$\boxed{\text{II-3}}$ Densities

Thm $\begin{cases} 0 < s < n \\ E \subset \mathbb{R}^n, \text{ is } \mathcal{H}^s\text{-measurable}, \ \mathcal{H}^s(E) < \infty \end{cases}$

$\Rightarrow$ (i) $\displaystyle \lim_{r \to 0} \frac{\mathcal{H}^s(B(x,r) \cap E)}{\alpha(s) r^s} = 0$ for $\mathcal{H}^s$-a.e. $x \in \mathbb{R}^n - E$

(ii) $\displaystyle \frac{1}{2^s} \le \limsup_{r \to 0} \frac{\mathcal{H}^s(B(x,r) \cap E)}{\alpha(s) r^s} \le 1$

for $\mathcal{H}^s$-a.e. $x \in E$

Rm

(i) It is possible to have

$$\begin{cases} \displaystyle \limsup_{r \to 0} \frac{\mathcal{H}^s(B(x,r) \cap E)}{\alpha(s) r^s} < 1 \\ \displaystyle \liminf_{r \to 0} \frac{\mathcal{H}^s(B(x,r) \cap E)}{\alpha(s) r^s} = 0 \end{cases}$$

for $\mathcal{H}^s$-a.e. $x \in E$ even if $0 < \mathcal{H}^s(E) < \infty$

Exercise 18: Examples

(ii) $\forall E \subset \mathbb{R}^n$ $\mathcal{L}^n$-measurable

$$\lim_{r \to 0} \frac{|B(x,r) \cap E|}{\underset{\alpha(n) r^n}{\underset{\shortparallel}{|B(x,r)|}}} = \begin{cases} 1 & \text{for } \mathcal{L}^n\text{-a.e. } x \in E \\ 0 & \text{for } \mathcal{L}^n\text{-a.e. } x \in \mathbb{R}^n - E \end{cases}$$

Ideas of Proof.

(i). Fix $t>0$

$$A_t \triangleq \left\{ x \in \mathbb{R}^n - E \;\middle|\; \limsup_{r\to 0} \frac{\mathcal{H}^s(B(x,r)\cap E)}{\alpha(s) r^s} > t \right\}$$

Claim $\mathcal{H}^s(A_t) = 0, \quad \forall\, t>0 \quad \to$ (i')

$\because$ $\mathcal{H}^s \llcorner E$ is a Radon measure

$\hookrightarrow$ $\forall \varepsilon>0, \;\exists\; K \subset\subset E$ s.t.

$$\mathcal{H}^s(E-K) \le \varepsilon.$$

Set $U \triangleq \mathbb{R}^n - K$ open

$\hookrightarrow$ $A_t \subset U$

Fix $\delta>0$.

$$\mathcal{F} \triangleq \left\{ B(x,r) \;\middle|\; B(x,r)\subset U,\; 0<r<\delta,\; \frac{\mathcal{H}^s(B(x,r)\cap E)}{\alpha(s) r^s} > t \right\}$$

$\xrightarrow[\text{Vitali CT}]{}$ $\exists\, \{B_i\}_{i=1}^\infty \subset \mathcal{F}$, $B_i \cap B_j = \phi$, $i \neq j$. s.t.

$$A_t \subset \bigcup_{i=1}^\infty \hat{B}_i, \qquad B_i = \overline{B(x_i, r_i)}$$

$\hookrightarrow$
$$\mathcal{H}^s_{10\delta}(A_t) \le \sum_{i=1}^\infty \alpha(s)(5r_i)^s \le \frac{5^s}{t} \sum_{i=1}^\infty \mathcal{H}^s(B(x_i, r_i)\cap E)$$
$$\le \frac{5^s}{t}\mathcal{H}^s(U\cap E) = \frac{5^s}{t}\mathcal{H}^s(E-K) \le \frac{5^s}{t}\varepsilon$$

$\xrightarrow{\delta\to 0}$
$$\mathcal{H}^s(A_t) \le 5^s t^{-1} \varepsilon \xrightarrow[\varepsilon\to 0]{} 0.$$

