Topics on Nonlinear Hyperbolic PDEs

Mathematical Institutes
Hilary Term 2022
February: 9th, 16th, 23rd
March: 2nd
Wednesdays 14:00-16:00

By Professor Gui-Qiang G. Chen

Lecture-1: 9th February 2022

Theory of Nonlinear Hyperbolic PDEs

is a large subject, which has close connections with the other areas of mathematics including

Analysis, Mechanics, Mathematical Physics, Differential Geometry/Topology, …

Besides its mathematical importance, it has a wide range of applications in

Engineering, Physics, Biology, Economics, …

- An introduction to most facets of the nonlinear theory
- No previous knowledge of hyperbolic PDEs is assumed
- However, some familiarity with linear PDE theory, analysis, and algebra is desirable.

Topics:

1. **Introduction:** Conservation laws, Euler equations; Connections with Einstein equations, calculus of variations, differential geometry, …; Hyperbolic systems, prototypes; Basic features and phenomena; ...

2. **One-Dimensional Theory:** Riemann problem, Cauchy problem; Elementary waves: shock waves, rarefaction waves, contact discontinuities; Lax entropy conditions; Glimm scheme, front-tracking, BV solutions; Compensated compactness, entropy analysis, L-p solutions; Vanishing viscosity methods; *Uniqueness and continuous dependence; ...

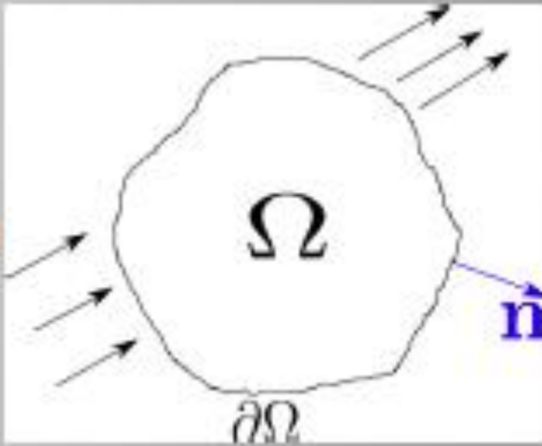
3. **Multidimensional Theory:** Basic features/phenomena (re-visit); Important models; Steady problems; Self-similar problems; Discontinuities and free boundary problems; Stability of shock waves, rarefaction waves, vortex sheets, entropy waves; *Divergence-measure fields; ...

4. **Further Connections and Applications**

References:

1. **R. Courant and D. Hilbert:** **Methods of Mathematical Physics,** Vol. II. Reprint of the 1962 original. John Wiley&Sons, Inc.: New York, 1989.
2. **C. M. Dafermos:** **Hyperbolic Conservation Laws in Continuum Physics,** Fourth edition. Springer-Verlag: Berlin, 2016.
3. **L. C. Evans:** **Partial Differential Equations,** Second edition. AMS: Providence, RI, 2010.
4. **L. Hormander:** **Lectures on Nonlinear Hyperbolic Differential Equations** Springer-Verlag: Berlin-Heidelberg, 1997.
5. **P. D. Lax:** **Hyperbolic Differential Equations,** AMS: Providence, 2000.
6. **G.-Q. Chen and M. Feldman:** **The Mathematics of Shock Reflection-Diffraction and von Neumann's Conjectures.** Annals of Mathematics Studies, 197. *Princeton University Press*: *Princeton, NJ,* 2018.
7. **D. Serre,** **Systems of Conservation Laws**, **Vols. I, II,** Cambridge University Press: Cambridge, 1999, 2000.
8. **C. D. Sogge,** **Lectures on Nonlinear Wave Equations**, Second edition. International Press, Boston, MA, 2008.

Conservation Laws:



Rate of change of the total amount of certain quantity contained in a fixed region Ω

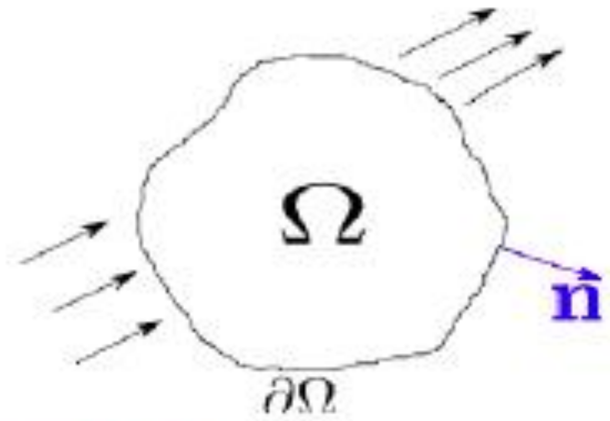
= Flux of this quantity across the boundary $\partial\Omega$ of the region

- The amount of such a quantity in any region can be measured by accounting for how much of it is currently present and how much of it enters or leaves the region in any fixed period of time.

Examples: Three Fundamental Laws of Nature

- Conservation Laws of Mass and Energy: Mass and Energy can be neither created nor destroyed.
- Conservation Law of Momentum: The total momentum of a closed system of objects remains constant through time.

Conservation Laws



Rate of Change of the Total Amount of Certain Quantity in a Fixed Region Ω
= Flux of the Quantity across the Boundary $\partial\Omega$.

Conservation Law via Calculus

$$\Longleftrightarrow \qquad \frac{d}{dt}\int_{\Omega} u d\mathbf{x} = -\int_{\partial\Omega} \mathbf{f}\cdot\mathbf{n}\, dS$$

u – Density of the Quantity
$\mathbf{n}$ – Outward Normal to Ω

$\mathbf{f}$ – Flux of the Quantity
dS – Surface Element on $\partial\Omega$

Calculus Manipulations $\Longrightarrow$

$$\partial_t u + \nabla_{\mathbf{x}}\cdot\mathbf{f}(u) = 0$$

Physical Systems with $m \geq 2$ Quantities – Density Functions

$\Longrightarrow$ **Systems of Conservation Laws**:

$$\partial_t\, \mathbf{u} + \nabla\cdot\mathbf{f}(\mathbf{u}) = 0$$

$$\mathbf{u} = (u_1, \cdots, u_m)^\top \qquad \mathbf{f}(\mathbf{u}) = (\mathbf{f}_1(\mathbf{u}), \cdots, \mathbf{f}_d(\mathbf{u}))$$

Euler Equations for Compressible Fluids

$$\begin{cases} \partial_t \rho + \nabla_{\mathbf{x}} \cdot (\rho \mathbf{v}) = 0 & \text{(conservation of mass)} \\ \partial_t(\rho \mathbf{v}) + \nabla_{\mathbf{x}} \cdot (\rho \mathbf{v} \otimes \mathbf{v}) + \nabla_{\mathbf{x}} p = 0 & \text{(conservation of momentum)} \\ \partial_t(\frac{1}{2}\rho|\mathbf{v}|^2 + \rho e) + \nabla_{\mathbf{x}} \cdot ((\frac{1}{2}\rho|\mathbf{v}|^2 + \rho e + p)\mathbf{v}) = 0 & \text{(conservation of energy)} \end{cases}$$

Constitutive Relations: $p = p(\rho, e)$

- ρ – density, $\mathbf{v} = (v_1, v_2, v_3)^\top$ – fluid velocity
- p – pressure, e – internal energy

Leonhard Euler

***Govern** the Flows when Convective Motions Dominate Diffusion/Dispersion, …

e.g. , shock waves in Gases, Elastic Fluids, Shallow Water, ……

Poisson, Challis, Stokes, Kelvin, Rayleigh, Airy, Earnshaw, Riemann, Rankine, Christoffel, Mach, Clausius, Kirchhoff, Gibbs, Hugoniot, Duhem, Hadamard, Jouguet, Zamplen, Weber, Taylor, Becker, Bethe, Weyl, von Neumann, Courant, Friedrichs, ……

George Stokes

Hyperbolicity

$$\partial_t \mathbf{u} + \nabla \cdot \mathbf{f}(\mathbf{u}) = 0, \quad \mathbf{u} \in \mathbb{R}^m, \ \mathbf{x} \in \mathbb{R}^d$$

Plane Wave Solutions: $\mathbf{u}(t, \mathbf{x}) = \mathbf{w}(t, \omega \cdot \mathbf{x})$

$\mathbf{w}(t, \xi)$ is determined by: $\partial_t \mathbf{w} + (\nabla_{\mathbf{w}} \mathbf{f}(\mathbf{w}) \cdot \omega)\, \partial_\xi \mathbf{w} = 0$

?? **Existence** of stable plane wave solutions ??

Hyperbolicity in D: For any $\omega \in S^{d-1}$, $\mathbf{u} \in D$,

$$(\nabla_{\mathbf{u}} \mathbf{f}(\mathbf{u}) \cdot \omega)_{m\times m}\, \mathbf{r}_j(\mathbf{u}, \omega) = \lambda_j(\mathbf{u}, \omega)\, \mathbf{r}_j(\mathbf{u}, \omega), \ 1 \le j \le m$$

$\lambda_j(\mathbf{u}, \omega)$ **are real**

Main Features:

Finiteness of Propagation Speeds;

Discontinuities of Solutions, $\cdots\cdots$

Well-Posedness: Existence, Uniqueness, Stability, $\cdots$

The Simplest Example

Scalar Conservation Laws

$$\partial_t u + \nabla \cdot \mathbf{f}(u) = 0, \quad u \in \mathbb{R},\ \mathbf{x} \in \mathbb{R}^d$$

$$\mathbf{f} : \mathbb{R} \to \mathbb{R}^d$$

Then

$$\lambda(u, \omega) = \mathbf{f}'(u) \cdot \omega, \qquad r(u, \omega) \equiv 1$$

$\Longrightarrow$ **Scalar conservation laws are always hyperbolic**

Hyperbolic Systems of Conservation Laws

$$\begin{cases} \frac{\partial}{\partial t} u_1 + \frac{\partial}{\partial x} f_1(u_1, \cdots, u_m) = 0, \\ \cdots\cdots \\ \frac{\partial}{\partial t} u_m + \frac{\partial}{\partial x} f_m(u_1, \cdots, u_m) = 0, \end{cases}$$

$$\mathbf{u}_t + \mathbf{f}(\mathbf{u})_x = 0$$

$\mathbf{u} = (u_1, \cdots, u_m)^\top \in \mathbb{R}^m$ conserved quantities

$\mathbf{f}(\mathbf{u}) = (f_1(\mathbf{u}), \cdots, f_m(\mathbf{u}))^\top$ fluxes

Hyperbolic Systems

$$\mathbf{u}_t + \mathbf{f}(\mathbf{u})_x = 0 \qquad \mathbf{u} = \mathbf{u}(t,x) \in \mathbb{R}^m$$

$$\mathbf{u}_t + \mathbf{A}(\mathbf{u})\mathbf{u}_x = 0 \qquad \mathbf{A}(\mathbf{u}) = \nabla \mathbf{f}(\mathbf{u})$$

