Geometric Measure Theory
And
Its Applications

http://people.maths.ox.ac.uk/chengq/teach/tcc12/tcc-GeoMeasure.html

tcc.maths.ox.ac.uk/material/Geometric Measure Theory.html

Mathematics Taught Course Centre
Michaelmas Term
October – December 2012
Tuesday 13:00-15:00

If you want to register this course, please email to:

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In order to receive credits, you should write a miniproject (5-10 pages) after the end of the course on some (your favorite) topic which the course will cover.

Oxford grades: pass/fail, or distinction for particularly good work.

Course Homepage:
http://people.maths.ox.ac.uk/chengq/teach/tcc12/tcc-GeoMeasure.html

III. Area and Coarea Formulas

"change of variables" formulas for

Lipschitz mappings $f: \mathbb{R}^n \to \mathbb{R}^m$

$m \geq n$ Area Formula

$m \leq n$ Coarea Formula

$\boxed{\text{III}-1}$ Lipschitz functions

Rademacher's Thm.

Globally Lipschitz $f: A \subset \mathbb{R}^n \longrightarrow \mathbb{R}^m$

$$|f(x)-f(y)| \leq C|x-y|, \quad \forall x, y \in A$$

$$\mathrm{Lip}(f) \triangleq \sup\left\{ \frac{|f(x)-f(y)|}{|x-y|} \,\middle|\, x, y \in A,\ x \neq y \right\}$$

Locally Lipschitz $f: A \subset \mathbb{R}^n \longrightarrow \mathbb{R}^m$

$\forall K \subset\subset A, \quad \exists\, C_K$ s.t.

$$|f(x)-f(y)| \leq C_K |x-y|, \quad \forall x, y \in K$$

Extension of Lipschitz Functions

$$\begin{cases} A \subset \mathbb{R}^n \\ f : A \longrightarrow \mathbb{R}^m \quad \text{Lipschitz} \end{cases}$$

Kirszbraum's Thm [Federer, §2.10.43]

$\exists$ a Lipschitz function $\bar{f} : \mathbb{R}^n \longrightarrow \mathbb{R}^m$

such that
$$\begin{cases} \bar{f} = f \quad \text{on } A \\ \mathrm{Lip}(\bar{f}) = \mathrm{Lip}(f) \end{cases}$$

Exercise 18

Differentiability $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is differentiable at $x \in \mathbb{R}^n$ if $\exists$ a linear mapping

$L : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ s.t.

$$\boxed{\lim_{y \to x} \frac{|f(y) - f(x) - L(x-y)|}{|x-y|} = 0}$$

$$\Updownarrow$$

$$\boxed{f(y) = f(x) + L(y-x) + o(|y-x|) \text{ as } y \to x}$$

↳ $\boxed{L \triangleq Df(x)}$

the derivative of f at x

Rademacher's Thm

Let $f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a locally Lipschitz function

$\hookrightarrow$ f is differentiable $\mathcal{L}^n$-a.e.

* The inequality $|f(x)-f(y)| \leq \mathrm{Lip}(f)\,|x-y|$ apparently says nothing about the possibility of locally approximating f by a linear map.

$\Rightarrow$

(i) Let $\begin{cases} f: \mathbb{R}^n \longrightarrow \mathbb{R}^m \text{ be locally Lipschitz} \\ Z \triangleq \{x \in \mathbb{R}^n \mid f(x)=0\} \end{cases}$

$\hookrightarrow$ $Df(x)=0 \qquad \mathcal{L}^n\text{-a.e. } x \in Z$

(ii) Let $\begin{cases} f, g: \mathbb{R}^n \longrightarrow \mathbb{R}^n \text{ be locally Lipschitz} \\ Y \triangleq \{x \in \mathbb{R}^n \mid g(f(x))=x\} \end{cases}$

$\hookrightarrow$ $Dg(f(x))\,Df(x) = I \qquad \mathcal{L}^n\text{-a.e. } x \in Y$