(ii)-1 claim $\limsup_{r\to 0} \frac{\mathcal{H}^s(B(x,r)\cap E)}{\alpha(s) r^s} \le 1$

for $\mathcal{H}^s$-a.e. $x\in E$

Fix $\varepsilon>0$, $t>1$.

$$B_t \triangleq \left\{ x\in E \;\middle|\; \limsup_{r\to 0} \frac{\mathcal{H}^s(B(x,r)\cap E)}{\alpha(s) r^s} > t \right\}$$

?? $\mathcal{H}^s(B_t)=0 \quad \forall\, t>1 \quad \rightarrow$ claim

$\because \mathcal{H}^s \llcorner E$ is Radon

$\hookrightarrow \exists$ open set $U\supset B_t$ s.t.

$$\mathcal{H}^s(U\cap E) \le \mathcal{H}^s(B_t)+\varepsilon$$

Define

$$\mathcal{F} \triangleq \left\{ B(x,r) \;\middle|\; B(x,r)\subset U,\ 0<r<\delta,\ \frac{\mathcal{H}^s(B(x,r)\cap E)}{\alpha(s) r^s} > t \right\}.$$

$\xrightarrow[\text{VCT-Corollary}]{}$ $\exists \{\underset{\substack{\parallel \\ B_i(x_i,r_i)}}{B_i}\}_{i=1}^{\infty} \subset \mathcal{F}$, $B_i\cap B_j=\phi$, $i\ne j$

s.t. $$B_t \subset \left(\bigcup_{i=1}^{m} B_i\right) \cup \left(\bigcup_{i=m+1}^{\infty} \widehat{B}_i\right),\quad \forall\, m=1,2\dots$$

$\hookrightarrow$
$$\mathcal{H}^s_{10\delta}(B_t) \le \sum_{i=1}^{m} \alpha(s) r_i^s + \sum_{i=m+1}^{\infty} \alpha(s)(5r_i)^s$$
$$\le \frac{1}{t}\sum_{i=1}^{m} \mathcal{H}^s(B_i\cap E) + \frac{5^s}{t}\sum_{i=m+1}^{\infty} \mathcal{H}^s(B_i\cap E).$$
$$\le \frac{1}{t}\mathcal{H}^s(U\cap E) + \frac{5^s}{t}\mathcal{H}^s\left(\bigcup_{i=m+1}^{\infty}(B_i\cap E)\right).$$

$\forall\, m=1,2,\dots$

$$\mathcal{H}^s_{10\delta}(B_t) \le \frac{1}{t}\mathcal{H}^s(U\cap E) + \frac{5^s}{t}\underbrace{\mathcal{H}^s\Big(\bigcup_{i=m+1}^{\infty}(B_i\cap E)\Big)}_{\xrightarrow{m\to\infty}\ 0}$$

$$\hookrightarrow \quad \mathcal{H}^s_{10\delta}(B_t) \le \frac{1}{t}\mathcal{H}^s(U\cap E) \le \frac{1}{t}\left(\mathcal{H}^s(B_t)+\varepsilon\right).$$

$\delta\to 0$ first $\qquad\qquad$ $\varepsilon \to 0$ second

$$\mathcal{H}^s(B_t) \quad \le \quad \frac{1}{t}\mathcal{H}^s(B_t) \qquad t>1$$

$$\mathcal{H}^s(B_t) \le \mathcal{H}^s(E) < \infty$$

$$\hookrightarrow \quad \mathcal{H}^s(B_t) = 0 \quad \forall\, t>1.$$

(ii)-2 claim $\displaystyle \limsup_{r\to 0} \frac{\mathcal{H}^s_\infty(B(x,r)\cap E)}{\alpha(s)r^s} \ge \frac{1}{2^s}$

for $\mathcal{H}^s$-a.e. $x\in E$

$\hookrightarrow$

$$\limsup_{r\to 0}\frac{\mathcal{H}^s(B(x,r)\cap E)}{\alpha(s)r^s}$$

$$\ge \limsup_{r\to 0}\frac{\mathcal{H}^s_\infty(B(x,r)\cap E)}{\alpha(s)r^s}$$

$$\ge \frac{1}{2^s}. \qquad\qquad \Rightarrow \square$$

For $\delta > 0$, $\tau \in (0,1)$

$$E(\delta,\tau) \triangleq \left\{ x \in E \,\middle|\, \begin{array}{l} \mathcal{H}^s_\delta(C \cap E) \le \tau \, \alpha(s) \left(\frac{\operatorname{diam} C}{2} \right)^s \\ \forall C \subset \mathbb{R}^n,\ x \in C,\ \operatorname{diam} C \le \delta \end{array} \right\}$$

