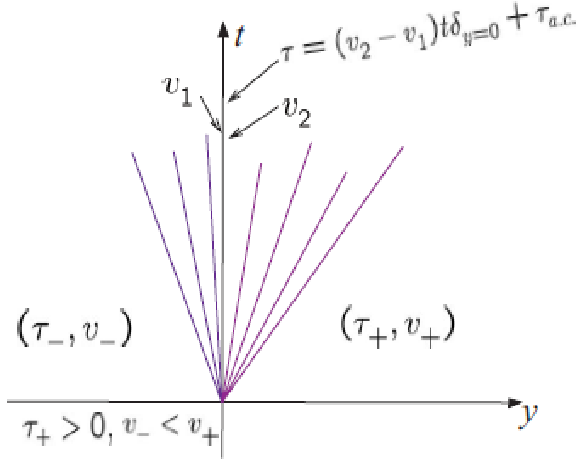
$$\eta_*(\rho, m) = \frac{1}{2}\frac{m^2}{\rho} + \rho e(\rho), \qquad q_*(\rho, m) = \frac{1}{2}\frac{m^3}{\rho^2} + m(e(\rho) + \frac{p}{\rho})$$

Cavitation and Concentration: Pressure $p(\rho) = a\rho^\gamma, \gamma > 1$

$$\partial_t \rho + \partial_x(\rho v) = 0, \qquad \partial_t(\rho v) + \partial_x(\rho v^2 + p(\rho)) = 0$$



$$(t,x) \to (t,y): \; y_t = \rho(t,x), y_x = -(\rho v)(t,x); \qquad \tau(t,y) = 1/\rho(t,x)$$



$$\partial_t \tau - \partial_y v = 0, \qquad \partial_t v + \partial_y p(1/\tau) = 0$$

Cavitation and Concentration: Pressure $p(\rho) = a\rho^\gamma, \gamma > 1$

$$\partial_t \rho + \partial_x(\rho v) = 0, \qquad \partial_t(\rho v) + \partial_x(\rho v^2 + p(\rho)) = 0$$

Theorem

Let the Cauchy initial data satisfy

$$0 \le \rho_0(x) \le C_0, \qquad |m_0(x)| \le C_0 \rho_0(x)$$

for some $C_0 > 0$. Then there exists a global entropy solution $(\rho, m)(t,x) = (\rho, \rho v)(t,x)$ of the Cauchy problem such that

$$0 \le \rho(t,x) \le C, \qquad |m(t,x)| \le C\rho(t,x),$$

where $C > 0$ is a constant depending only on $\gamma > 1$, $a > 0$, and $C_0 > 0$.

DiPerna: $\gamma = \frac{N+2}{N}, N \ge 5$ odd, Ding-Luo & Chen: $\gamma \in (1, \frac{5}{3}]$,
Lions-Perthame-Tadmor: $\gamma \ge 3$, Lions-Perthame-Souganidis: $\gamma \in (\frac{5}{3}, 3)$,
Chen-LeFloch: General pressure laws

Entropy Pairs and the Young Measure-Valued Solution $\nu_{t,r}$

Weak entropy pairs are represented as

$$\eta^\psi(\rho,\rho v)=\int_{\mathbb{R}}\chi(s)\psi(s)\,\mathrm{d}s,\quad q^\psi(\rho,\rho v)=\int_{\mathbb{R}}(\theta s+(1-\theta)v)\chi(s)\psi(s)\,\mathrm{d}s$$

by *compactly supported* C^2-test functions $\psi(s)$, for

$$\chi(s):=[\rho^{2\theta}-(u-s)^2]_+^\lambda,\qquad \theta=\frac{\gamma-1}{2},\lambda=\frac{3-\gamma}{2(\gamma-1)}.$$

Let $\nu_{t,r}$ be the Young measure determined by the solutions of the Navier-Stokes equations. Then $\nu_{t,r}$ **is confined by the following communtative relations**:

$$(\gamma-1)(s_2-s_1)\big(\overline{\chi(s_1)\chi(s_2)}-\overline{\chi(s_1)}\ \overline{\chi(s_2)}\big)$$
$$=(3-\gamma)\big(\overline{v\chi(s_2)}\ \overline{\chi(s_1)}-\overline{v\chi(s_1)}\ \overline{\chi(s_2)}\big)\qquad \text{for a.e. } s_1,s_2\in\mathbb{R}$$

where $\overline{f(s)}:=\langle\nu_{t,r},f(s;\rho,v)\rangle$.

- **If supp** $\nu_{t,r}$ **is bounded**, then $\nu_{t,r}(\boldsymbol{\lambda})=\nu_{(\rho(t,r),m(t,r))}(\boldsymbol{\lambda})$.

  DiPerna: $\gamma=\frac{N+2}{N}$, $N\geq 5$ odd, Ding-Luo & Chen: $\gamma\in(1,\frac{5}{3}]$,
  Lions-Perthame-Tadmor: $\gamma\geq 3$, Lions-Perthame-Souganidis: $\gamma\in(\frac{5}{3},3)$,
  Chen-LeFloch: General pressure laws

Measure-Valued Solution: Reduction for $\gamma = 3$ (cf. LPT)

When $\gamma = 3$, then the commutation relation becomes

$$\overline{\chi(s_1)\chi(s_2)} = \overline{\chi(s_1)}\ \overline{\chi(s_2)},$$

which implies

$$\overline{\chi(s)}^2 = \overline{\chi(s)^2},$$

by taking $s_1 = s_2$. That is,

$$\langle \nu_{t,r}, \left(\chi(s) - \overline{\chi(s)}\right)^2 \rangle = 0 \qquad \text{for any } s \in \mathbb{R}$$

This implies that ν must be a Dirac mass on the set $\{\rho > 0\}$ or be supported completely in the vacuum $V = \{\rho = 0\}$, that is, the measure-valued solution $\nu_{t,r}$ is a Dirac mass in the phase coordinates (ρ, m):

$$\nu_{t,r}(\rho, m) = \delta_{(\rho(t,r), m(t,r))}(\rho, m).$$

Bounded Supported Measure-Valued Solution for $\gamma > 3$ —-Arguments by Lions-Perthame-Tadmor

Let $A := \cup \{(u - \rho^\theta, \rho^\theta + u) : (\rho, u) \in \operatorname{supp} \nu\}$ be open set

Let $J = (s_-, s_+)$ be any bounded connected component of A

Note that $\operatorname{supp} \chi(s) = \{(\rho, u) : u - \rho^\theta \le s \le u + \rho^\theta\}$.

By definition of J, $\overline{\chi(s)} > 0$ for *a.e.* $s \in J$, so that

$$\frac{1-\theta}{\theta} \frac{1}{s_2 - s_1} \left(\frac{\overline{u\chi(s_2)}}{\overline{\chi(s_2)}} - \frac{\overline{u\chi(s_1)}}{\overline{\chi(s_1)}} \right) = \frac{\overline{\chi(s_1)\chi(s_2)}}{\overline{\chi(s_1)}\;\overline{\chi(s_2)}} - 1$$

- Taking $s_1, s_2 \to s$ yields $\frac{\partial}{\partial s}\left(\frac{\overline{u\chi(s)}}{\overline{\chi(s)}} \right) = \frac{1-\theta}{\theta} \left(\frac{\overline{\chi^2(s)}}{\left(\overline{\chi(s)}\right)^2} - 1 \right) \le 0.$

  Then the function $\frac{\overline{u\chi(s)}}{\overline{\chi(s)}}$ is non-increasing on J.

- Set $u_0 = \frac{s_- + s_+}{2}$. Then $\lim_{s \to s_+} \frac{\overline{u\chi(s)}}{\overline{\chi(s)}} \ge u_0 \ge \lim_{s \to s_-} \frac{\overline{u\chi(s)}}{\overline{\chi(s)}}$.
- Then $\frac{\overline{u\chi(s)}}{\overline{\chi(s)}}$ is constant, which implies $\overline{\chi(s)^2} = \overline{\chi(s)}^2$.

Measure-Valued Solution: Any Connected Component J of the Support Is Bounded for $\gamma > 3$

Strategy: On the contrary, let $\inf\{s : s \in J\} = -\infty$.
Fix $M_0 > 0$ such that $M_0 + 1 \in J$ and restrict $s_2 \in (M_0, M_0 + 1)$;
Choose $s_1 \le -2|M_0| \ll -1$ to reach the contradiction.

New Observation: $\int_{M_0}^{M_0+1} \frac{\overline{\chi(s_1)\chi(s_2)}}{\overline{\chi(s_1)}} ds_2 \le C(\lambda)|s_1|^{\lambda}, \qquad \lambda < 0.$

By LPT's argument: $\frac{\overline{\chi(s_1)\chi(s_2)}}{\overline{\chi(s_1)}} \ge \overline{\chi(s_2)}$ a.e. $s_1, s_2 \in J, s_1 < s_2,$

$$\Longrightarrow \int_{M_0}^{M_0+1} \frac{\overline{\chi(s_1)\chi(s_2)}}{\overline{\chi(s_1)}} ds_2 \ge \int_{M_0}^{M_0+1} \overline{\chi(s_2)} ds_2 = C(M_0, \lambda) > 0$$

Combining the TWO facts, we have

$$0 < C(M_0, \lambda) = \int_{M_0}^{M_0+1} \frac{\overline{\chi(s_1)\chi(s_2)}}{\overline{\chi(s_1)}} ds_2 \le C(\lambda)|s_1|^{\lambda},$$

which is a contradiction when $s_1 \to -\infty$.

*The case when J is unbounded from above can be treated similarly.