The system is **strictly hyperbolic** if each $m \times m$ matrix $\mathbf{A}(\mathbf{u})$ has real distinct eigenvalues

$$\lambda_1(\mathbf{u}) < \lambda_2(\mathbf{u}) < \cdots < \lambda_m(\mathbf{u})$$

Right eigenvectors $\mathbf{r}_1(\mathbf{u}), \cdots, \mathbf{r}_m(\mathbf{u})$ (column vectors)

Left eigenvectors $\mathbf{l}_1(\mathbf{u}), \cdots, \mathbf{l}_m(\mathbf{u})$ (row vectors)

$$\mathbf{A}\mathbf{r}_i = \lambda_i \mathbf{r}_i \qquad \mathbf{l}_i \mathbf{A} = \lambda_i \mathbf{l}_i$$

Choose the bases so that

$$\mathbf{l}_i \cdot \mathbf{r}_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Invariance of Hyperbolicity under Change of Coordinates

Theorem

- *Let* $\mathbf{u}$ *be a smooth solution of the strictly hyperbolic system*

$$\mathbf{u}_t + \mathbf{A}(\mathbf{u})\mathbf{u}_x = 0 \qquad \text{in } \mathbb{R}_+ \times \mathbb{R}$$

- *Assume* $\Phi : \mathbb{R}^m \to \mathbb{R}^m$ *is a smooth diffeomorphism, with inverse* Ψ

Then $\mathbf{w} := \Phi(\mathbf{u})$ *solves the strictly hyperbolic system*

$$\mathbf{w}_t + \mathbf{B}(\mathbf{w})\mathbf{w}_x = 0 \qquad \text{in } \mathbb{R}_+ \times \mathbb{R}$$

for $\quad \mathbf{B}(w) := \nabla\Phi(\Psi(\mathbf{w}))\mathbf{A}(\Psi(\mathbf{w}))\nabla\Psi(\mathbf{w}) \qquad \mathbf{w} \in \mathbb{R}^m$

Dependence of Eigenvalues and Eigenvectors on $\mathbf{u}$

Theorem

Assume that the matrix function $\mathbf{A}(\mathbf{u})$ *is smooth, strictly hyperbolic. Then*

- *The eigenvalues* $\lambda_k(\mathbf{u})$ *depend smoothly on* $\mathbf{u} \in \mathbb{R}^m, k = 1, \cdots, m$
- *We can select the right eigenvectors* $\mathbf{r}_k(\mathbf{u})$ *and left eigenvector* $\mathbf{l}_k(\mathbf{u})$ *to depend smoothly on* $\mathbf{u} \in \mathbb{R}^m$ *and satisfy the normalization*

$$|\mathbf{r}_k(\mathbf{u})|, |\mathbf{l}_k(\mathbf{u})| = 1, \qquad k = 1, \cdots, m.$$

*We are not only globally and smoothly defining the eigenvalues and eigenspaces of $\mathbf{A}(\mathbf{u})$, but also globally providing the eigenspaces of $\mathbf{A}(\mathbf{u})$ with an orientation.

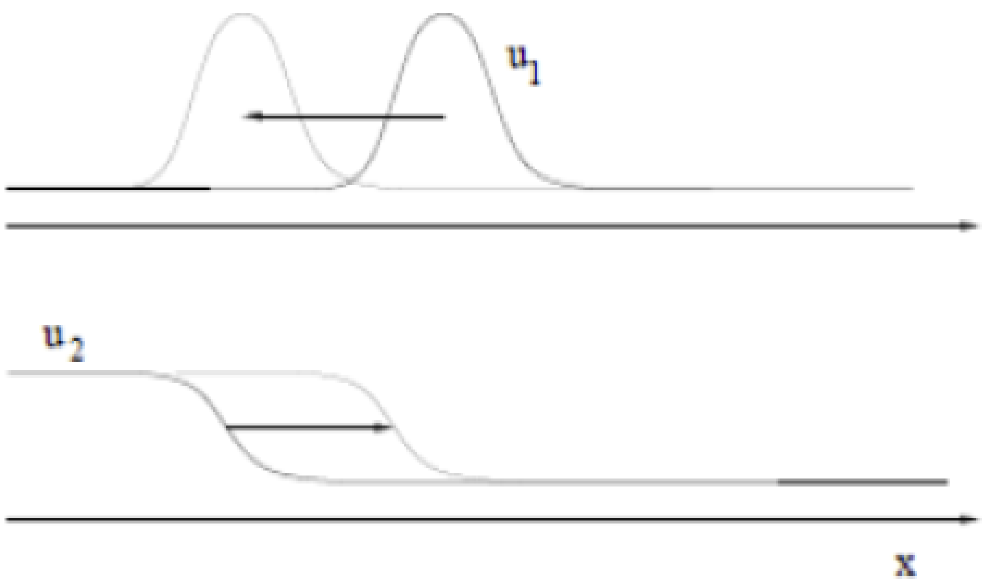
Linear Hyperbolic Systems

$$\mathbf{u}_t + \mathbf{A}\mathbf{u}_x = 0 \qquad \mathbf{u}(0,x) = \phi(x)$$

$\lambda_1 < \cdots < \lambda_m$ eignevalues $\qquad \mathbf{r}_1, \cdots, \mathbf{r}_m$ eigenvectors

Explicit solutions: **Linear superposition of travelling waves**

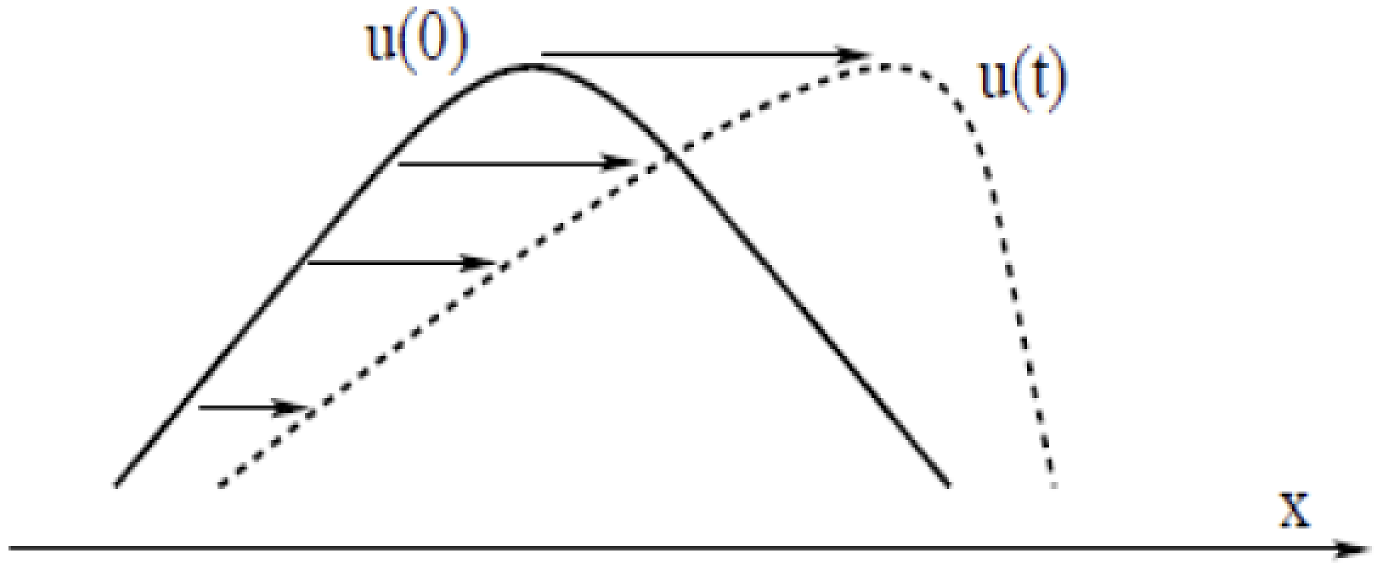
$$\mathbf{u}(t,x) = \sum_i \phi_i(x - \lambda_i t)\,\mathbf{r}_i \qquad \phi_i(s) = \mathbf{l}_i \cdot \phi(s)$$