$\rightarrow$ If $\{C_j\}_{j=1}^\infty \subset \mathbb{R}^n$, $\operatorname{diam} C_j \le \delta$,

$C_j \cap E(\delta,\tau) \neq \phi$. $E(\delta,\tau) \subset \bigcup_{j=1}^\infty C_j$

$\rightarrow$

$$\begin{aligned} \mathcal{H}^s_\delta(E(\delta,\tau)) &\le \sum_{j=1}^\infty \mathcal{H}^s_\delta(C_j \cap E(\delta,\tau)) \\ &\le \sum_{j=1}^\infty \mathcal{H}^s_\delta(C_j \cap E) \\ &\le \tau \sum_{j=1}^\infty \alpha(s) \left(\frac{\operatorname{diam} C_j}{2} \right)^s \end{aligned}$$

$\rightarrow$ $\mathcal{H}^s_\delta(E(\delta,\tau)) \le \tau \mathcal{H}^s_\delta(E(\delta,\tau))$ $\le \mathcal{H}^s_\delta(E) \le \mathcal{H}^s(E) < \infty$

$\xrightarrow{\tau \in (0,1)}$ $\mathcal{H}^s_\delta(E(\delta,\tau)) = 0$

$\rightarrow$ $\boxed{\mathcal{H}^s(E(\delta, 1-\delta)) = 0}$

Set

$$D=\left\{x\in E \;\middle|\; \limsup_{r\to 0}\frac{\mathcal{H}^s_\infty(B(x,r)\cap E)}{\alpha(s)r^s}<\frac{1}{2^s}\right\}$$

If we can show

$$\underline{\mathcal{H}^s(D)=0} \quad \longrightarrow \quad \underline{\text{claim}}$$

$\forall\, x\in D,\ \exists\, \delta>0$ s.t.

$$\frac{\mathcal{H}^s_\infty(B(x,r)\cap E)}{\alpha(s)r^s}\le\frac{1-\delta}{2^s} \qquad \forall\, 0<r\le\delta \quad (*)$$

If $x\in C\cap D$, $\operatorname{diam} C\le\delta$

$$\mathcal{H}^s_\delta(C\cap E)=\mathcal{H}^s_\infty(C\cap E)\le\mathcal{H}^s_\infty(B(x,\operatorname{diam}C)\cap E)$$

$$\overset{(*)}{\le}(1-\delta)\,\alpha(s)\left(\frac{\operatorname{diam}C}{2}\right)^s.$$

$\hookrightarrow \quad x\in E(\delta, 1-\delta)$

$$\Longrightarrow \quad D\subset\bigcup_{k=1}^\infty E\left(\frac{1}{k}, 1-\frac{1}{k}\right).$$

$\hookrightarrow \quad \mathcal{H}^s(D)=0$ $\square$

$\boxed{\text{II-4}}$ Elementary Properties of Functions via Hausdorff Measures.

A. Lipschitz Mappings

$f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is called Lipschitz

if $\exists\ C > 0$, s.t.

$$|f(x) - f(y)| \leq C|x-y|, \quad \forall x, y \in \mathbb{R}^n$$

$$\mathrm{Lip}(f) \triangleq \sup\left\{ \frac{|f(x)-f(y)|}{|x-y|} \;\middle|\; x, y \in \mathbb{R}^n, \ x \neq y \right\}$$

Thm. $0 \leq s < \infty$

(i) $\mathcal{H}^s(f(A)) \leq (\mathrm{Lip}(f))^s \mathcal{H}^s(A), \quad \forall A \subset \mathbb{R}^n$