Measure-Valued Solution: Any Connected Component J of the Support Is Bounded for $\gamma \in (1,3)$, I: Strategy

On the contrary, suppose that J is unbounded from below.
Let $M_0 = \sup\{s : s \in J\} \in (-\infty, \infty]$.
Let $s_1, s_2, s_3 \in (-\infty, M_0)$ with $s_1 < s_2 < s_3$. The commutation relation $\Longrightarrow$

$$(s_2 - s_1)\frac{\overline{\chi(s_1)\chi(s_2)}}{\overline{\chi(s_1)}} + (s_3 - s_2)\frac{\overline{\chi(s_3)\chi(s_2)}}{\overline{\chi(s_3)}} = (s_3 - s_1)\overline{\chi(s_2)}\frac{\overline{\chi(s_1)\chi(s_3)}}{\overline{\chi(s_1)}\,\overline{\chi(s_3)}}.$$

Differentiating this equation in s_2 and dividing by $(s_3 - s_1)$, we obtain

$$\frac{s_2 - s_1}{s_3 - s_1}\frac{\overline{\chi(s_1)\chi'(s_2)}}{\overline{\chi(s_1)}} + \frac{s_3 - s_2}{s_3 - s_1}\frac{\overline{\chi(s_3)\chi'(s_2)}}{\overline{\chi(s_3)}} + \frac{1}{s_3 - s_1}\frac{\overline{\chi(s_1)\chi(s_2)}}{\overline{\chi(s_1)}}$$
$$-\frac{1}{s_3 - s_1}\frac{\overline{\chi(s_3)\chi(s_2)}}{\overline{\chi(s_3)}} = \overline{\chi'(s_2)}\frac{\overline{\chi(s_1)\chi(s_3)}}{\overline{\chi(s_1)}\,\overline{\chi(s_3)}}.$$

Strategy: Take $s_1 \to -\infty$ and show that the left-hand side has a smaller order than the right-hand side $\Longrightarrow$ Contradiction.

Measure-Valued Solution: Any Connected Component J of the Support Is Bounded for $\gamma \in (1,3)$, II: Steps

- As before: For any $s_1, s_3 \in J$, $\frac{\overline{\chi(s_1)\chi(s_3)}}{\overline{\chi(s_1)}\;\overline{\chi(s_3)}} \geq 1$;
- $\overline{\chi(s)} \geq 0$ is not identically zero and $\overline{\chi(s)} \to 0$ as $s \to \inf J, \sup J$,
  $\Longrightarrow$ there exists s_2 such that $\overline{\chi'(s_2)} > 0, \quad \overline{\chi(s_2)} > 0$.
- Let $s_3 > s_2$ be points such that $\overline{\chi(s_3)} > 0$ and let $s_1 \to -\infty$. From the 1st identity, $\frac{\overline{\chi(s_1)\chi(s_2)}}{\overline{\chi(s_1)}} = \overline{\chi(s_2)}\frac{\overline{\chi(s_1)\chi(s_3)}}{\overline{\chi(s_1)}\;\overline{\chi(s_3)}} + o(1)$, as $s_1 \to -\infty$.
- From the 2nd equation, by throwing away the negative terms, we obtain

$$\overline{\chi'(s_2)}\frac{\overline{\chi(s_1)\chi(s_3)}}{\overline{\chi(s_1)}\;\overline{\chi(s_3)}} \leq \Big(\frac{2\lambda}{s_2 - s_1} + \frac{1}{s_3 - s_1}\Big)\frac{\overline{\chi(s_1)\chi(s_2)}}{\overline{\chi(s_1)}} + o(1).$$

$$\Longrightarrow \qquad \Big(\overline{\chi'(s_2)} - \frac{2\lambda\overline{\chi(s_2)}}{s_2 - s_1} - \frac{\overline{\chi(s_2)}}{s_3 - s_1}\Big)\frac{\overline{\chi(s_1)\chi(s_3)}}{\overline{\chi(s_1)}\;\overline{\chi(s_3)}} \leq o(1).$$

Contradiction as $s_1 \to -\infty$.

*The proof above is much simpler than that by LeFloch-Westdickenberg 2009.

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Hyperbolic Conservation Laws

$$\partial_t \mathbf{u} + \nabla \cdot \mathbf{f}(\mathbf{u}) = 0$$

$\mathbf{u} = (u_1, \cdots, u_m)^\top$, $\mathbf{x} = (x_1, \cdots, x_d)$, $\nabla = (\partial_{x_1}, \cdots, \partial_{x_d})$

$\mathbf{f} = (\mathbf{f}_1, \cdots, \mathbf{f}_d) : \mathbb{R}^m \to (\mathbb{R}^m)^d$ is a nonlinear mapping

$\mathbf{f}_i : \mathbb{R}^m \to \mathbb{R}^m$ for $i = 1, \cdots, d$

$$\partial_t \mathbf{A}(\mathbf{u}, \mathbf{u}_t, \nabla \mathbf{u}) + \nabla \cdot \mathbf{B}(\mathbf{u}, \mathbf{u}_t, \nabla \mathbf{u}) = 0$$

$\mathbf{A}, \mathbf{B} : \mathbb{R}^m \times \mathbb{R}^m \times (\mathbb{R}^m)^d \to \mathbb{R}^m$ are nonlinear mappings

Connections and Applications:

- Fluid Mechanics and Related: Euler Equations and Related Equations
  Gas, shallow water, elastic body, reacting gas, plasma, ...
- Special Relativity: Relativistic Euler Equations and Related Equations
  General Relativity: Einstein Equations and Related Equations
- Differential Geometry: Isometric Embeddings, Nonsmooth Manifolds..
- ......

Convex Entropy and Hyperbolicity

Entropy: $\eta : \mathbb{R}^m \to \mathbb{R}$ if there exists $\mathbf{q}$:

$$\mathbf{q} = (q_1, \cdots, q_d) : \mathbb{R}^m \to \mathbb{R}^d,$$

satisfying $\nabla q_i(\mathbf{u}) = \nabla \eta(\mathbf{u}) \nabla \mathbf{f}_i(\mathbf{u}), \;\; i = 1, \cdots, d$

Convex entropy $\eta(\mathbf{u})$: $\nabla^2 \eta(\mathbf{u}) \geq 0$

Strictly convex entropy $\eta(\mathbf{u})$: $\nabla^2 \eta(\mathbf{u}) > 0$

Entropy inequality: For any convex $(\eta, \mathbf{q}) \in C^2$.

$$\partial_t \eta(\mathbf{u}) + \nabla \cdot \mathbf{q}(\mathbf{u}) \leq 0 \qquad \mathcal{D}'$$

Theorem. If system (*) is endowed with a strictly convex entropy η in a state domain D, then system (*) must be hyperbolic and symmetrizable in D.

Proof —I: Sketch

1. Taking $\nabla_{\mathbf{u}}$ both sides: $\nabla_{\mathbf{u}}\eta(\mathbf{u})\,\nabla_{\mathbf{u}}\mathbf{f}_k(\mathbf{u}) = \nabla_{\mathbf{u}}\mathbf{q}_k(\mathbf{u})$, $k = 1, 2, \cdots, d$, to obtain

$$\nabla^2_{\mathbf{u}}\eta(\mathbf{u})\,\nabla_{\mathbf{u}}\mathbf{f}_k(\mathbf{u}) + \nabla_{\mathbf{u}}\eta(\mathbf{u})\,\nabla^2_{\mathbf{u}}\mathbf{f}_k(\mathbf{u}) = \nabla^2_{\mathbf{u}}\mathbf{q}_k(\mathbf{u}).$$

Using the **symmetry** of the matrices $\nabla_{\mathbf{u}}\eta(\mathbf{u})\nabla^2_{\mathbf{u}}\mathbf{f}_k(\mathbf{u})$ and $\nabla^2_{\mathbf{u}}\mathbf{q}_k(\mathbf{u})$, we find that, for fixed $k = 1, 2, \cdots, d$,

$$\nabla^2_{\mathbf{u}}\eta(\mathbf{u})\nabla_{\mathbf{u}}\mathbf{f}_k(\mathbf{u}) \qquad \textbf{is symmetric.}$$

2. Multiplying system (*) by $\nabla^2_{\mathbf{u}}\eta(\mathbf{u})$ yields

$$\nabla^2_{\mathbf{u}}\eta(\mathbf{u})\partial_t\mathbf{u} + \nabla^2_{\mathbf{u}}\eta(\mathbf{u})\nabla_{\mathbf{u}}\mathbf{f}(\mathbf{u}) \cdot \nabla_{\mathbf{x}}\mathbf{u} = 0. \qquad (**)$$

Since $\nabla^2_{\mathbf{u}}\eta(\mathbf{u}) > 0$, the hyperbolicity of (*) and the hyperbolicity of (**) is equivalent.

The hyperbolicity of (**) is equivalent to:

For any $\omega \in S^{d-1}$, all zeros of the determinant

$$|\lambda\nabla^2_{\mathbf{u}}\eta(\mathbf{u}) - \nabla^2_{\mathbf{u}}\eta(\mathbf{u})\nabla_{\mathbf{u}}\mathbf{f}(\mathbf{u}) \cdot \omega| \qquad \text{are real.}$$

Proof —II: Sketch

3. Since $\nabla^2\eta(\mathbf{u})$ is a real symmetric, positive definite matrix, then there exists a matrix $C(\mathbf{u})$ such that

$$\nabla^2\eta(\mathbf{u}) = C(\mathbf{u})C(\mathbf{u})^\top.$$

Then it is equivalent to showing that,

For any $\omega \in S^{d-1}$, the eigenvalues of the following matrix

$$C(\mathbf{u})^{-1}\left(\nabla^2\eta(\mathbf{u})\nabla\mathbf{f}(\mathbf{u})\cdot\omega\right)\left(C(\mathbf{u})^{-1}\right)^\top \qquad \text{are real.}$$

This is TRUE since the matrix is real and symmetric.

Remarks.

1. The proof is taken from Friedrich-Lax 1971
2. A system of conservation laws is endowed with a strictly convex entropy if and only if the system is conservatively symmetrizable.

   Friedrich-Lax 1971

   Godunov 1961, 1978, 1987; Boillat 1965; Mock (Sever) 1980;

Conservatively Symmetrizable: Godunov 1961

There exists an invertible change of variables $\mathbf{u} = \Phi(\mathbf{w}) \in \mathbb{R}^m$, with inverse $\mathbf{w} = \Psi(\mathbf{u})$, such that

- $\Phi(\mathbf{w})$ is the gradient (with respect to $\mathbf{w}$) of a scalar map $a_0 : \mathbb{R}^m \to \mathbb{R}$ with $\nabla_{\mathbf{w}}\Phi(\mathbf{w}) = \nabla^2_{\mathbf{w}} a_0(\mathbf{w})$ strictly positive definite.
- $\mathbf{f}(\Phi(\mathbf{w}))$ is the gradient (with respect to $\mathbf{w}$) of a vector may $(a_1, \cdots, a_d) : \mathbb{R}^m \to \mathbb{R}^d$.