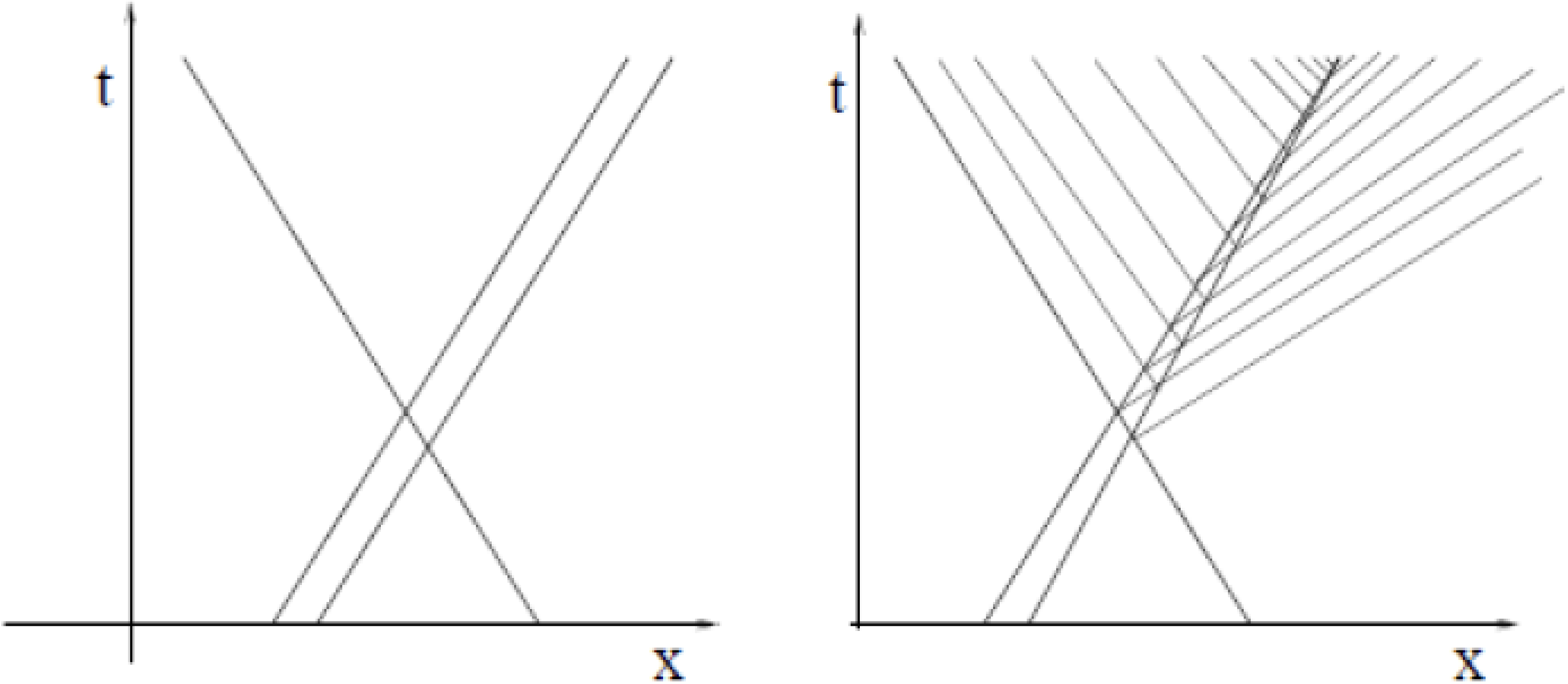
Nonlinear Effects

$$u_t + A(u)u_x = 0$$

eigenvalues depend on u $\Longrightarrow$ waves change shape



eigenvectors depend on u $\Longrightarrow$ nontrivial wave interactions

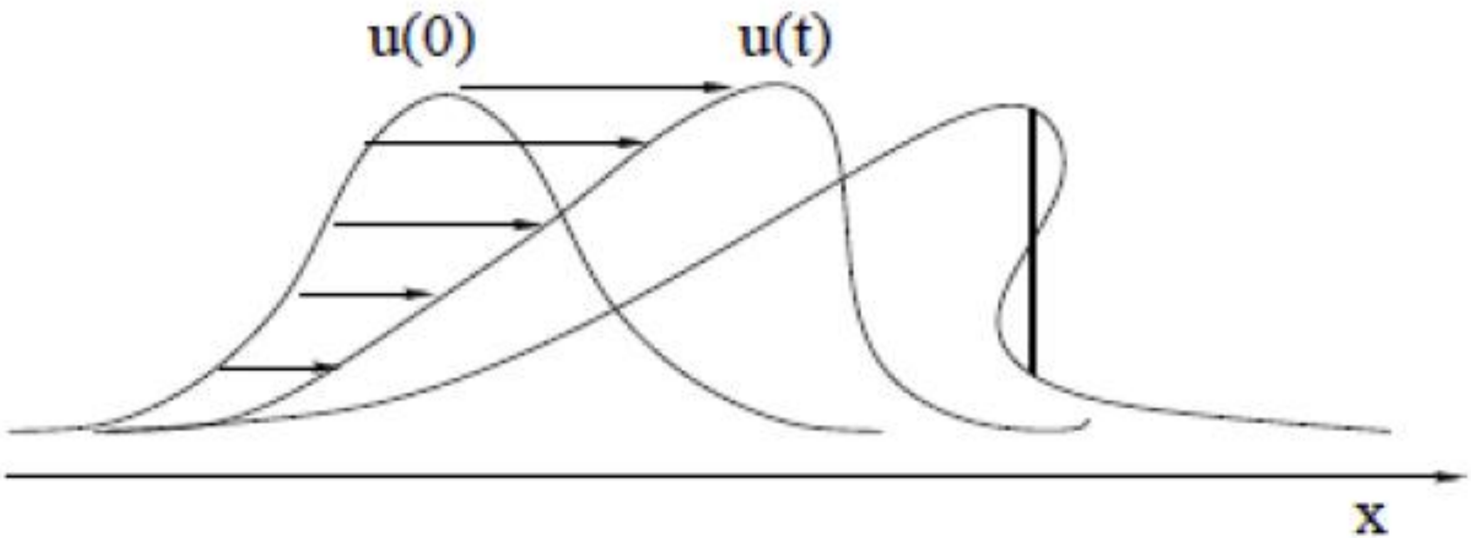


linear nonlinear

Loss of Regularity

$$u_t + (u^2/2)_x = 0 \qquad\qquad u_t + uu_x = 0$$

$$f(u) = u^2/2 \qquad\qquad \text{characteristic speed: } f'(u) = u$$

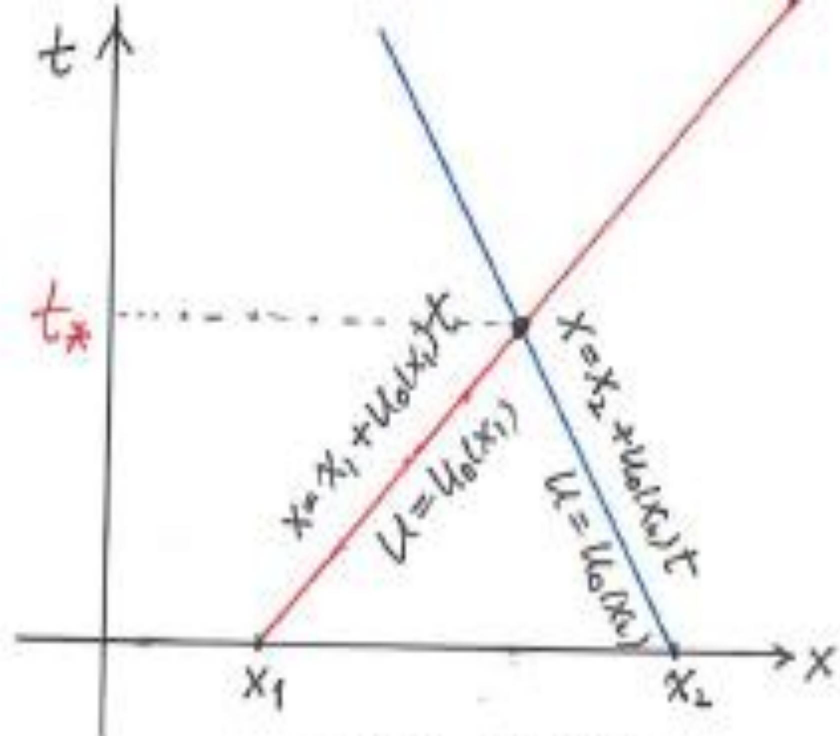


Global solutions only in a space of discontinuous functions

1-D Example

$$\begin{cases} u_t + \left(\frac{u^2}{2}\right)_x = 0 \\ u|_{t=0} = u_0(x) \end{cases}$$

$$u_t + u\,u_x = 0$$



$u_0(x_1) > u_0(x_2)$

$$\begin{cases} \frac{dx}{dt} = u, \longrightarrow x = x_j + u_0(x_j)t \\ \frac{du}{dt} = 0 \longrightarrow u = u_0(x_j), \; j = 1, 2 \end{cases}$$

When $t \to t_* = \frac{x_2 - x_1}{u_0(x_1) - u_0(x_2)} > 0$, $u(t,x)$ is Multi-Valued

Smooth Solutions – Evolution of Wave Components

$$\mathbf{u}_t = -\mathbf{A}(\mathbf{u})\mathbf{u}_x$$

$\lambda_i(\mathbf{u}) = i$-th eigenvalue, $\quad \mathbf{l}_i(\mathbf{u}), \mathbf{r}_i(\mathbf{u}) = i$-th eigenvectors

$u_x^i := \mathbf{l}_i \cdot \mathbf{u}_x = [i\text{-th component of } \mathbf{u}_x] = [\text{density of } i\text{-waves in } \mathbf{u}]$

$$\mathbf{u}_x = \sum_{i=1}^{m} u_x^i \, \mathbf{r}_i(\mathbf{u}) \qquad \mathbf{u}_t = -\sum_{i=1}^{m} \lambda_i(\mathbf{u}) u_x^i \, \mathbf{r}_i(\mathbf{u})$$

Differentiate the 1st equation w.r.t. t and the 2nd w.r.t x

$\Longrightarrow$ Evolution equation for scalar components u_x^i:

$$\left(u_x^i\right)_t + \left(\lambda_i u_x^i\right)_x = \sum_{j>k} (\lambda_j - \lambda_k)\Big(\mathbf{l}_i \cdot [\mathbf{r}_j, \mathbf{r}_k]\Big) u_x^j u_x^k$$

Source Terms

$$(\lambda_j - \lambda_k)\Big(\mathbf{l}_i \cdot [\mathbf{r}_j, \mathbf{r}_k]\Big) u_x^j u_x^k$$

$=$amount of i-waves produced by the interaction of j-waves with k-waves

$\lambda_j - \lambda_k$ $=$[difference in speed]

$=$[rate at which j-waves and k-waves cross each other]

$u_x^j u_x^k$$=$ [density of j-waves]$\times$ [density of k-waves]

$[\mathbf{r}_j, \mathbf{r}_k] = (\nabla \mathbf{r}_k)\mathbf{r}_j - (\nabla \mathbf{r}_j)\mathbf{r}_k$ (Lie bracket)

$=$ [directional derivative of $\mathbf{r}_k$ in the direction of $\mathbf{r}_j$]

$-$[directional derivative of $\mathbf{r}_j$ in the direction of $\mathbf{r}_k$]

$\mathbf{l}_i \cdot [\mathbf{r}_j, \mathbf{r}_k]$$=$ i-th component of the Lie bracket $[\mathbf{r}_j, \mathbf{r}_k]$ along the basis of eigenvectors $\{\mathbf{r}_1, \cdots, \mathbf{r}_m\}$

Shock Solutions

$$\mathbf{u}_t + \mathbf{f}(\mathbf{u})_x = 0$$

$$\mathbf{u}(t,x) = \begin{cases} \mathbf{u}^- & \text{if} \quad x < \lambda t \\ \mathbf{u}^+ & \text{if} \quad x > \lambda t \end{cases}$$ is a weak solution