(ii) $\begin{cases} n > k \\ P: \mathbb{R}^n \longrightarrow \mathbb{R}^k \quad \text{the usual projection} \end{cases}$

$\hookrightarrow \ \mathcal{H}^s(P(A)) \leq \mathcal{H}^s(A)$

(i) $\Rightarrow$ (ii)

$\boxed{\mathrm{Lip}(P) = 1}$

Proof of (i)

Fix $\delta > 0$

Choose $\{C_i\}_{i=1}^{\infty} \subset \mathbb{R}^n$ s.t. $\operatorname{diam} C_i \leq \delta$, $A \subset \bigcup_{i=1}^{\infty} C_i$

$$\Rightarrow \begin{cases} \operatorname{diam} f(C_i) \leq \operatorname{Lip}(f) \operatorname{diam} C_i \leq \operatorname{Lip}(f)\,\delta \\ f(A) \subset \bigcup_{i=1}^{\infty} f(C_i) \end{cases}$$

$$\Rightarrow \mathcal{H}^s_{\operatorname{Lip}(f)\delta}(f(A)) \leq \sum_{i=1}^{\infty} \alpha(s) \left(\frac{\operatorname{diam} f(C_i)}{2} \right)^s$$

$$\leq \left(\operatorname{Lip}(f)\right)^s \sum_{i=1}^{\infty} \alpha(s) \left(\frac{\operatorname{diam} C_i}{2} \right)^s$$

Take infima over all such sets $\{C_i\}_{i=1}^{\infty}$

$$\hookrightarrow \mathcal{H}^s_{\operatorname{Lip}(f)\delta}(f(A)) \leq (\operatorname{Lip}(f))^s \mathcal{H}^s_\delta(A)$$

$$\downarrow \delta \to 0 \qquad\qquad \downarrow \delta \to 0$$

$$\mathcal{H}^s(f(A)) \leq (\operatorname{Lip}(f))^s \mathcal{H}^s(A)$$

B. Graphs of Lipschitz functions

$f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$, $A \subset \mathbb{R}^n$

The graphs of f over A.

$$G(f;A) \triangleq \{(x, f(x)) \mid x \in A\} \subset \mathbb{R}^n \times \mathbb{R}^m = \mathbb{R}^{n+m}$$

Thm Assume $f: \mathbb{R}^n \to \mathbb{R}^m$, $\mathcal{L}^n(A) > 0$

↳ (i) $\mathcal{H}\text{-dim}(G(f;A)) \geq n$.

(ii) If f is Lipschitz,

↳ $\boxed{\mathcal{H}\text{-dim}(G(f;A)) = n}$

* $\mathcal{H}^n(G(f;A))$ can be computed via the Area formula according to the usual rules of Calculus.

Proof.

(i) Let $P: \mathbb{R}^{n+m} \to \mathbb{R}^n$ be the projection

↳ $\mathcal{H}^n(G(f;A)) \geq \mathcal{H}^n(A) = \mathcal{L}^n(A) > 0.$

$$\mathcal{H}^n(G(f;A)) \geq \mathcal{H}^n(P(G(f;A))) = \mathcal{H}^n(A)$$

↳ $\mathcal{H}\text{-}\dim(G(f;A)) \geq n$

(ii) $Q \subset \mathbb{R}^n$ unit cube (side length 1)

$Q = \bigcup_{j=1}^{k^n} Q_j$, $\operatorname{diam} Q_j = \frac{\sqrt{n}}{k}$, side length of $Q_j = \frac{1}{k}$.

$a_j^i = \min_{x \in Q_j} f^i(x)$, $b_j^i = \max_{x \in Q_j} f^i(x)$, $i = 1, \cdots, m$, $j = 1, \cdots, k^n$

↳ $|b_j^i - a_j^i| \underset{\substack{\uparrow \\ f \text{ is Lipschitz}}}{\leq} \operatorname{Lip}(f) \operatorname{diam} Q_j = \operatorname{Lip}(f) \frac{\sqrt{n}}{k}$

↳ $\{(x, f(x)) \mid x \in Q_j \cap A\} \subset C_j \triangleq Q_j \times \prod_{i=1}^{m} (a_j^i, b_j^i)$.