Then system (*) can be written as

$$\partial_t(\nabla_{\mathbf{w}} a_0(\mathbf{w})) + \sum_{j=1}^{d} \partial_{x_j}(\nabla_{\mathbf{w}} a_j(\mathbf{w})) = 0,$$

or, equivalently, as $\mathbf{A}_0(\mathbf{w})\partial_t \mathbf{w} + \sum_{j=1}^{d} \mathbf{A}_j(\mathbf{w})\partial_{x_j}\mathbf{w} = 0$
where the $m \times m$ matrices $\mathbf{A}_i, i = 0, 1, \cdots, d$, are symmetric Jacobians.

Remarks:

- $\mathbf{u} \to \mathbf{w} = \Psi(\mathbf{u}) = \nabla_{\mathbf{u}}\eta(\mathbf{u})$
- $\eta(\mathbf{u}) = \mathbf{u} \cdot \Psi(\mathbf{u}) - a_0(\Psi(\mathbf{u})), \quad \mathbf{q}(\mathbf{u}) = \mathbf{u} \cdot \Psi(\mathbf{u}) - (a_1, \cdots, a_j)(\Psi(\mathbf{u}))$

Applications I: Local Existence and Stability

- **Local Existence of Classical Solutions**

$$\mathbf{u}_0 \in H^s \cap L^\infty, s > \frac{d}{2} + 1 \Longrightarrow \mathbf{u} \in C([0,T]; H^s) \cap C^1([0,T]; H^{s-1})$$

Kato 1975, Majda 1984

Makino-Ukai-Kawashima 1986, Chemin 1990, $\cdots$

- **Local Existence and Stability of Shock Front Solutions**

$$\mathbf{u}(\mathbf{x}) = \begin{cases} \mathbf{u}^+(t,\mathbf{x}), & (t,\mathbf{x}) \in S^+, \\ \mathbf{u}^-(t,\mathbf{x}), & (t,\mathbf{x}) \in S^- \end{cases}$$

Majda 1983, Métivier 1990, $\cdots$

The symmetry plays an essential role in the following situation:

$$\begin{aligned} &2\mathbf{u}^\top \nabla^2\eta(\mathbf{v})\nabla\mathbf{f}_k(\mathbf{v})\partial_{x_k}\mathbf{u} \\ &= \partial_{x_k}(\mathbf{u}^\top\nabla^2\eta(\mathbf{v})\nabla\mathbf{f}_k(\mathbf{v})\mathbf{u}) - \mathbf{u}^\top\partial_{x_k}(\nabla^2\eta(\mathbf{v})\nabla\mathbf{f}_k(\mathbf{v}))\mathbf{u} \end{aligned}$$

to get the first energy estimate (the L^2 estimate)

Applications II: Stability of Lipschitz Solutions–1

$\mathbf{v} \in K$ is a Lipschitz solution on $[0, T)$ with initial data $\mathbf{v}_0(\mathbf{x})$
$\mathbf{u} \in K$ is any entropy solution on $[0, T)$ with initial data $\mathbf{u}_0(\mathbf{x})$

$$\int_{|\mathbf{x}|<R} |\mathbf{u}(t,\mathbf{x}) - \mathbf{v}(t,\mathbf{x})|^2 d\mathbf{x} \le C(T) \int_{|\mathbf{x}|<R+Lt} |\mathbf{u}_0(\mathbf{x}) - \mathbf{v}_0(\mathbf{x})|^2 d\mathbf{x}$$

Sketch of Proof: Assume that $\nabla^2_{\mathbf{u}}\eta(\mathbf{u}) \ge c_0 > 0$

1. Use the Dafermos relative entropy and entropy flux pair:

$$\bar{\eta}(\mathbf{u},\mathbf{v}) = \eta(\mathbf{u}) - \eta(\mathbf{v}) - \nabla\eta(\mathbf{v})(\mathbf{u}-\mathbf{v}) \ge c_0(\mathbf{u}-\mathbf{v})^2,$$
$$\bar{\mathbf{q}}(\mathbf{u},\mathbf{v}) = \mathbf{q}(\mathbf{u}) - \mathbf{q}(\mathbf{v}) - \nabla\eta(\mathbf{v})(\mathbf{f}(\mathbf{u}) - \mathbf{f}(\mathbf{v}))$$

and compute to find

$$\partial_t\bar{\eta}(\mathbf{u},\mathbf{v}) + \nabla_{\mathbf{x}} \cdot \bar{\mathbf{q}}(\mathbf{u},\mathbf{v})$$
$$\le -\{\partial_t(\nabla\eta(\mathbf{v}))(\mathbf{u}-\mathbf{v}) + \sum_{k=1}^{d} \partial_{x_k}(\nabla\eta(\mathbf{v}))(\mathbf{f}_k(\mathbf{u}) - \mathbf{f}_k(\mathbf{v}))\}.$$

Applications III: Stability of Lipschitz Solutions–2

2. Since $\mathbf{v}$ is a classical solution, we use the symmetry property with the strictly convex entropy η to have

$$\partial_t(\nabla\eta(\mathbf{v})) = (\partial_t\mathbf{v})^\top\nabla^2\eta(\mathbf{v}) = -\sum_{k=1}^{d}(\partial_{x_k}\mathbf{f}_k(\mathbf{v}))^\top\nabla^2\eta(\mathbf{v})$$

$$= -\sum_{k=1}^{d}(\partial_{x_k}\mathbf{v})^\top(\nabla\mathbf{f}_k(\mathbf{v}))^\top\nabla^2\eta(\mathbf{v}) = -\sum_{k=1}^{d}(\partial_{x_k}\mathbf{v})^\top(\nabla^2\eta(\mathbf{v})\nabla\mathbf{f}_k(\mathbf{v}))^\top$$

$$= -\sum_{k=1}^{d}(\partial_{x_k}\mathbf{v})^\top\nabla^2\eta(\mathbf{v})\nabla\mathbf{f}_k(\mathbf{v}).$$

$$\Longrightarrow \qquad \partial_t\bar{\eta}(\mathbf{u},\mathbf{v}) + \nabla_{\mathbf{x}}\cdot\bar{\mathbf{q}}(\mathbf{u},\mathbf{v})$$

$$\leq -\sum_{k=1}^{d}(\partial_{x_k}\mathbf{v})^\top\nabla^2\eta(\mathbf{v})\big(\mathbf{f}_k(\mathbf{u}) - \mathbf{f}_k(\mathbf{v}) - \nabla\mathbf{f}_k(\mathbf{v})(\mathbf{u}-\mathbf{v})\big)$$

Integrating over a set $\{(\tau,\mathbf{x}) : 0 \leq \tau \leq t \leq T, |\mathbf{x}| \leq R + L(t-\tau)\}$ for $L \gg 0$ and employing the Gronwall inequality to conclude the result.

Applications III: Remarks

1. **The proof is taken from Dafermos 2002**
   Also Dafermos 1979 and DiPerna 1979

2. **The stability of rarefaction waves for the Euler equations for multidimensional compressible fluids also holds**:
   G.-Q. Chen & J. Chen: JHDE 2007

3. **Multidimensional hyperbolic systems of conservation laws with partially convex entropies and involutions**: Dafermos 2002
   Also Dafermos 1986, Boillat 1988.

4. **For multidimensional hyperbolic systems of conservation laws without a strictly convex entropy, it is possible to enlarge the system so that the enlarged system is endowed with a globally defined, strictly convex entropy.**

   Elastodynamics: Isentropic Model

   Electromagnetism: Born-Infeld Nonlinear Model

Strict Hyperbolicity

Lax 1982, Friedland-Robin-Sylvester 1984:

For $d = 3$, there are no strictly hyperbolic systems when

$$m \equiv \pm 2, \pm 3, \pm 4 \pmod 8$$

Theorem. Let A, B, C be the three matrices such that

$$\alpha A + \beta B + \gamma C$$

has real eigenvalues for any real α, β, γ.
When

$$m \equiv \pm 2, \pm 3, \pm 4 \pmod 8,$$

then there exist $(\alpha_0, \beta_0, \gamma_0), \alpha_0^2 + \beta_0^2 + \gamma_0^2 \neq 0$ such that

$$\alpha_0 A + \beta_0 B + \gamma_0 C$$

is degenerate, that is, there are two eigenvalues of the matrix which coincide.

Proof—I: We prove only the case $m \equiv 2 (\bmod 4)$

1. Denote $\mathcal{M}$ the set of all real $m \times m$ matrices with real eigenvalues

Denote $\mathcal{N}$ the set of nondegenerate matrices that have m distinct real eigenvalues in $\mathcal{M}$

The normalized eigenvectors $\mathbf{r}_j$ of $N \in \mathcal{N}$

$$N\mathbf{r}_j = \lambda_j \mathbf{r}_j, \qquad |\mathbf{r}_j| = 1, j = 1, 2, \cdots, m,$$

are determined up to a factor ± 1.

2. Let $N(\theta), 0 \leq \theta \leq 2\pi$, be a closed curve in $\mathcal{N}$ (if exists!).

If we fix $\mathbf{r}_j(0)$, then $\mathbf{r}_j(\theta)$ can be determined uniquely by requiring continuous dependence on θ. Since $N(2\pi) = N(0)$, then

$$\mathbf{r}_j(2\pi) = \tau_j \mathbf{r}_j(0), \qquad \tau_j = \pm 1.$$

Clearly,

(i) Each τ_j is a homotopy invariant of the closed curve;

(ii) Each $\tau_j = 1$ when $N(\theta)$ is constant.

Proof—II: $m \equiv 2 \pmod 4$

3. Suppose now that the theorem is false. Then

$$N(\theta) = cos\theta A + sin\theta B$$

is a closed curve in $\mathcal{N}$ and $\lambda_1(\theta) < \lambda_2(\theta) < \cdots < \lambda_m(\theta)$.