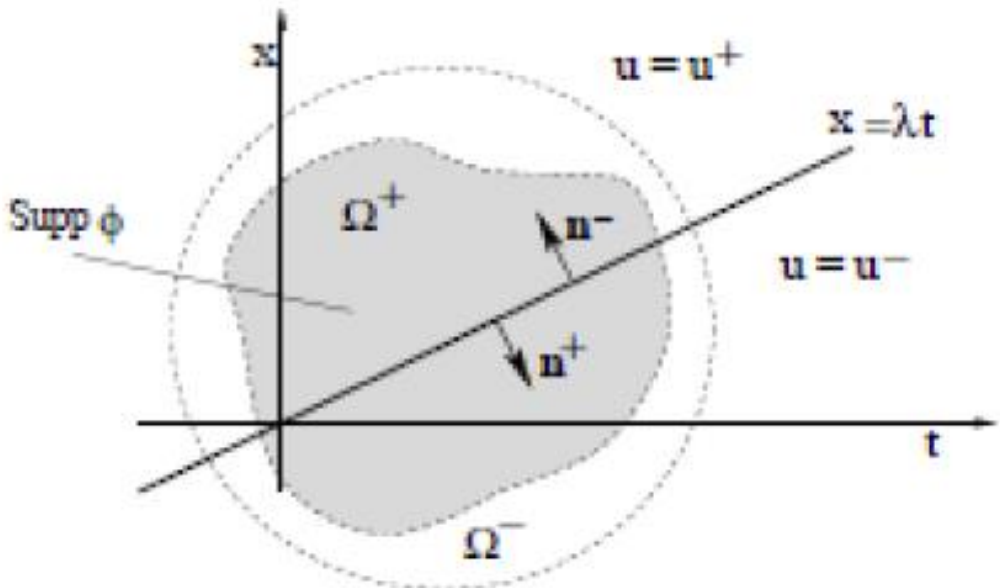
if and only if the **Rankine-Hugoniot Equations** hold:

$$\lambda \left[\mathbf{u}^+ - \mathbf{u}^-\right] = \mathbf{f}(\mathbf{u}^+) - \mathbf{f}(\mathbf{u}^-)$$

[Speed of the shock]$\times$**[Jump in the state]** $=$ **[Jump in the flux]**

Derivation of the Rankine - Hugoniot Equations

$$\begin{aligned} 0 &= \iint \Big\{ u\phi_t + f(u)\phi_x \Big\}\, dxdt = \iint_{\Omega^+\cup\Omega^-} \text{div}\ (u\phi,\, f(u)\phi)\, dxdt \\ &= \int_{\partial\Omega^+} \mathbf{n}^+ \cdot \mathbf{v}\, ds + \int_{\partial\Omega^-} \mathbf{n}^- \cdot \mathbf{v}\, ds \\ &= \int \Big[\lambda(u^+ - u^-) - (f(u^+) - f(u^-)) \Big]\, \phi(t, \lambda t)\, dt\,. \end{aligned}$$



$$\mathbf{v} \doteq \Big(u\phi\,,\ f(u)\phi \Big)$$

Alternative Formulation

$$
\begin{aligned}
\lambda(u^+ - u^-) &= f(u^+) - f(u^-) \\
&= \int_0^1 \nabla f(\theta u^+ + (1-\theta) u^-)(u^+ - u^-)\, d\theta \\
&= A(u^+, u^-)(u^+ - u^-)
\end{aligned}
$$

$A(u, v) := \int_0^1 \nabla f(\theta u + (1-\theta) v)\, d\theta$

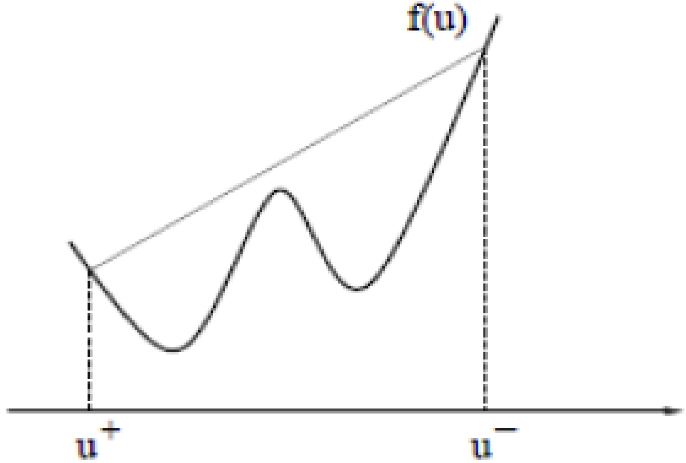
[averaged Jacobian matrix]

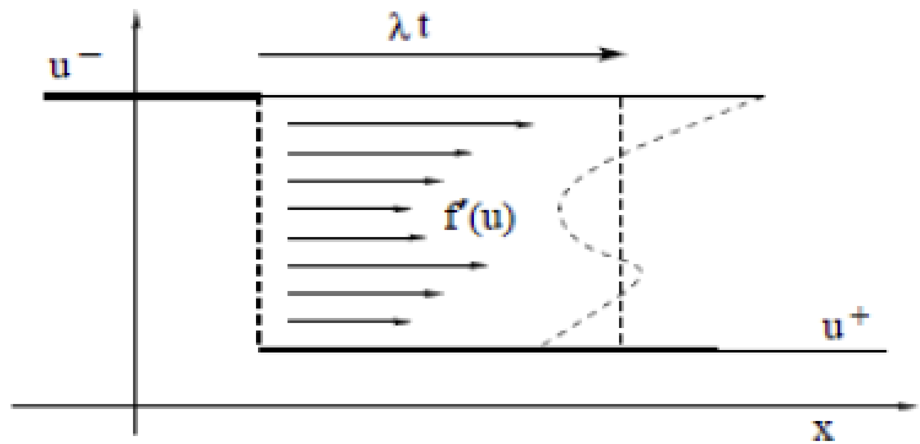
The Rankine-Hugoniot conditions hold if and only if

$$\lambda(u^+ - u^-) = A(u^+, u^-)(u^+ - u^-)$$

- **The jump $u^+ - u^-$ is an eigenvector of the averaged matrix $A(u^+, u^-)$**
- **The speed λ coincides with the corresponding eigenvalue**

The Rankine-Hugoniot condition for the scalar conservation law $\quad u_t + f(u)_x = 0$





$$\lambda \;=\; \frac{f(u^+) - f(u^-)}{u^+ - u^-} \;=\; \frac{1}{u^+ - u^-}\int_{u^-}^{u^+} f'(s)\,ds$$

[speed of the shock] $=$ [slope of secant line through u^-, u^+ on the graph of f]

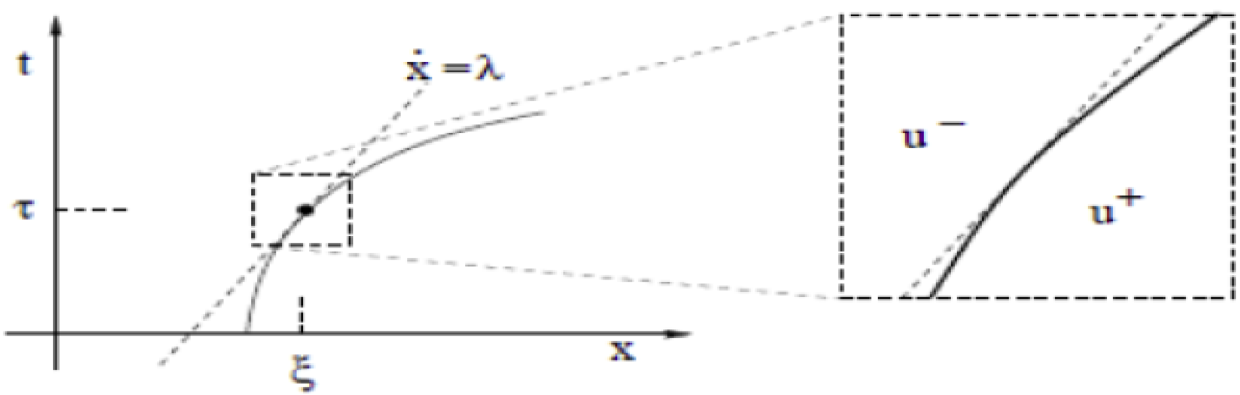
$=$ [average of the characteristic speeds between u^- and u^+]

Points of Approximate Jump

The function $\mathbf{u} = \mathbf{u}(t, x)$ has an approximate jump at a point (τ, ξ) if there exists states $\mathbf{u}^- = \mathbf{u}^+$ and a speed λ such that, setting

$$U(t,x) := \begin{cases} \mathbf{u}^- & \text{if} \quad x < \lambda t \\ \mathbf{u}^+ & \text{if} \quad x > \lambda t \end{cases}$$

there holds: $\lim_{\rho \to 0+} \frac{1}{\rho^2} \int_{\tau-\rho}^{\tau+\rho} \int_{\xi-\rho}^{\xi+\rho} \left| \mathbf{u}(t,x) - U(t-\tau, x-\xi) \right| dxdt = 0$



Theorem

If $\mathbf{u}$ is a weak solution to the system $\mathbf{u}_t + \mathbf{f}(\mathbf{u})_x = 0$, then the Rankine-Hugoniot equations hold at each point of approximate jump.

Construction of Shock Waves

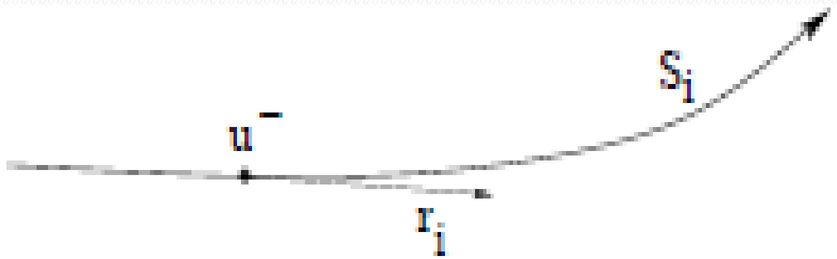
Problem: Given $\mathbf{u}^- \in \mathbb{R}^m$, find the states $\mathbf{u}^+ \in \mathbb{R}^m$ which, for some speed λ, satisfy the Rankine-Hugoniot equations:

$$\lambda(\mathbf{u}^+ - \mathbf{u}^-) = \mathbf{f}(\mathbf{u}^+) - \mathbf{f}(\mathbf{u}^-) = \mathbf{A}(\mathbf{u}^-, \mathbf{u}^+)(\mathbf{u}^+ - \mathbf{u}^-)$$

Alternative Formulation: Fix $i \in \{1, \cdots, m\}$. The jump $\mathbf{u}^+ - \mathbf{u}^-$ is a (right) i-eigenvector of the avergaed matrix $\mathbf{A}(\mathbf{u}^-, \mathbf{u}^+)$ if and only if it is orthogonal to all (left) eigenvectors $\mathbf{l}_j(\mathbf{u}^+, \mathbf{u}^-)$ of $\mathbf{A}(\mathbf{u}^-, \mathbf{u}^+)$:

$$\mathbf{l}_j(\mathbf{u}^-, \mathbf{u}^+) \cdot (\mathbf{u}^+ - \mathbf{u}^-) = 0 \qquad \text{for all } j \neq i$$

Implicit Function Theorem $\Longrightarrow$ For each i, there exists a curve $s \to S_i(s)(\mathbf{u}^-)$ of pints that satisfy $(RH)_i$.