↳ $\boxed{G(f; A \cap Q) \subset \bigcup_{j=1}^{k^n} C_j}$ $\operatorname{diam} C_j \leq \frac{C}{k}$

↳ $\mathcal{H}^n_{\frac{C}{k}}(G(f; A \cap Q)) \leq \sum_{j=1}^{k^n} \alpha(n) \left(\frac{\operatorname{diam} C_j}{2}\right)^n$

$\qquad \leq k^n \alpha(n) \left(\frac{C}{2k}\right)^n \leq \alpha(n) \left(\frac{C}{2}\right)^n$

$\downarrow k \to \infty$

$\mathcal{H}^n(G(f; A \cap Q)) < \infty$

↳ $\mathcal{H}\text{-}\dim(G(f; A \cap Q)) \leq n \longrightarrow \mathcal{H}\text{-}\dim(G(f;A)) \leq n$

$\square$

C. The Set where a Summable Function is Large

Thm Let $f \in L^1_{loc}(\mathbb{R}^n)$. $0 \leq s < n$

$$\Lambda_s \triangleq \left\{ x \in \mathbb{R}^n \,\middle|\, \limsup_{r\to 0} \frac{1}{r^s} \int_{B(x,r)} |f| dy > 0 \right\}$$

$$\Longrightarrow \quad \mathcal{H}^s(\Lambda_s) = 0$$

Proof. W.O.L.G. we assume $f \in L^1(\mathbb{R}^n)$

1. Lebesgue-Besicovitch Differentiation Thm

$\hookrightarrow \quad \lim_{r\to 0} \fint_{B(x,r)} |f| dy = |f(x)| \quad \mathcal{L}^n\text{-a.e}$

$\hookrightarrow_{0\leq s<n} \quad \lim_{r\to 0} \frac{1}{r^s} \int_{B(x,r)} |f| dy = 0 \quad \mathcal{L}^n\text{-a.e.}$

$\Longrightarrow \quad \mathcal{L}^n(\Lambda_s) = 0$

2. Fix $\varepsilon > 0$, $\delta > 0$, $\sigma > 0$

$f \in L^1(\mathbb{R}^n) \Longrightarrow \exists\, \eta > 0$, s.t. $\mathcal{L}^n(U) \leq \eta$

$\hookrightarrow \int_U |f| dx < \sigma$

$$\Lambda_s^\varepsilon \triangleq \left\{ x \in \mathbb{R}^n \,\middle|\, \limsup_{r\to 0} \frac{1}{r^s} \int_{B(x,r)} |f| dy > \varepsilon \right\}$$

$\hookrightarrow \mathcal{L}^n(\Lambda_s^\varepsilon) = 0$

$\hookrightarrow \exists$ open $U \supset \Lambda_s^\varepsilon$ s.t $\mathcal{L}^n(U) < \eta$.

3. Set

$$\mathcal{J} \triangleq \left\{ B(x,r) \;\middle|\; \begin{array}{l} x \in \Lambda_s^{\varepsilon},\ 0<r<\delta,\ B(x,r) \subset U \\ \int_{B(x,r)} |f|\,dy > \varepsilon r^s \end{array} \right\}$$

$$\xRightarrow{\text{Vitali CT}} \exists \{B_j\}_{j=1}^{\infty} \subset \mathcal{J},\quad B_i \cap B_j = \phi,\ i \neq j$$

s.t. $\Lambda_s^{\varepsilon} \subset \bigcup_{j=1}^{\infty} \hat{B}_j$, $\quad B_j = \overline{B_j(x_j, r_j)}$

$$\hookrightarrow \mathcal{H}_{10\delta}^{s}(\Lambda_s^{\varepsilon}) \leq \sum_{i=1}^{\infty} \alpha(s)(5r_i)^s$$

$$\leq \frac{\alpha(s)5^s}{\varepsilon} \sum_{i=1}^{\infty} \int_{B_i} |f|\,dy$$

$$\Big\downarrow {\scriptstyle \delta \to 0}$$

$$\mathcal{H}^{s}(\Lambda_s^{\varepsilon}) \leq \frac{\alpha(s)5^s}{\varepsilon} \int_U |f|\,dy$$