Since $N(\pi) = -N(0)$, we have

$$\lambda_j(\pi) = -\lambda_{m-j+1}(0), \quad \mathbf{r}_j(\pi) = \rho_j \mathbf{r}_{m-j+1}(0), \qquad \rho_j = \pm 1.$$

Since the ordered basis $\{\mathbf{r}_1(\theta), \mathbf{r}_2(\theta), \cdots, \mathbf{r}_m(\theta)\}$ is defined continuously, it retains its orientation. Then the ordered bases

$$\{\mathbf{r}_1(0), \mathbf{r}_2(0), \cdots, \mathbf{r}_m(0)\} \quad \text{and} \quad \{\rho_1 \mathbf{r}_m(0), \rho_2 \mathbf{r}_{m-1}(0), \cdots, \rho_m \mathbf{r}_1(0)\}$$

have the same orientation.
Since $m \equiv 2 \pmod 4$, reversing the order reverses the orientation of an ordered basis, which implies $\Pi_{j=1}^{m} \rho_j = -1$ (exercise?). Then there exists k such that

$$\rho_k \rho_{m-k+1} = -1.$$

Proof—III: $m \equiv 2 (\mathrm{mod}\, 4)$

Since $N(\theta + \pi) = -N(\theta)$, then

$$\lambda_j(\theta + \pi) = -\lambda_{m-j+1}(\theta),$$

which implies $\mathbf{r}_j(2\pi) = \rho_j \mathbf{r}_{m-j+1}(\pi) = \rho_j \rho_{m-j+1} \mathbf{r}_{m-j+1}(0).$
Therefore, we have

$$\tau_j = \rho_j \rho_{m-j+1}.$$

Then Step 3 implies $\tau_k = -1$, which yields that the curve

$$N(\theta) = cos\theta A + sin\theta B \qquad \text{is not homotopic to a point.}$$

4. Suppose that all matrices of form

$$\alpha A + \beta B + \gamma C, \quad \alpha^2 + \beta^2 + \gamma^2 = 1, \qquad \text{belong } \mathcal{N}.$$

Then, since the sphere is simply connected, the curve $N(\theta)$ could be contracted to a point, contracting $\tau_k = -1$.
This completes the proof.

Isentropic Euler Equations

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho \mathbf{v}) = 0 \\ \partial_t (\rho \mathbf{v}) + \nabla \cdot (\rho \mathbf{v} \otimes \mathbf{v}) + \nabla p = 0 \end{cases}$$

where the pressure is regarded as a function of density with constant S_0:

$$p = p(\rho, S_0)$$

For a polytropic gas,

$$p(\rho) = \kappa_0 \rho^\gamma, \qquad \gamma > 1,$$

where $\kappa_0 > 0$ is any constant under scaling

Isentropic Euler Equations

Case $d = 2, m = 3$: Strictly hyperbolic

$$\lambda_- < \lambda_0 < \lambda_+, \qquad \text{when } \rho > 0$$

$$\lambda_0 = \omega_1 u_1 + \omega_2 u_2, \quad \lambda_\pm = \omega_1 u_1 + \omega_2 u_2 \pm \sqrt{p'(\rho)}$$

Case $d = 3, m = 4$: Nonstrictly hyperbolic since

$$\lambda_0 = \omega_1 u_1 + \omega_2 u_2 + \omega_3 u_3$$

has double multiplicity, with

$$\lambda_\pm = \omega_1 u_1 + \omega_2 u_2 + \omega_3 u_3 \pm \sqrt{p'(\rho)}$$

Full Euler Equations

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho \mathbf{v}) = 0, \\ \partial_t (\rho \mathbf{v}) + \nabla \cdot (\rho \mathbf{v} \otimes \mathbf{v}) + \nabla p = 0, \\ \partial_t (\rho E) + \nabla \cdot (\rho \mathbf{v}(E + \frac{p}{\rho})) = 0 \end{cases} \quad (t, \mathbf{x}) \in \mathbb{R}^{d+1}_+ := (0, \infty) \times \mathbb{R}^d$$

Constitutive Relations: $p = p(\rho, e), \quad E = \frac{1}{2}|\mathbf{v}|^2 + e$

$\tau = \frac{1}{\rho}$ —Deformation gradient (specific volume for fluids, strain for solids)

$\mathbf{v} = (v_1, \cdots, v_d)^\top$ —Fluid velocity with $\mathbf{m} = \rho \mathbf{v}$ the momentum vector

p —Scalar pressure

E —Total energy with e the internal energy which is a given function of (τ, p) or (ρ, p) defined through thermodynamical relations

The notation $\mathbf{a} \otimes \mathbf{b}$ denotes the tensor product of the vectors $\mathbf{a}$ and $\mathbf{b}$

Full Euler Equations

Case $d = 2$, $m = 4$: Nonstrictly hyperbolic since

$$\lambda_0 = \omega_1 u_1 + \omega_2 u_2$$

has double multiplicity, with

$$\lambda_\pm = \omega_1 u_1 + \omega_2 u_2 \pm \sqrt{\gamma p / \rho}$$

Case $d = 3$, $m = 5$: Nonstrictly hyperbolic since

$$\lambda_0 = \omega_1 u_1 + \omega_2 u_2 + \omega_3 u_3$$

has triple multiplicity, with

$$\lambda_\pm = \omega_1 u_1 + \omega_2 u_2 + \omega_3 u_3 \pm \sqrt{\gamma p / \rho}$$

Genuine Nonlinearity

$$\nabla_{\mathbf{u}}\lambda_j(\mathbf{u};\omega)\cdot\mathbf{r}_j(\mathbf{u};\omega)\neq 0 \qquad \text{for any } \omega\in S^{d-1}$$

Theorem. Any scalar quasilinear conservation law in d-space dimension ($d\geq 2$) is never genuinely nonlinear in all directions.

In this case, $\lambda(u;\omega)=\mathbf{f}'(u)\cdot\omega$ and $r=1$,

$$\lambda'(u;\omega)r\equiv\mathbf{f}'(u)\cdot\omega$$

Impossible to make this never equals to zero.

Generalization: Genuine Nonlinearity:

$$|\{u\,:\,\tau+\mathbf{f}'(u)\cdot\omega=0\}|=0 \qquad \text{for any } (\tau,\omega)\in S^{d+1}$$

Under this strong nonlinearity:

(i) Solution operators are compact:

Lions-Perthame-Tadmor 1994, Tao-Tadmor 2007

(ii) Decay of periodic solutions: Chen-Frid 1999

(iii) Trace of entropy solutions: Chen-Rascle 2000, Vasseur 2001, $\cdots$

(iv) Structure of L^∞ entropy solutions: Otto-DeLellis-Westdickenberg 2003

Genuine Nonlinearity

Theorem (Lax 1984). Every real, strictly hyperbolic quasilinear system for

$$d = 2, \ \ m = 2k, \qquad k \geq 1 \text{ odd},$$

is linearly degenerate in some direction.

Proof. We prove only for the case $m = 2$.
1. For fixed $\mathbf{u} \in \mathbb{R}^m$, define $C(\theta; \mathbf{u}) = \nabla \mathbf{f}_1(\mathbf{u})\, cos\theta + \nabla \mathbf{f}_2(\mathbf{u})\, sin\theta$.
Denote the eigenvalues of $C(\theta; \mathbf{u})$ by $\lambda_\pm(\theta; \mathbf{u})$: $\lambda_-(\theta; \mathbf{u}) < \lambda_+(\theta; \mathbf{u})$ with

$$C(\theta; \mathbf{u})\mathbf{r}_\pm(\theta; \mathbf{u}) = \lambda_\pm(\theta; \mathbf{u})\mathbf{r}_\pm(\theta; \mathbf{u}), \quad |\mathbf{r}_\pm(\theta; \mathbf{u})| = 1.$$

This still leaves an arbitrary factor ± 1, which we fix arbitrarily at $\theta = 0$.
For all other $\theta \in [0, 2\pi]$ by requiring $\mathbf{r}_\pm(\theta; \mathbf{u})$ to vary continuously with θ.
2. Since $C(\theta + \pi; \mathbf{u}) = -C(\theta; \mathbf{u})$,

$$\lambda_+(\theta + \pi; \mathbf{u}) = -\lambda_-(\theta; \mathbf{u}), \ \ \lambda_-(\theta + \pi; \mathbf{u}) = -\lambda_+(\theta; \mathbf{u}).$$

It follows from this and $|\mathbf{r}_\pm| = 1$ that

$$\mathbf{r}_+(\theta + \pi; \mathbf{u}) = \sigma_+ \mathbf{r}_-(\theta; \mathbf{u}), \ \ \mathbf{r}_-(\theta + \pi; \mathbf{u}) = \sigma_- \mathbf{r}_+(\theta; \mathbf{u}), \ \ \text{with } \sigma_\pm = 1 \text{ or } -1.$$

Genuine Nonlinearity

3. Since $\mathbf{r}_\pm(\theta; \mathbf{u})$ were chosen to be continuous functions of θ, we have

(i) $\sigma_\pm$ are also continuous functions of θ and, thus, they must be constant since $\sigma_\pm = \pm 1$;

(ii) The orientation of the ordered basis: $\{\mathbf{r}_-(\theta; \mathbf{u}),\ \mathbf{r}_+(\theta; \mathbf{u})\}$ does not change and, hence, the bases

$$\{\mathbf{r}_-(0; \mathbf{u}),\ \mathbf{r}_+(0; \mathbf{u})\} \text{ and } \{\mathbf{r}_-(\pi; \mathbf{u}),\ \mathbf{r}_+(\pi; \mathbf{u})\}$$

have the same orientation.