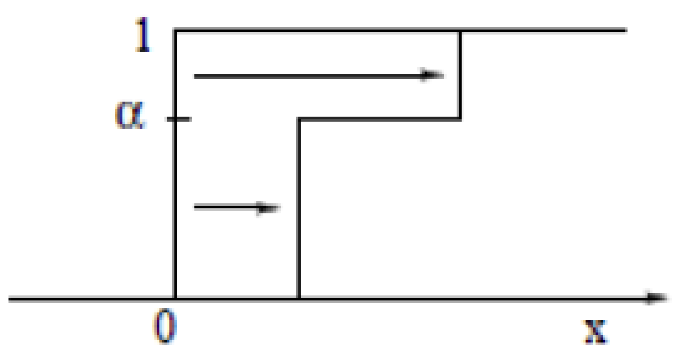
Non-uniqueness of Weak solutions

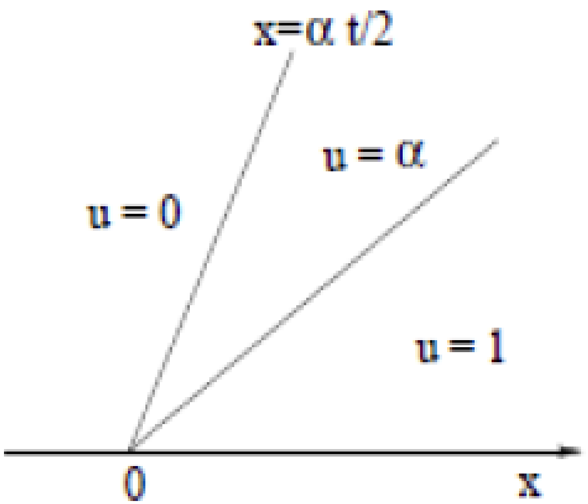
Example: a Cauchy problem for Burgers' equation

$$u_t + (u^2/2)_x = 0 \qquad u(0,x) = \begin{cases} 1 & \text{if} \quad x \geq 0 \\ 0 & \text{if} \quad x < 0 \end{cases}$$

Each $\alpha \in [0,1]$ yields a weak solution

$$u_\alpha(t,x) = \begin{cases} 0 & \text{if} \quad x < \alpha t/2 \\ \alpha & \text{if} \quad \alpha t/2 < x < (1+\alpha)t/2 \\ 1 & \textit{if} \quad x \geq (1+\alpha)t/2 \end{cases}$$





Admissibility Conditions on Shocks

$$\mathbf{u}_t + \mathbf{f}(\mathbf{u})_x = 0$$

- Solutions should be stable w.r.t. small initial perturbations
- Solutions should be limits of suitable approximations and/or physical regularisations (Vanishing viscosity, relaxation, $\cdots$)
- Any convex entropy should not increase
- $\cdots\cdots$

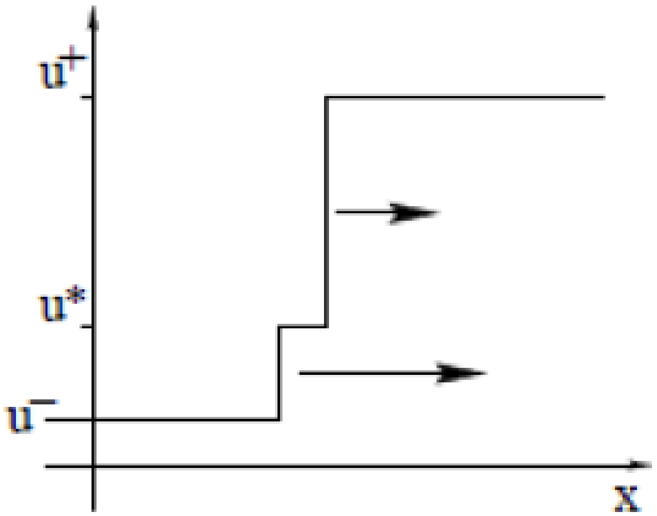
Stability conditions: the scalar case

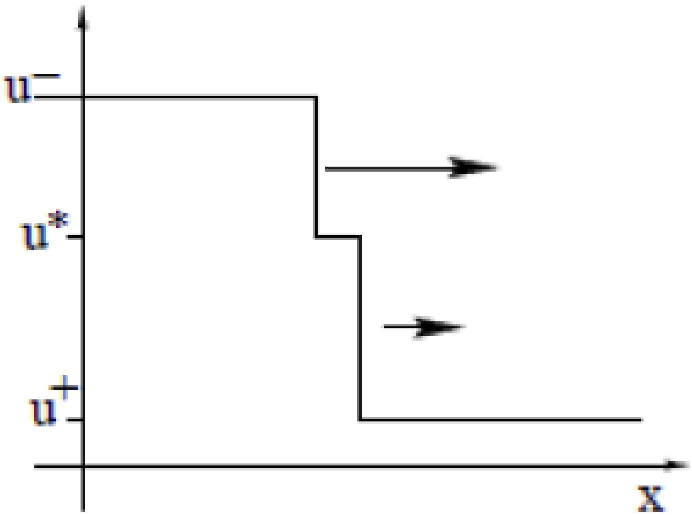
Perturb the shock with left and right states u^-, u^+ by inserting an intermediate state $u^* \in [u^-, u^+]$

Initial shock is stable $\Longleftrightarrow$

[speed of jump behind] $\leq$ [speed of jump ahead]

$$\frac{f(u^*) - f(u^-)}{u^* - u^-} \geq \frac{f(u^+) - f(u^*)}{u^+ - u^*}$$





Stability conditions: the scalar case

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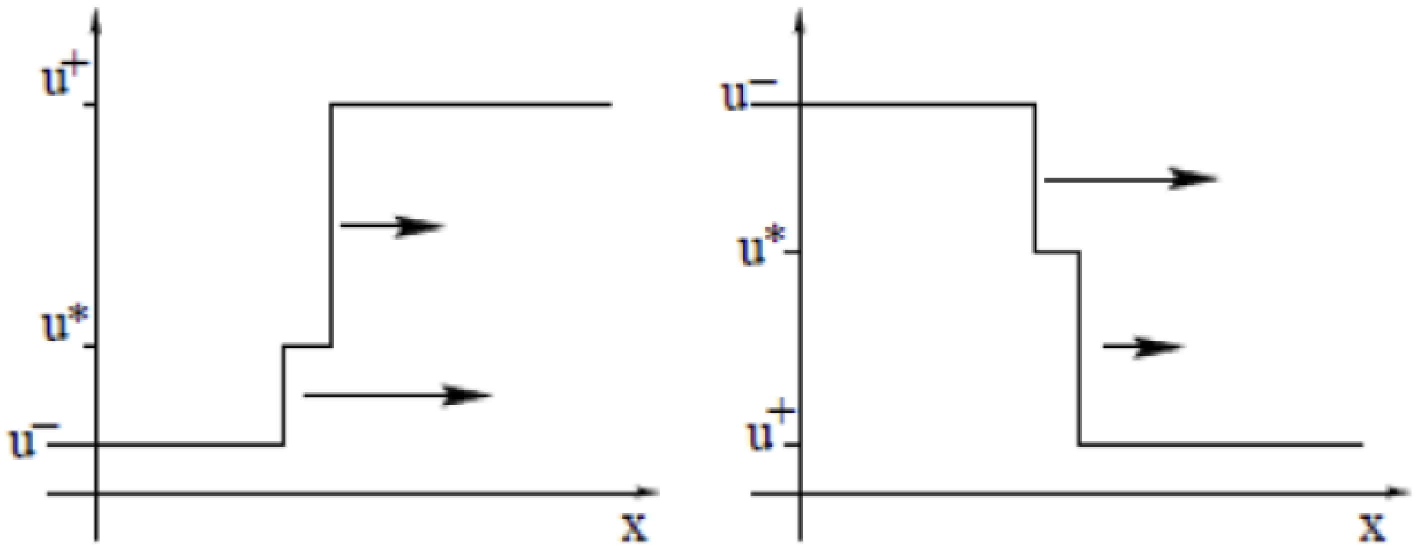
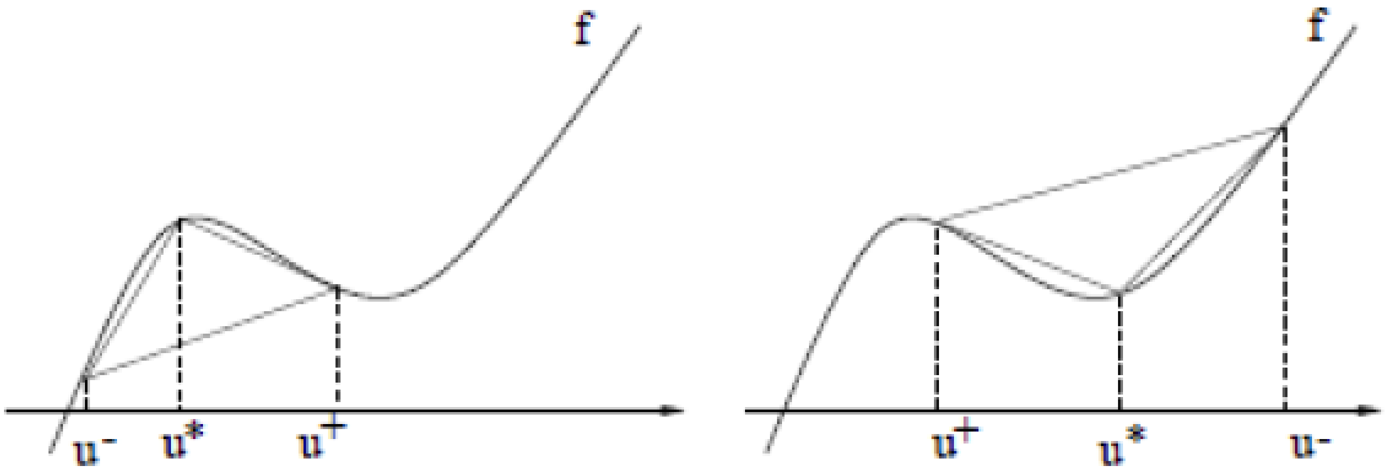


speed of a shock = slope of a secant line to the graph of f



Stability conditions:

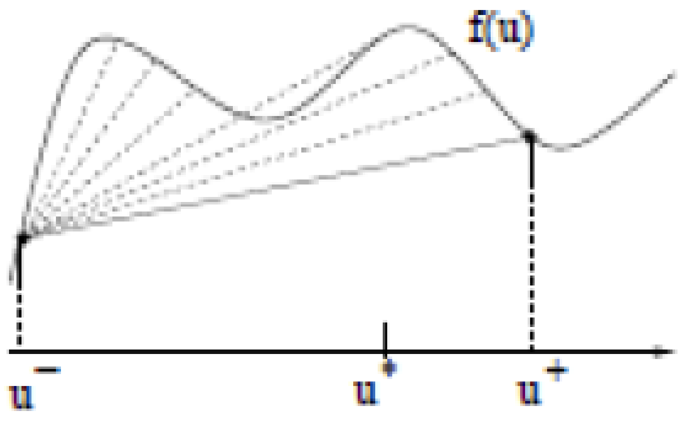
- when $u^- < u^+$ the graph of f should remain above the secant line
- when $u^- > u^+$, the graph of f should remain below the secant line