$$\leq \frac{\alpha(s)5^s}{\varepsilon}\,\sigma \xrightarrow{\sigma \to 0} 0$$

$$\hookrightarrow \boxed{\mathcal{H}^{s}(\Lambda_s^{\varepsilon}) = 0}$$

Geometric Measure Theory And Its Applications

http://people.maths.ox.ac.uk/chengq/teach/tcc12/tcc-GeoMeasure.html

tcc.maths.ox.ac.uk/material/Geometric Measure Theory.html

Mathematics Taught Course Centre
Michaelmas Term
October – December 2012
Tuesday 13:00-15:00

III. Area and Coarea Formulas

"change of variables" formulas for

Lipschitz mappings $f: \mathbb{R}^n \to \mathbb{R}^m$

$m \geq n$ Area Formula

$m \leq n$ Coarea Formula

$\boxed{\text{III}-1}$ Lipschitz functions

Rademacher's Thm.

Globally Lipschitz $f: A \subset \mathbb{R}^n \longrightarrow \mathbb{R}^m$

$$|f(x) - f(y)| \leq C|x-y|, \quad \forall x, y \in A$$

$$\mathrm{Lip}(f) \triangleq \sup\left\{ \frac{|f(x)-f(y)|}{|x-y|} \,\middle|\, x, y \in A, \ x \neq y \right\}$$

Locally Lipschitz $f: A \subset \mathbb{R}^n \longrightarrow \mathbb{R}^m$

$\forall K \subset\subset A, \quad \exists C_K$ s.t.

$$|f(x) - f(y)| \leq C_K |x-y|, \quad \forall x, y \in K$$

Extension of Lipschitz Functions

$$\begin{cases} A \subset \mathbb{R}^n \\ f : A \to \mathbb{R}^m \quad \text{Lipschitz} \end{cases}$$

Kirszbraum's Thm [Federer, §2.10.43]

$\exists$ a Lipschitz function $\bar{f} : \mathbb{R}^n \to \mathbb{R}^m$

such that
$$\begin{cases} \bar{f} = f \quad \text{on } A \\ \operatorname{Lip}(\bar{f}) = \operatorname{Lip}(f) \end{cases}$$

Exercise 18

Differentiability $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $x \in \mathbb{R}^n$ if $\exists$ a linear mapping

$$L : \mathbb{R}^n \to \mathbb{R}^m \quad \text{s.t.}$$

$$\boxed{\lim_{y \to x} \frac{|f(y) - f(x) - L(x-y)|}{|x-y|} = 0}$$

$$\Updownarrow$$

$$\boxed{f(y) = f(x) + L(y-x) + o(|y-x|) \text{ as } y \to x}$$

$\hookrightarrow$ $\boxed{L \triangleq Df(x)}$

the derivative of f at x

Rademacher's Thm

Let $f: \mathbb{R}^n \to \mathbb{R}^m$ be a locally Lipschitz function

$\hookrightarrow$ f is differentiable $\mathcal{L}^n$-a.e.

* The inequality $|f(x)-f(y)| \le Lip(f)|x-y|$ apparently says nothing about the possibility of locally approximating f by a linear map.

$\Rightarrow$

(i) Let $\begin{cases} f: \mathbb{R}^n \to \mathbb{R}^m \text{ be locally Lipschitz} \\ Z \triangleq \{x \in \mathbb{R}^n \mid f(x) = 0\} \end{cases}$

$\hookrightarrow$ $Df(x) = 0 \quad \mathcal{L}^n\text{-a.e. } x \in Z$

(ii) Let $\begin{cases} f, g: \mathbb{R}^n \to \mathbb{R}^n \text{ be locally Lipschitz} \\ Y \triangleq \{x \in \mathbb{R}^n \mid g(f(x)) = x\} \end{cases}$

$\hookrightarrow$ $Dg(f(x))\,Df(x) = I \quad \mathcal{L}^n\text{-a.e. } x \in Y$

$\boxed{\text{Exercise 19}}$