Therefore, by Step 2,

$$\{\mathbf{r}_-(0; \mathbf{u}),\ \mathbf{r}_+(0; \mathbf{u})\} \text{ and } \{\sigma_-\mathbf{r}_+(0; \mathbf{u}),\ \sigma_+\mathbf{r}_-(0; \mathbf{u})\}$$

have the same orientation. Then

$$\sigma_+\sigma_- = -1, \quad \mathbf{r}_+(2\pi; \mathbf{u}) = \sigma_+\mathbf{r}_-(\pi; \mathbf{u}) = \sigma_+\sigma_-\mathbf{r}_+(0, \mathbf{u}) = -\mathbf{r}_+(0, \mathbf{u}).$$

Similarly, we have

$$\mathbf{r}_-(2\pi; \mathbf{u}) = -\mathbf{r}_-(0; \mathbf{u}).$$

Genuine Nonlinearity

4. Since the eigenvalues $\lambda_\pm(\theta; \mathbf{u})$ are periodic functions of θ with period 2π for fixed $\mathbf{u} \in \mathbb{R}^2$, so are their gradients. Then

$$\nabla_\mathbf{u}\lambda_\pm(2\pi; \mathbf{u}) \cdot \mathbf{r}_\pm(2\pi; \mathbf{u}) = -\nabla_\mathbf{u}\lambda_\pm(0; \mathbf{u}) \cdot \mathbf{r}_\pm(0; \mathbf{u}).$$

Noticing that

$$\nabla_\mathbf{u}\lambda_\pm(\theta; \mathbf{u}) \cdot \mathbf{r}_\pm(\theta; \mathbf{u})$$

varies continuously with θ for any fixed $\mathbf{u} \in \mathbb{R}^2$, we conclude that there exists $\theta_\pm \in (0, 2\pi)$ such that

$$\nabla_\mathbf{u}\lambda_\pm(\theta_\pm; \mathbf{u}) \cdot \mathbf{r}_\pm(\theta_\pm; \mathbf{u}) = 0.$$

This completes the proof.

Exercise: Give a detailed proof for the general case $m = 2k$, $k \geq 1$ odd.

Euler Equations: $d = 2$

Isentropic Euler Equations: $m = 3$

$$\lambda_0 = \omega_1 u_1 + \omega_2 u_2, \quad \lambda_\pm = \omega_1 u_1 + \omega_2 u_2 \pm \sqrt{p'(\rho)},$$

$$\mathbf{r}_0 = (-\omega_2, \omega_1, 0)^\top, \quad \mathbf{r}_\pm = (\pm\omega_1, \pm\omega_2, \frac{\rho}{\sqrt{p'(\rho)}})^\top,$$

which implies

$$\nabla\lambda_0 \cdot \mathbf{r}_0 \equiv 0, \quad \nabla\lambda_\pm \cdot \mathbf{r}_\pm = \pm\frac{\rho p''(\rho) + 2p'(\rho)}{2p'(\rho)}.$$

Full Euler Equations: $m = 4$

$$\lambda_0 = \omega_1 u_1 + \omega_2 u_2, \quad \lambda_\pm = \omega_1 u_1 + \omega_2 u_2 \pm \sqrt{\gamma p/\rho},$$

$$\mathbf{r}_0 = (-\omega_2, \omega_1, 0, 1)^\top, \quad \mathbf{r}_\pm = (\pm\omega_1, \pm\omega_2, \sqrt{\gamma p \rho}, \rho\frac{\rho}{\gamma p})^\top,$$

which implies

$$\nabla\lambda_0 \cdot \mathbf{r}_0 \equiv 0, \quad \nabla\lambda_\pm \cdot \mathbf{r}_\pm = \pm\frac{\gamma + 1}{2} \neq 0.$$

Remark

Quite often, linear degeneracy results from the loss of strict hyperbolicity.

For example, even in the one-dimensional case:

If there exists $j \neq k$ such that

$$\lambda_j(\mathbf{u}) = \lambda_k(\mathbf{u}) \qquad \text{for all } \mathbf{u} \in K,$$

then Boillat (1972) proved that

the $j-$ and $k-$characteristic families are linearly degenerate in K.

Singularities $\Longrightarrow$ Discontinuous/Singular Solutions

Cauchy Problem in $\mathbb{R}^3$ for polytropic gases with smooth initial data:

$$(\rho, \mathbf{v}, S)|_{t=0} = (\rho_0, \mathbf{v}_0, S_0)(\mathbf{x}), \quad \rho_0(\mathbf{x}) > 0, \ \mathbf{x} \in \mathbb{R}^3,$$

satisfying

$$(\rho_0, \mathbf{v}_0, S_0)(\mathbf{x}) = (\bar{\rho}, 0, \bar{S}) \qquad \text{for} \quad |\mathbf{x}| \geq R, \tag{1}$$

where $\bar{\rho} > 0$, $\bar{S}$, and R are given constants.

The support of the smooth disturbance $(\rho_0(\mathbf{x}) - \bar{\rho}, \mathbf{v}_0(\mathbf{x}), S_0(\mathbf{x}) - \bar{S})$ propagates with speed at most $\sigma = \sqrt{p_\rho(\bar{\rho}, \bar{S})}$ (the sound speed), that is,

$$(\rho, \mathbf{v}, S)(t, \mathbf{x}) = (\bar{\rho}, 0, \bar{S}), \quad \text{if} \quad |\mathbf{x}| \geq R + \sigma t. \tag{2}$$

Singularities

$$P(t) = \int_{\mathbb{R}^3} \left(\rho(t,\mathbf{x}) \exp(S(t,\mathbf{x})/\gamma) - \bar{\rho} \exp(\bar{S}/\gamma)\right) d\mathbf{x},$$

$$F(t) = \int_{\mathbb{R}^3} \mathbf{x} \cdot (\rho \mathbf{v})(t,\mathbf{x}) d\mathbf{x}$$

Theorem (Sideris 1985). Suppose that $(\rho, \mathbf{v}, S)(t, \mathbf{x})$ is a C^1 solution for $0 < t < T$ and

$$P(0) \geq 0, \quad F(0) > \frac{16\pi}{3} \sigma R^4 \max_{\mathbf{x}} \{\rho_0(\mathbf{x})\}. \tag{3}$$

Then the lifespan T of the C^1 solution is finite.

Remark. Condition (3) can be replaced by the condition: $S_0(\mathbf{x}) \geq \bar{S}$ and, for some $0 < R_0 < R$,

$$\int_{|\mathbf{x}|>r} |\mathbf{x}|^{-1} (|\mathbf{x}| - r)^2 (\rho_0(\mathbf{x}) - \bar{\rho}) d\mathbf{x} > 0,$$

$$\int_{|\mathbf{x}|>r} |\mathbf{x}|^{-3} (|\mathbf{x}|^2 - r^2)\, \mathbf{x} \cdot (\rho_0 \mathbf{v}_0)(\mathbf{x}) d\mathbf{x} \geq 0 \qquad \text{for } R_0 < r < R.$$

Singularities: Proof —1: $M(t) = \int_{\mathbb{R}^3}(\rho(t,\mathbf{x}) - \bar{\rho})d\mathbf{x}$

Using (2), equations (E-1), and integration by parts yields

$$M'(t) = -\int_{\mathbb{R}^3} \nabla\cdot(\rho\mathbf{v})d\mathbf{x} = 0, \qquad P'(t) = -\int_{\mathbb{R}^3} \nabla\cdot(\rho\mathbf{v}\exp(S/\gamma))d\mathbf{x} = 0,$$

which implies $\qquad M(t) = M(0), \;\; P(t) = P(0).$

$$F'(t) = \int_{\mathbb{R}^3} \left(\rho|\mathbf{v}|^2 + 3(p - \bar{p})\right) d\mathbf{x} = \int_{B(t)} \left(\rho|\mathbf{v}|^2 + 3(p - \bar{p})\right) d\mathbf{x}, \qquad (4)$$

where $B(t) = \{\mathbf{x} \in \mathbb{R}^3 \,:\, |\mathbf{x}| \le R + \sigma t\}$.

From Hölder's inequality and (3)–(4), one has

$$\begin{aligned} \int_{B(t)} p\,d\mathbf{x} &\ge \frac{1}{|B(t)|^{\gamma-1}}\Big(\int_{B(t)} p^{1/\gamma}d\mathbf{x}\Big)^{\gamma} \\ &= \frac{1}{|B(t)|^{\gamma-1}}\Big(P(0) + \int_{B(t)} \bar{p}^{1/\gamma}d\mathbf{x}\Big)^{\gamma} \ge \int_{B(t)} \bar{p}\,d\mathbf{x}. \end{aligned}$$

$$\Longrightarrow \qquad F'(t) \ge \int_{\mathbb{R}^3} \rho|\mathbf{v}|^2 d\mathbf{x} \ge 0. \qquad (5)$$

Proof —2: By the Cauchy-Schwarz inequality and (4)

(i) $F(0) > 0 \implies F(t) > 0$ for $0 < t < T$.

(ii)
$$F(t)^2 = \left(\int_{B(t)} \mathbf{x}\cdot\rho\mathbf{v}\,d\mathbf{x}\right)^2 \le \int_{B(t)} \rho|\mathbf{v}|^2 d\mathbf{x} \int_{B(t)} \rho|\mathbf{x}|^2 d\mathbf{x}$$
$$\le (R+\sigma t)^2 \int_{B(t)} \rho|\mathbf{v}|^2 d\mathbf{x}\left(M(t) + \int_{B(t)} \bar{\rho}\,d\mathbf{x}\right)$$
$$\le (R+\sigma t)^2 \int_{B(t)} \rho|\mathbf{v}|^2 d\mathbf{x}\left(\int_{B(t)} (\rho_0(\mathbf{x}) - \bar{\rho})d\mathbf{x} + \int_{B(t)} \bar{\rho}\,d\mathbf{x}\right)$$
$$\le \frac{4\pi}{3}(R+\sigma t)^5 \max_{\mathbf{x}}\{\rho_0(\mathbf{x})\} \int_{B(t)} \rho|\mathbf{v}|^2 d\mathbf{x}$$
$$\le \frac{4\pi}{3}(R+\sigma t)^5 \max_{\mathbf{x}}\{\rho_0(\mathbf{x})\}\, F'(t).$$

Dividing by $F(t)^2$ above and integrating from 0 to T yields

$$F(0)^{-1} > F(0)^{-1} - F(T)^{-1} \ge \frac{R^{-4} - (R+\sigma T)^{-4}}{\frac{16}{3}\pi\sigma \max\{\rho_0(\mathbf{x})\}}$$

$$\implies \quad (R+\sigma T)^4 < \frac{R^4 F(0)}{F(0) - \frac{16}{3}\pi\sigma R^4 \max\{\rho_0(\mathbf{x})\}}$$

Singularities: Remarks

1. The method of the proof above applies equally well in 1– and 2–space dimensions. In the isentropic case (S is a constant), the condition $P(0) \geq 0$ reduces to $M(0) \geq 0$.