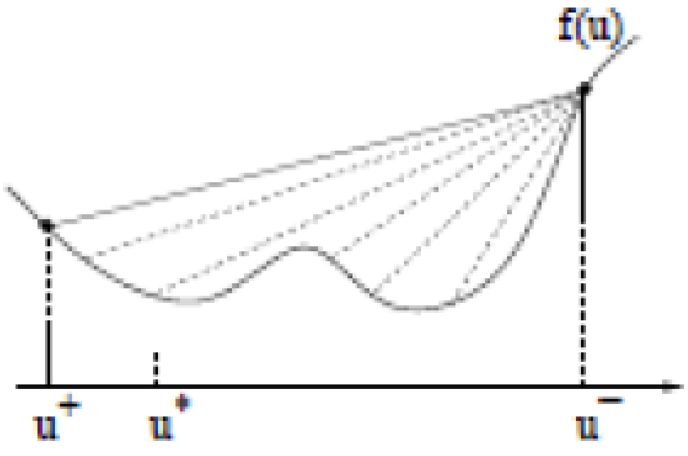
General stability conditions

Scalar case: stability holds if and only if

$$\frac{f(u^*) - f(u^-)}{u^* - u^-} \geq \frac{f(u^+) - f(u^-)}{u^+ - u^-}$$

for every intermediate state $u^* \in [u^-, u^+]$





Vector Valued Case: $\mathbf{u}^+ = S_i(\sigma)(\mathbf{u}^-)$ for some $\sigma \in \mathbb{R}$

Admissibility Condition (T.-P. Liu)

The speed $\lambda(\sigma)$ of the shock joining $\mathbf{u}^-$ with $\mathbf{u}^+$ must be less or equal to the speed of every smaller shock, joining $\mathbf{u}^-$ with an intermediate state $\mathbf{u}^* = S_i(s)(\mathbf{u}^-), s \in [0, \sigma]$:

$$\lambda(\mathbf{u}^-, \mathbf{u}^+) \leq \lambda(\mathbf{u}^-, \mathbf{u}^*)$$

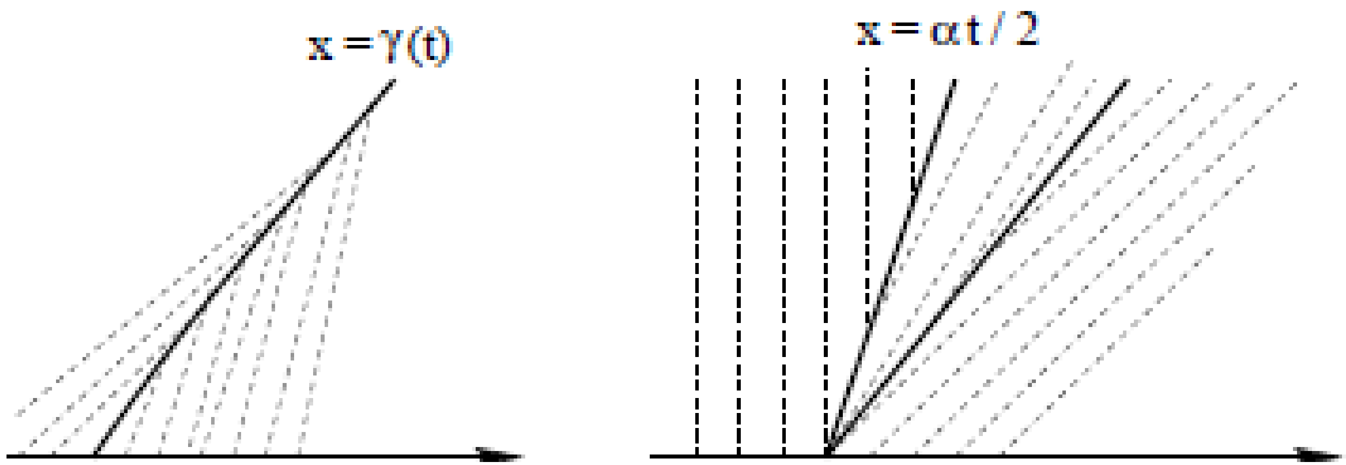
- The Liu condition singles out precisely the solutions which are limits of vanishing viscosity approximations

$$\mathbf{u}_t^\varepsilon + \mathbf{f}(\mathbf{u}^\varepsilon)_x = \varepsilon \mathbf{u}_{xx}^\varepsilon \qquad \mathbf{u}^\varepsilon \to \mathbf{u} \quad \text{as } \varepsilon \to 0$$

Admissibility Condition (P. Lax)

A shock connecting the states u^-, u^+, travelling with speed $\lambda = \lambda_i(u^-, u^+)$ is *admissible* if

$$\lambda_i(u^-) \;\geq\; \lambda_i(u^-, u^+) \;\geq\; \lambda_i(u^+)$$



- Geometric meaning: characteristics flow toward the shock from both sides
- The Liu condition implies the Lax condition

Mathematical Entropy – Entropy Flux

$$\mathbf{u}_t + \mathbf{f}(\mathbf{u})_x = 0$$

Definition: A function $\eta : \mathbb{R}^m \to \mathbb{R}$ is called an **Entropy**, with **Entropy Flux** $q : \mathbb{R}^m \to \mathbb{R}$ if

$$\nabla \eta(\mathbf{u}) \nabla \mathbf{f}(\mathbf{u}) = \nabla q(\mathbf{u})$$

For **smooth** solutions $\mathbf{u} = \mathbf{u}(t, x)$, this implies

$$\begin{aligned} \eta(\mathbf{u})_t + q(\mathbf{u})_x &= \nabla \eta(\mathbf{u})\, \mathbf{u}_t + \nabla q(\mathbf{u})\, \mathbf{u}_x \\ &= -(\nabla \eta(\mathbf{u}) \nabla \mathbf{f}(\mathbf{u}))\mathbf{u}_x + \nabla q(\mathbf{u})\, \mathbf{u}_x = 0 \end{aligned}$$

$\Longrightarrow$ $\eta(\mathbf{u})$ is an additional conserved quantity,
with flux $q(\mathbf{u})$

Existence of Entropy – Entropy Flux Pairs

$$\nabla \eta(\mathbf{u}) \, \nabla \mathbf{f}(\mathbf{u}) = \nabla q(\mathbf{u}).$$

$$\left(\frac{\partial \eta}{\partial u_1} \cdots \frac{\partial \eta}{\partial u_m}\right) \begin{pmatrix} \frac{\partial f_1}{\partial u_1} & \cdots & \frac{\partial f_1}{\partial u_m} \\ & \cdots & \\ \frac{\partial f_m}{\partial u_1} & \cdots & \frac{\partial f_m}{\partial u_m} \end{pmatrix} = \left(\frac{\partial q}{\partial u_1} \cdots \frac{\partial q}{\partial u_m}\right)$$

- A systems of m equations for 2 unknown functions: $\eta(\mathbf{u})$ and $q(\mathbf{u})$
- Over-determined if $m > 2$
- However, most of physical systems (described by several conservation laws) are endowed with natural entropies

Entropy Admissibility Condition

A weak solution u of the hyperbolic system $\mathrm{u}_t + \mathrm{f}(\mathrm{u})_x = 0$ is **Entropy Admissible** if

$$\eta(\mathrm{u})_t + q(\mathrm{u})_x \leq 0$$

in the sense of distributions, for every entropy-entropy flux pair (η, q) with $\nabla^2\eta(\mathrm{u}) \geq 0$, i.e. convex.

$$\iint \left\{\eta(\mathrm{u})\varphi_t + q(\mathrm{u})\varphi_x\right\} dxdt \geq 0 \qquad \varphi \in C_c^\infty,\ \varphi \geq 0$$

- Smooth solutions conserve all entropies
- Solutions with shocks are admissible if they dissipate all convex entropies

Consistency with Vanishing Viscosity Approximations

$$\mathbf{u}_t^\varepsilon + \mathbf{f}(\mathbf{u}^\varepsilon)_x = \varepsilon \mathbf{u}_{xx}^\varepsilon \qquad \mathbf{u}^\varepsilon \to \mathbf{u} \quad \text{as } \varepsilon \to 0$$

For any entropy-entropy flux pair

$$(\eta(\mathbf{u}), q(\mathbf{u})) \qquad \nabla^2 \eta(\mathbf{u}) \geq 0,$$

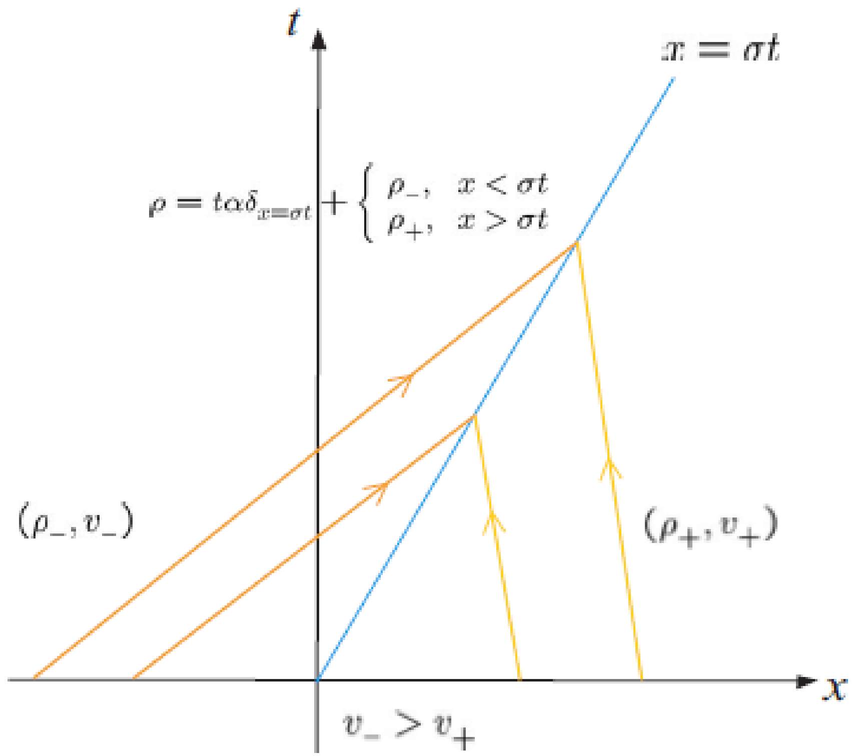
multiply $\nabla \eta(\mathbf{u}^\varepsilon)$ both sides of the system yields

$$\begin{aligned} \eta(\mathbf{u}^\varepsilon)_t + q(\mathbf{u}^\varepsilon)_x &= \varepsilon \eta(\mathbf{u}^\varepsilon)_{xx} - \varepsilon (\mathbf{u}_x)^\top \nabla^2 \eta(\mathbf{u}^\varepsilon) \mathbf{u}_x \\ &\leq \varepsilon \eta(\mathbf{u}^\varepsilon)_{xx} \to 0 \end{aligned}$$

in the sense of distributions.