2. To illustrate a way in which the conditions in (3) may be satisfied, consider the case: $\rho_0 = \bar{\rho}$, $S_0 = \bar{S}$. Then (3) holds (with $P(0) = 0$) if

$$\int_{|\mathbf{x}|<R} \mathbf{x} \cdot \mathbf{v}_0(\mathbf{x}) d\mathbf{x} > \frac{16\pi}{3} \sigma R^4.$$

Comparing both sides, one finds that the initial velocity must be supersonic in some region relative to the sound speed at infinity. The formation of a singularity is detected as the disturbance overtakes the wave front forcing the front to propagate with supersonic speed.

3. The result indicates that the C^1 regularity of solutions breaks down in a finite time. It is believed that in fact only $\nabla\rho$ and $\nabla\mathbf{v}$ blow up in most cases [Alinhac 1993: Axisymmetric initial data in $\mathbb{R}^2$.]

4. D. Christodoulou, 2007: The formation of shocks in 3-dimensional relativistic perfect fluids: Nature of breakdown...

BV or L^1 Bounds for Multi-D Case?

Case $d = 1, m \geq 2$: Glimm's BV theory: 1965

$$\|\mathbf{u}(t,\cdot)\|_{BV} \leq C\|\mathbf{u}_0(\cdot)\|_{BV}$$

as long as $\|\mathbf{u}_0(\cdot)\|_{BV}$ is small enough.

Case $d = 1, m = 2$: L^∞ Bounds

$$\|\mathbf{u}(t,\cdot) - \bar{\mathbf{u}}\|_{L^\infty} \leq C\|\mathbf{u}_0 - \bar{\mathbf{u}}\|_{L^\infty}$$

for the Isentropic Euler equations [DiPerna, Ding-Chen-Luo, Chen, Lions-Perthame-Tadmor, Lions-Perthame-Souganidis, Chen-LeFloch].

The first test should be to investigate whether entropy solutions for the multidimensional case satisfy the relatively modest stability estimate:

$$\|\mathbf{u}(t,\cdot) - \bar{\mathbf{u}}\|_{L^p} \leq C_p\|\mathbf{u}_0 - \bar{\mathbf{u}}\|_{L^p}, \qquad (*)$$

or

$$\|\mathbf{u}(t,\cdot)\|_{BV} \leq C\|\mathbf{u}_0\|_{BV}.$$

Since we assume that the system is endowed with a strictly convex entropy, then we conclude that the L^2–estimate holds.

Question: ?? L^p–estimate for any $p \neq 2$??

The case $p = 1$ and $p = \infty$ is of particular interest.

BV or L^1 Bounds for Multi-D Case?

Rauch (1987): The necessary condition for the system to be held is

$$\nabla \mathbf{f}_k \nabla \mathbf{f}_l = \nabla \mathbf{f}_l \nabla \mathbf{f}_k, \qquad k, l = 1, \cdots, d. \qquad (**)$$

Dafermos (1995): When $m = 2$, the necessary condition (**) is also sufficient for (*) for any $1 \le p \le 2$ and, under additional assumptions on the system, even for $p = \infty$.

The analysis suggests that only systems in which the commutativity relation (**) holds offer any hope for treatment in the framework of L^1.

This special case includes the scalar case $m = 1$ and the case of single space dimension $d = 1$. Beyond that, it contains very few systems of (even modest) physical interest. An example is the system with fluxes:

$$\mathbf{f}_k(\mathbf{u}) = \phi(|\mathbf{u}|^2)\mathbf{u}, \qquad k = 1, 2, \cdots, d,$$

which governs the flow of a fluid in an anisotropic porous medium.

L. Ambrosio and C. De Lellis 2003: $\exists\ \mathbf{u}(t, \mathbf{x}) \in L^\infty$ for $t > 0$

C. De Lellis: Duke Math. J. 2005: $u_0 \in BV$, but $u(t, \mathbf{x}) \notin BV$ for $t > 0$

Question: ?? L^1–Stability??

Commutativity Relation (**) vs Linear Stability

The reason why the relation (**) is the necessary condition for (*) is based on the linear theory by Brenner 1966 who proved the following:

Consider the linear symmetric hyperbolic system

$$\partial_t \mathbf{u} + \sum_{k=1}^{d} A_k(t, \mathbf{x}) \partial_{x_k} \mathbf{u} = 0. \qquad (***)$$

Then the following three statements are equivalent:

(i) (*) is satisfied for some $p \neq 2$;

(ii) (*) holds for all $1 \leq p \leq \infty$;

(iii) A_k commute:

$$A_k A_l = A_l A_k, \qquad \text{for all } l, k = 1, 2, \cdots, d.$$

Nonuniquess for the Isentropic Euler Equations

Camillo De Lellis and László Székelyhidi Jr.: 2010:

Theorem

Let $d \geq 2$. Then, for any given function $p = p(\rho)$ with $p'(\rho) > 0$ when $\rho > 0$, there exist bounded initial data $(\rho_0, \mathbf{v}_0)$ with $\rho_0(\mathbf{x}) \geq c_0 > 0$ for which there exist infinitely many bounded solutions $(\rho, \mathbf{v})$ with $\rho \geq c > 0$, satisfying the energy identity in the sense of distributions:

$$\partial_t\Big(\rho(\frac{|\mathbf{v}|^2}{2} + e(\rho))\Big) + \nabla_{\mathbf{x}} \cdot \Big(\rho\mathbf{v}(\frac{|\mathbf{v}|^2}{2} + e + \frac{p}{\rho})\Big) = 0.$$

Point: Vortex Sheets, Vorticity Waves, Entropy Waves, ...,

Discontinuities of Solutions

$$\partial_t \mathbf{u} + \nabla \cdot \mathbf{f}(\mathbf{u}) = 0, \qquad \mathbf{x} = (x_1, \cdots, x_d)^\top \in \mathbb{R}^d$$

An oriented surface $\mathbf{\Gamma}(t)$ with unit normal $\mathbf{n} = (n_t, \cdots, n_d)^\top \in \mathbb{R}^d$ in the $(t, \mathbf{x})$-space is a discontinuity of a piecewise smooth entropy solution U with

$$\mathbf{u}(t, \mathbf{x}) = \begin{cases} \mathbf{u}^+(t, \mathbf{x}), & (t, \mathbf{x}) \cdot \mathbf{n} > 0, \\ \mathbf{u}^-(t, \mathbf{x}), & (t, \mathbf{x}) \cdot \mathbf{n} < 0, \end{cases}$$

if the Rankine-Hugoniot Condition is satisfied

$$(\mathbf{u}^+ - \mathbf{u}^-, \mathbf{f}(\mathbf{u}^+) - \mathbf{f}(\mathbf{u}^-)) \cdot \mathbf{n} = \mathbf{0} \qquad \text{along } \mathbf{\Gamma}(\mathbf{t}).$$

The surface $(\mathbf{\Gamma}(\mathbf{t}), \mathbf{u})$ is called a Shock Wave if the Entropy Condition (i.e., the Second Law of Thermodynamics) is satisfied:

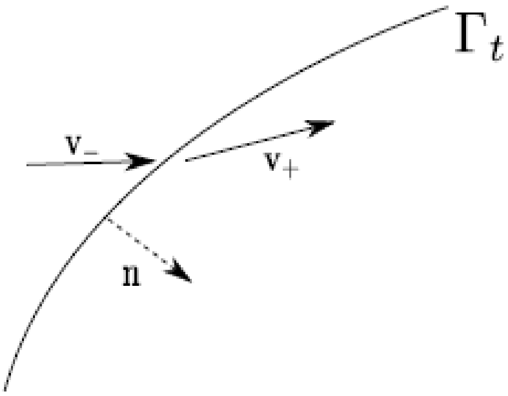
$$(\eta(\mathbf{u}^+) - \eta(\mathbf{u}^-), \mathbf{q}(\mathbf{u}^+) - \mathbf{q}(\mathbf{u}^-)) \cdot \mathbf{n} > \mathbf{0} \qquad \text{along } \mathbf{\Gamma}(\mathbf{t}),$$

for some $(\eta(\mathbf{u}), \mathbf{q}(\mathbf{u}))$: $\nabla^2 \eta(\mathbf{u}) \geq 0$, $\nabla q_j(\mathbf{u}) = \nabla \eta(\mathbf{u}) \mathbf{f}_j(\mathbf{u})$, $j = 1, \cdots, d$

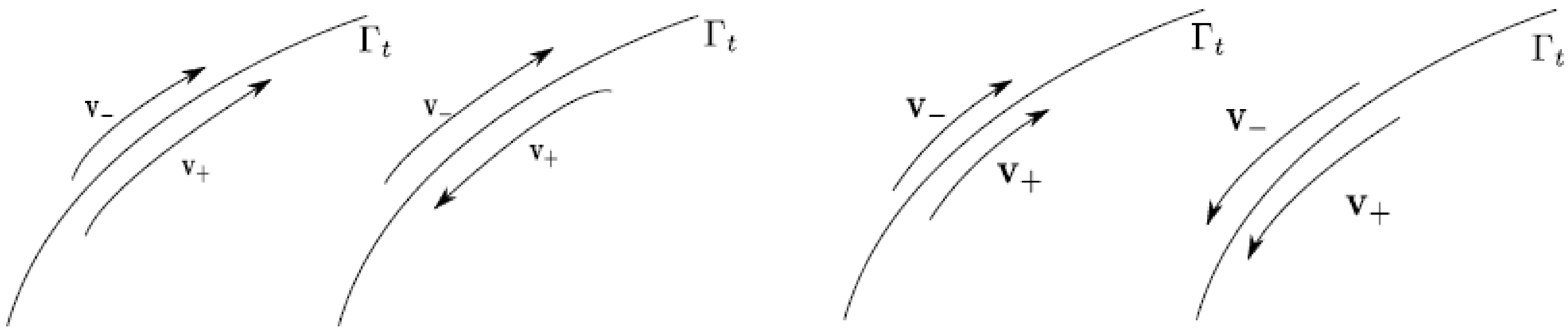
Example: For the full Euler equations: $(\eta(\mathbf{u}), \mathbf{q}(\mathbf{u})) = (-\rho S, -\rho \mathbf{v} S)$.

Two Types of Discontinuities

Noncharacteristic Discontinuities: Shock Waves:



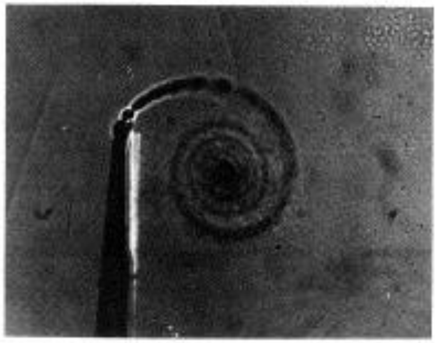
Characteristic Discontinuities: Vortex Sheets/Entropy Waves



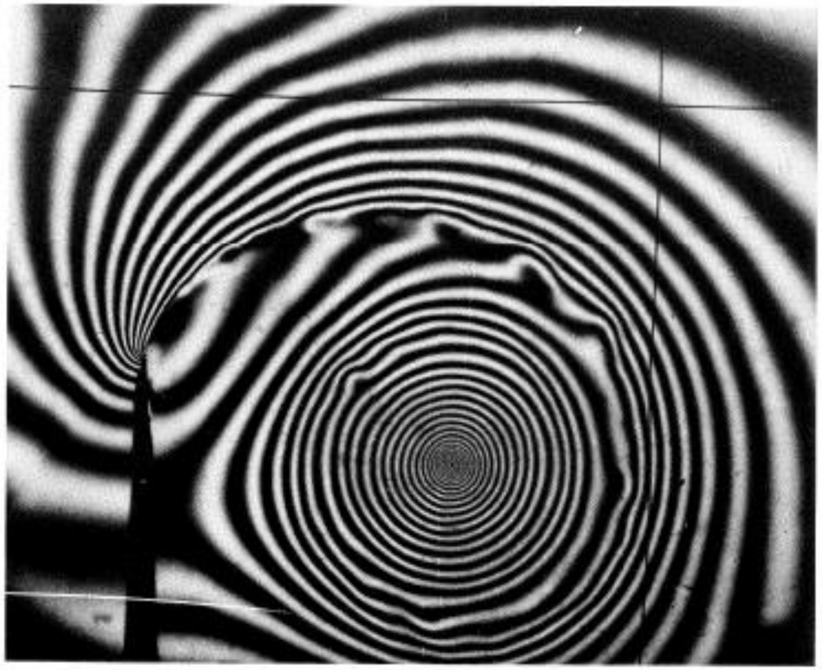
(i) $(p_+, \rho_+) = (p_-, \rho_-)$, $v_+ \neq v_-$

(ii) $(p_+, v_+) = (p_-, v_-)$, $\rho_+ \neq \rho_-$

Vortex from a Wedge

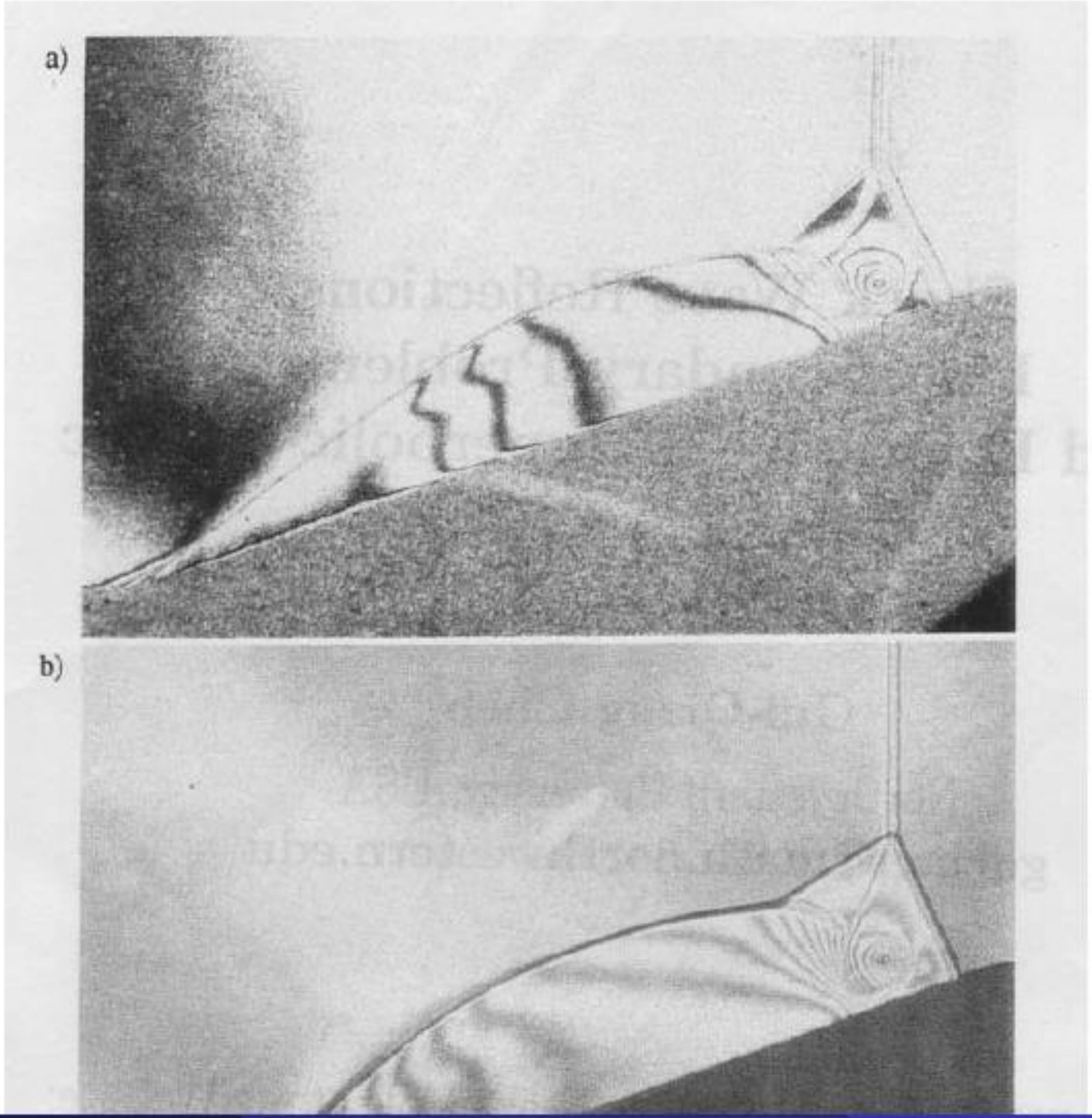


82. Vortex from a wedge in a shock tube. This schlieren photograph shows the vortex that spirals from the tip of a thin wedge after the air is set in motion normal to it by the passage of a weak plane shock wave, which is out of sight to the right. Other photographs show that the flow pattern is "conical" or "pseudo-stationary," remaining always similar to itself but growing in size in proportion to the time. *Photograph by Walker Bleakney*

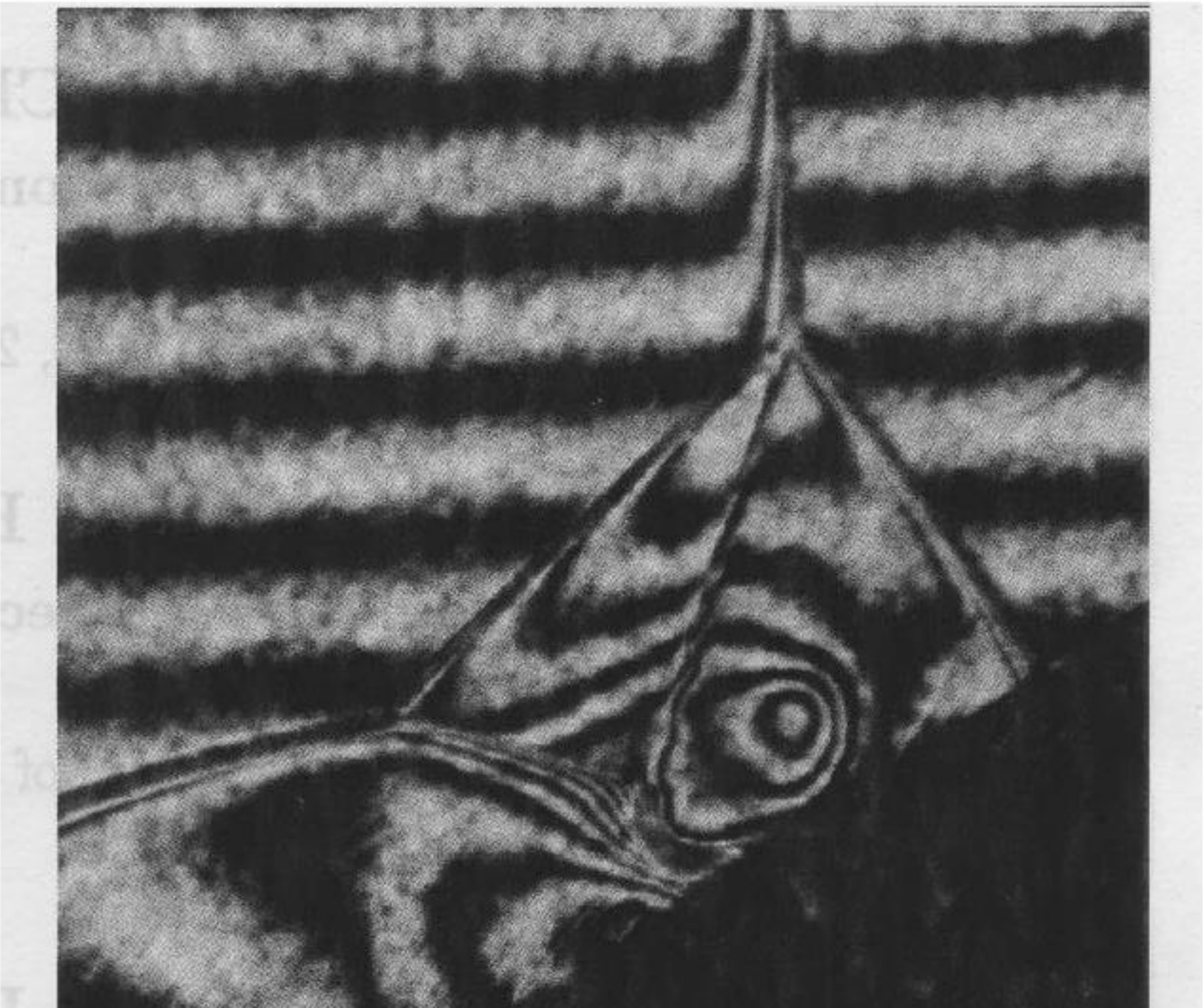


83. Density in a vortex from a wedge. A quite different view of the phenomenon above is given by this infinite-fringe interferogram, which shows lines of constant density. A striking feature is the almost perfectly circular density distribution about the center of the vortex, extending nearly to the wedge. *Photograph by Walker Bleakney*