Pressureless Euler Equations

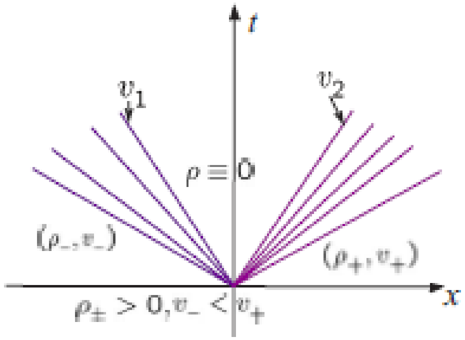
$$\partial_t \rho + \partial_x(\rho v) = 0, \qquad \partial_t(\rho v) + \partial_x(\rho v^2) = 0$$



$$\alpha = \frac{1}{\sqrt{1+\sigma^2}}(\sigma[\rho] - [\rho v]) > 0, \qquad \sigma = \frac{\sqrt{\rho_+}v_+ + \sqrt{\rho_-}v_-}{\sqrt{\rho_+} + \sqrt{\rho_-}} \in (v_+, v_-)$$

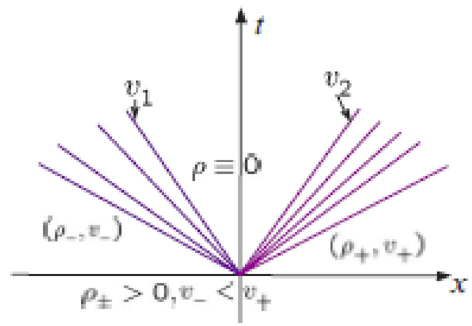
Isentropic Euler Equations Pressure Function $p(\rho) = \kappa\rho^\gamma$

$$\partial_t \rho + \partial_x(\rho v) = 0, \qquad \partial_t(\rho v) + \partial_x(\rho v^2 + p(\rho)) = 0$$

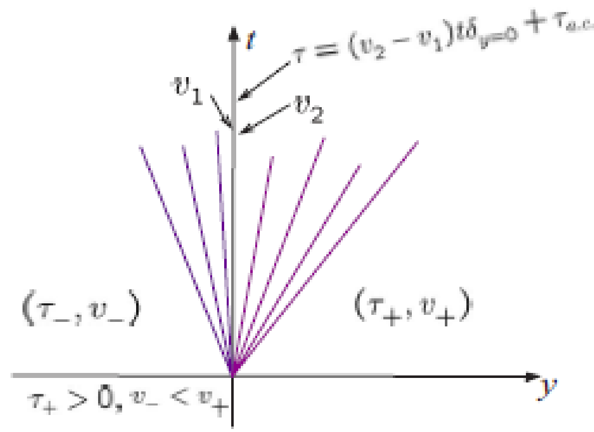


Isentropic Euler Equations Pressure Function $p(\rho) = \kappa\rho^\gamma$

$$\partial_t \rho + \partial_x(\rho v) = 0, \qquad \partial_t(\rho v) + \partial_x(\rho v^2 + p(\rho)) = 0$$



$$(t, x) \to (t, y): \; y_t = \rho(t, x), y_x = -(\rho v)(t, x); \qquad \tau(t, y) = 1/\rho(t, x)$$



Scalar Conservation Laws

$$(*) \quad \begin{cases} u_t + f(u)_x = 0 \\ u|_{t=0} = u_0(x) \in L^\infty \end{cases} \qquad \begin{array}{l} f(0)=0 \\ f''(u) \geq c_0 > 0 \end{array}$$

Hamilton-Jacobi Equations

$$(**) \quad \begin{cases} W_t + f(W_x) = 0 \\ W|_{t=0} = h(x) = \int_0^x u_0(y)\,dy \quad \text{Lipschitz} \end{cases}$$

$(**) \Longrightarrow (*)$ $\qquad u(t,x) = W_x(t,x)$

$(*) \Longrightarrow (**)$ $\qquad W(t,x) = \int_0^x u(t,y)\,dy - \int_0^t f(u(\tau,0))\,d\tau$

Legendre Transform $\quad L = f^*$

$$L(q) = f^*(q) = \sup_{p \in \mathbb{R}} \{ qp - f(p) \}$$

$$\hookrightarrow f(p) = L^*(p) = \sup_{q \in \mathbb{R}} \{ pq - L(q) \}$$

Theorem (Convex Duality)

$$\begin{cases} f: p \longmapsto f(p) \text{ is convex} \\ \lim_{|p| \to \infty} \frac{f(p)}{|p|} = \infty \end{cases}$$

$$\Longleftrightarrow$$

$$\begin{cases} L: q \longmapsto L(q) \text{ is convex} \\ \lim_{|q| \to \infty} \frac{L(q)}{|q|} = \infty \end{cases}$$

$\boxed{*}$ $L = f^*, \quad f = L^*$

Hamilton-Jacobi Equations

$$(**)\begin{cases} W_t + f(W_x) = 0 & f'' \geq c_0 > 0 \\ W|_{t=0} = h(x) & \text{Lipschitz} \end{cases}$$

Motivation: Dynamic Programming in Control Theory

cf. Evans, Ch. 10.3

The Value Function

$$(A)\quad W(t,x) = \inf\left\{ \int_0^t f^*(\dot{v}(s))\,ds + h(y) \;\middle|\; \begin{array}{l} v \in C^1 \\ v(0) = y \\ v(t) = x \end{array} \right\}$$

is a solution of $(**)$

⇒ **Hopf-Lax Formula**. For $x \in \mathbb{R}$, $t > 0$

$$(B)\quad W(t,x) = \min_{y \in \mathbb{R}} \left\{ t f^*\left(\frac{x-y}{t}\right) + h(y) \right\}$$

Proof (A) $\Rightarrow$ (B)

1. Fix any $y \in \mathbb{R}$. Define

$$v(s) := y + \frac{s}{t}(x-y), \quad 0 \le s \le t$$

(A) $\Longrightarrow$ $$w(t,x) \le \int_0^t f^*(\dot v(s))\,ds + h(y)$$

$$= t f^*\left(\frac{x-y}{t}\right) + h(y)$$

$\hookrightarrow$ $$w(t,x) \le \inf_{y \in \mathbb{R}} \left\{ t f^*\left(\frac{x-y}{t}\right) + h(y) \right\}$$

2. If $v(\cdot)$ is any C^1-function satisfying $v(t)=x$, by the Convexity of f^*

$$f^*\left(\frac{1}{t}\int_0^t \dot v(s)\,ds\right) \le \frac{1}{t}\int_0^t f^*(\dot v(s))\,ds$$

// $v(0)=y$ $\Downarrow$

$$t\, f^*\left(\frac{x-y}{t}\right) + h(y) \le \int_0^t f^*(\dot v(s))\,ds + h(y)$$

$\hookrightarrow$ $$\inf_{y \in \mathbb{R}} \left\{ t f^*\left(\frac{x-y}{t}\right) + h(y) \right\} \le u(t,x)$$

3. $$u(t,x) = \inf_{y \in \mathbb{R}} \left\{ t f^*\left(\frac{x-y}{t}\right) + h(y) \right\}$$

The infimum can be achieved!

Hopf-Lax Formula: For $x \in \mathbb{R}$, $t > 0$

$$W(t,x) = \min_{y \in \mathbb{R}} \left\{ t f^*\left(\frac{x-y}{t}\right) + h(y) \right\}$$

$\Rightarrow$

(i) $W(t,x)$ is Lipschitz Continuous $\xrightarrow{\text{Rademacher's Thm}}$ Differentiable a.e.

(ii) $W_t(t,x) + f(W_x(t,x)) = 0$ a.e.

(iii) $W(0,x) = h(x)$.

(iv) $W(t, x+z) - 2W(t,x) + W(t, x-z) \leq C\left(1+\frac{1}{t}\right)|z|^2$

$\forall\, x, z \in \mathbb{R}$, $t > 0$.

for some $C > 0$.

(v) $W(t,x)$ is unique in the class of weak solutions that satisfy (i)–(iv).

Scalar Conservation Laws

$$\begin{cases} u_t + f(u)_x = 0 \\ u|_{t=0} = u_0(x) \in L^\infty \end{cases}$$

Candidate.

Differentiable a.e.

$$u(t,x) := \frac{\partial}{\partial x}\left[\min_{y\in\mathbb{R}}\left\{t f^*\left(\frac{x-y}{t}\right) + h(y)\right\}\right]$$

$$h(y) = \int_0^y u_0(x)\,dx$$

Lax-Oleinik Formula

$$\begin{cases} f'' \geq c_0 > 0 \\ u_0 \in L^\infty(\mathbb{R}) \end{cases}$$

(i) For each $t>0$, for all but at most countably many values of $x \in \mathbb{R}$, $\exists$ a unique $y(t,x)$ such that

$$\min_{y \in \mathbb{R}} \left\{ t f^*\left(\frac{x-y}{t}\right) + h(y) \right\} = t f^*\left(\frac{x-y(t,x)}{t}\right) + h(y(t,x))$$

(ii) The mapping $x \longmapsto y(t,x)$ is nondecreasing

(iii) For each $t>0$

$$u(t,x) = (f')^{-1}\left(\frac{x-y(t,x)}{t}\right) \quad \text{for a.e. } x$$

(iv) $u(t,x)$ is a weak solution satisfying

$$u(t,x+z) - u(t,x) \leq \frac{C}{t} z \qquad \forall\, t>0,\ x, z \in \mathbb{R},\ z \geq 0$$

for some $C>0$.

(v) $u(t,x)$ is unique in the class of weak solutions that satisfy (iv).