Mach Reflection-Diffraction I

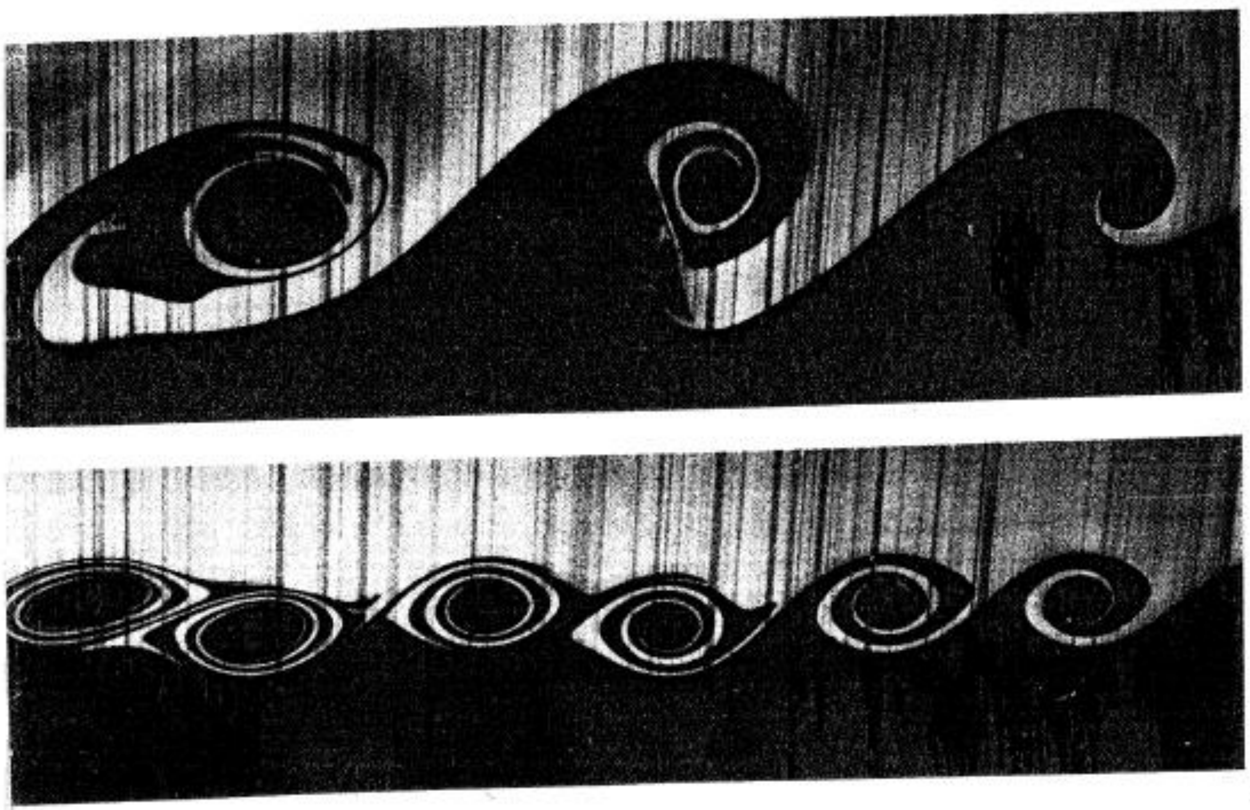


Mach Reflection-Diffraction II

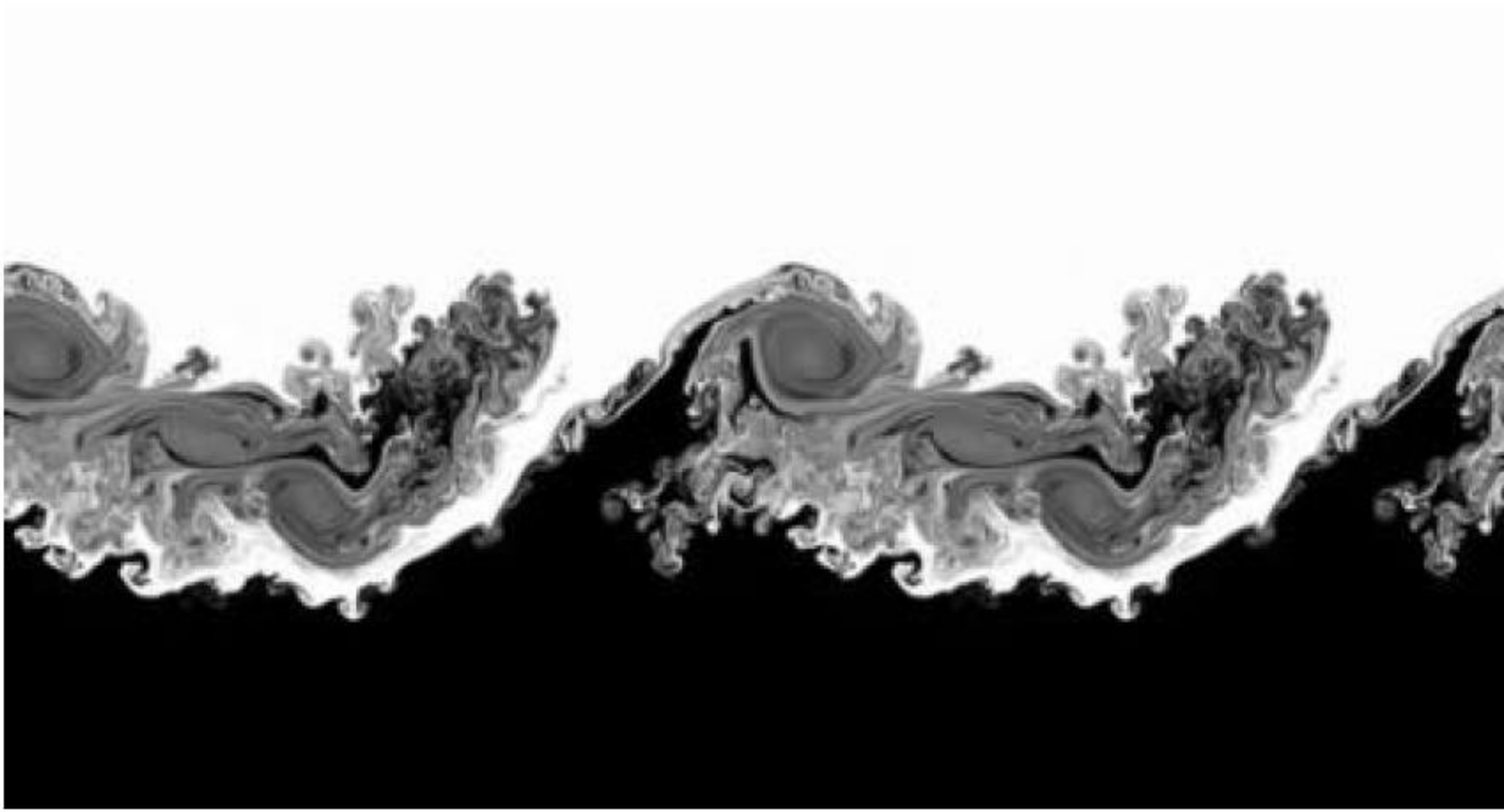


Kelvin-Helmholtz Instability I: Clouds over San Francisco

Kelvin-Helmholtz Instability II



Kelvin-Helmholtz Instability III



Good Frameworks for Studying Entropy Solutions of Multidimensional Conservation Laws?

One of such candidates may be derived from the theory of divergence-measure fields, which is based on the following class of Entropy Solutions:

(i) $\mathbf{u}(t,\mathbf{x}) \in \mathcal{M}, L^p, 1 \le p \le \infty$;

(ii) For any convex entropy pair $(\eta, \mathbf{q})$,

$$\partial_t \eta(\mathbf{u}) + \nabla_{\mathbf{x}} \cdot \mathbf{q}(\mathbf{u}) \le 0 \qquad \mathcal{D}'$$

as long as $(\eta(\mathbf{u}(t,\mathbf{x})), \mathbf{q}(\mathbf{u}(t,\mathbf{x}))) \in \mathcal{D}'$

Then Schwartz lemma tells us that

$$\mathrm{div}_{(t,\mathbf{x})}(\eta(\mathbf{u}(t,\mathbf{x})), \mathbf{q}(\mathbf{u}(t,\mathbf{x}))) \in \mathcal{M}$$

$\Longrightarrow$

The vector field $(\eta(\mathbf{u}(t,\mathbf{x})), \mathbf{q}(\mathbf{u}(t,\mathbf{x})))$ is a divergence measure field.

Approaches and Strategies: Proposal

Diverse Approaches in Sciences:

- Experimental data
- Large and small scale computing by a search for effective numerical methods
- Modelling (Asymptotic and Qualitative)
- Rigorous proofs for prototype problems and an understanding of the solutions

Two Strategies as a first step:

- Study good, simpler nonlinear models with physical motivations;
- Study special, concrete nonlinear problems with physical motivations

Meanwhile, extend the results and ideas to:

- Study the Euler equations in gas dynamics and elasticity
- Study nonlinear systems that the Euler equations are the main subsystem or describe the dynamics of macroscopic variables such as MHD, Euler-Poisson Equations, Combustion, Relativistic Euler Equations, ...
- Study more general hyperbolic systems and related problems
- Develop further new mathematical ideas, techniques, approaches, as well as new mathematical theories

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