Proof of Lax-Oleinik formula

1. $y \mapsto t f^*\left(\frac{x-y}{t}\right) + h(y)$ has a minimum at $y = y(t,x)$

↳ $z \mapsto t f^*\left(\frac{x - y(t,z)}{t}\right) + h(y(t,z))$

has a minimum at $z = x$

↳ $-(f^*)'\left(\frac{x-y(t,x)}{t}\right) y_x(t,x) + \frac{\partial}{\partial x} h(y(t,x)) = 0$

2. $u(t,x) = \frac{\partial}{\partial x}\left[t f^*\left(\frac{x-y(t,x)}{t}\right) + h(y(t,x))\right]$

$$= (f^*)'\left(\frac{x-y(t,x)}{t}\right)(1 - y_x(t,x)) + \frac{\partial}{\partial x} h(y(t,x))$$

$$\overset{(1)}{=} (f^*)'\left(\frac{x-y(t,x)}{t}\right) = (f')^{-1}\left(\frac{x-y(t,x)}{t}\right)$$

$$f^*(q) = \max_{p\in\mathbb{R}} (qp - f(p)) = q p^* - f(p^*)$$

$$q = f'(p^*), \quad p^* = (f')^{-1}(q).$$

$$(f^*)'(q) = p^* + q p^*_q - \underbrace{f'(p^*)}_{q} p^*_q$$

$$= p^*(q) = (f')^{-1}(q)$$

3. Oleinik E-Condition

Note that, in computing the minimum for $y(t,x)$, we need only consider those y s.t. $\left|\frac{x-y}{t}\right| \le C$, for some C

$\Rightarrow$ W.O.L.G. We may assume that $(f')^{-1}$ is globally Lipschitz Continuous.

Note $\quad y(t,\cdot)\nearrow, \quad (f')^{-1}\nearrow$

For any $\delta > 0$

$$
\begin{aligned}
u(t,x) &= (f')^{-1}\left(\frac{x-y(t,x)}{t}\right) \\
&\ge (f')^{-1}\left(\frac{x-y(t,x+\delta)}{t}\right) \\
&\ge (f')^{-1}\left(\frac{x+\delta-y(t,x+\delta)}{t}\right) - \frac{Lip((f')^{-1})\delta}{t} \\
&= u(t,x+\delta) - \frac{Lip((f')^{-1})\delta}{t}.
\end{aligned}
$$

Oleinik E-Condition

$$(E) \quad u(t, x+z) - u(t, x) \le \frac{C}{t} z. \qquad \forall t>0,\ x, z \in \mathbb{R},\ z \ge 0$$

Lax Entropy Condition

$$u(t, x-0) \ge u(t, x+0)$$

$(E) \Rightarrow \quad x \longrightarrow u(t,x) - \frac{C}{t}x$ is nonincreasing

↳ $\exists$ left and right hand limits at each point

↳ $u(t, x-0) - \frac{C}{t}x \ge u(t, x+0) - \frac{C}{t}x$

↳ $u(t, x-0) \ge u(t, x+0)$

Applications

I. Riemann Problem

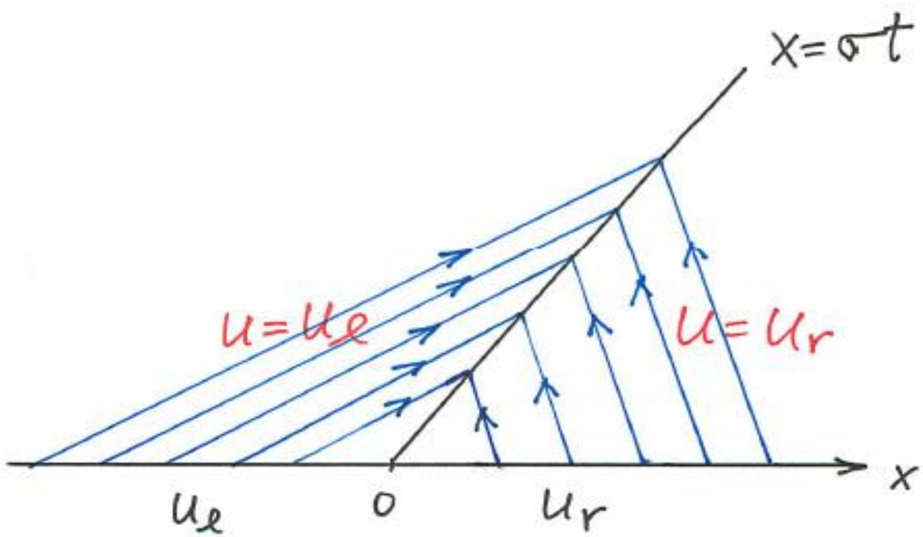
$$u_0(x) = \begin{cases} u_l & x<0 \\ u_r & x>0 \end{cases}$$

(i) $u_l > u_r \Rightarrow$ Shock Wave Solution

$$u(t,x) = \begin{cases} u_l & \frac{x}{t} < \sigma \\ u_r & \frac{x}{t} > \sigma \end{cases}$$

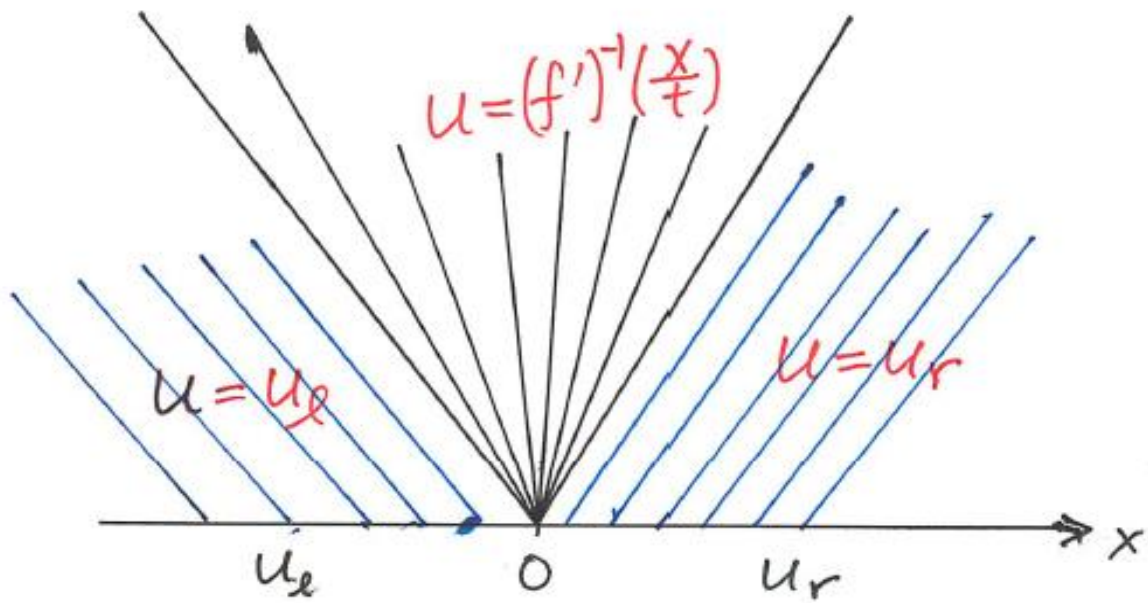
where

$$\sigma = \frac{[f]}{[u]} = \frac{f(u_r)-f(u_l)}{u_r - u_l}$$ shock speed



(ii) $u_l < u_r \Longrightarrow$ Rarefaction Wave Solution

$$u(t,x) = \begin{cases} u_l & \frac{x}{t} \le f'(u_l) \\ (f')^{-1}\left(\frac{x}{t}\right) & f'(u_l) < \frac{x}{t} < f'(u_r) \\ u_r & \frac{x}{t} \ge f'(u_r) \end{cases}$$



II. Decay in L^∞

If $\begin{cases} f(0)=0 \\ u_0 \in L^1 \cap L^\infty(\mathbb{R}) \end{cases}$

$\Rightarrow$ $\exists\, C>0$ such that

$$|u(t,x)| \le \frac{C}{t^{1/2}} \quad \forall x\in\mathbb{R},\ t>0$$

* Different from the linear Case
* The decay rate $t^{-1/2}$ is optimal.

III. Decay to the N-Wave in L^1

If $\begin{cases} f(0)=0 \\ u_0 \text{ has compact support} \end{cases}$

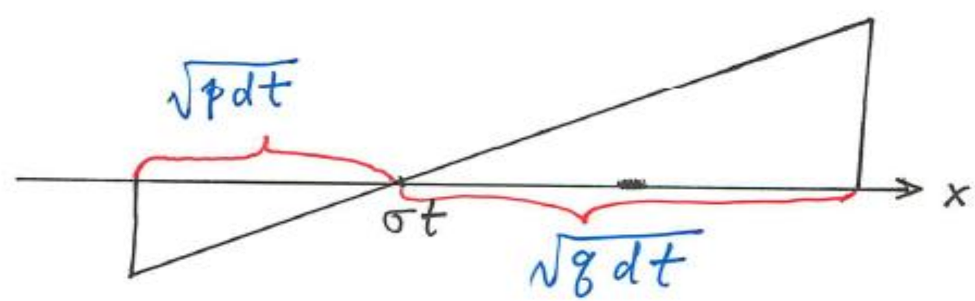
$\Longrightarrow$

$$\| u(t,\cdot) - N(t,\cdot) \|_{L^1(\mathbb{R})} \leq \frac{C}{t^{1/2}} \qquad \forall t>0$$

$N(t,\cdot) =$

$$\begin{cases} \frac{1}{d}\left(\frac{x}{t}-\sigma\right) & -\sqrt{pdt} < x - \ t < \sqrt{qdt} \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{cases} p = -2 \min_{y\in\mathbb{R}} \left(\int_{-\infty}^{y} u_0(x)\,dx \right) \geq 0 \\ q = 2 \max_{y\in\mathbb{R}} \left(\int_{y}^{\infty} u_0(x)\,dx \right) \geq 0 \\ d = f''(0) > 0 \\ \sigma = f'(0) \end{cases}$$



N- Wave

Ex

$$\begin{cases} u_t + \left(\frac{u^2}{2}\right)_x = 0 \\ u|_{t=0} = u_0(x) = \begin{cases} 0 & x<0 \\ 1 & 0 \le x \le 1 \\ 0 & x>1 \end{cases} \end{cases}$$

Solution

$0 \le t \le 2$

$$u(t,x) = \begin{cases} 0 & x<0 \\ \frac{x}{t} & 0<x<t \\ 1 & t<x<1+\frac{t}{2} \\ 0 & x>1+\frac{t}{2} \end{cases}$$

$t \ge 2$

$$u(t,x) = \begin{cases} 0 & x<0 \\ \frac{x}{t} & 0<x<\sqrt{2t} \\ 0 & x>\sqrt{2t} \end{cases}$$

L^∞:

$$|u(t,x)| \le \left|\frac{x}{t}\right| \le \frac{\sqrt{2t}}{t} = \frac{\sqrt{2}}{\sqrt{t}}$$

L^1:

$$N(t,x) = \begin{cases} \frac{x}{t} & 0<x<\sqrt{2t} \\ 0 & \text{otherwise.} \end{cases}$$

$$[\sigma=0,\ d=1,\quad p=0,\ \gamma=2]$$