$\boxed{\text{Exercise 19}}$

$\boxed{\text{III-2}}$ Review: Linear Maps / Jacobians

↖ Linear Algebra

Definitions

1. A Linear Map $O : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is orthogonal if $(Ox)\cdot(Oy) = x\cdot y \quad \forall\, x, y \in \mathbb{R}^n$

2. A Linear Map $S : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ is symmetric if $x\cdot(Sy) = (Sx)\cdot y \quad \forall\, x, y \in \mathbb{R}^n$

3. A Linear Map $D : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is diagonal if $\exists\, d_1, \cdots, d_n \in \mathbb{R}$ s.t.
$$Dx = (d_1 x_1, \cdots, d_n x_n) \quad \forall\, x \in \mathbb{R}^n$$

4. Let $A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be linear. The adjoint of A is the linear map $A^* : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ s.t.
$$x\cdot(A^* y) = (Ax)\cdot y \quad \forall\, x \in \mathbb{R}^n, y \in \mathbb{R}^m$$

Routine Facts from Linear Algebra

- $A^{**} = A$
- $(A \circ B)^* = B^* \circ A^*$
- $O^* = O^{-1}$ if $O : \mathbb{R}^n \to \mathbb{R}^n$ is orthogonal
- $S^* = S$ if $S : \mathbb{R}^n \to \mathbb{R}^n$ is symmetric
- If $S : \mathbb{R}^n \to \mathbb{R}^n$ is symmetric

$$\hookrightarrow \quad \exists \begin{cases} \text{an orthogonal map } O : \mathbb{R}^n \to \mathbb{R}^n \\ \text{a diagonal map } D : \mathbb{R}^n \to \mathbb{R}^n \end{cases}$$

s.t. $\quad S = O \circ D \circ O^{-1}$

- If $O : \mathbb{R}^n \to \mathbb{R}^m$ is orthogonal

$$\hookrightarrow \quad \begin{cases} n \leq m \\ O^* \circ O = I \quad \text{on } \mathbb{R}^n \\ O \circ O^* = I \quad \text{on } O(\mathbb{R}^n) \subset \mathbb{R}^m \end{cases}$$

Polar Decomposition

$L : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ a linear mapping

(i) If $n \leq m$

↳ $\exists \begin{cases} \text{a symmetric map} & S : \mathbb{R}^n \to \mathbb{R}^n \\ \text{an orthogonal map} & O : \mathbb{R}^n \to \mathbb{R}^m \end{cases}$

s.t. $L = O \circ S$

(ii) If $n \geq m$

↳ $\exists \begin{cases} \text{a symmetric map} & S : \mathbb{R}^m \to \mathbb{R}^m \\ \text{an orthogonal map} & O : \mathbb{R}^m \to \mathbb{R}^n \end{cases}$

s.t. $L = S \circ O^*$

Jacobian of $L : \mathbb{R}^n \longrightarrow \mathbb{R}^m$

(i) If $n \leq m$.

↳ The Jacobian of L: $[\![L]\!] = |\det S|$

(ii) If $n \geq m$

↳ The Jacobian of L: $[\![L]\!] = |\det S|$.

Rms

① $[\![L]\!] \cancel{\sim}$ choices of O and S

② $[\![L]\!] = [\![L^*]\!]$

Thm

(i) If $n \le m$. $\rightarrow$ $\llbracket L \rrbracket^2 = \det(L^* \circ L)$

(ii) If $n \ge m$ $\rightarrow$ $\llbracket L \rrbracket = \det(L \circ L^*)$

Notations

(i) If $n \le m$, we define

$$\Lambda(m,n) \triangleq \{\lambda : \{1,\cdots,n\} \to \{1,\cdots,m\} \mid \lambda \text{ is increasing}\}$$

(ii) For each $\lambda \in \Lambda(m,n)$, we define $P_\lambda : \mathbb{R}^m \to \mathbb{R}^n$ by

$$P_\lambda(x_1,\cdots,x_m) \triangleq (x_{\lambda(1)},\cdots,x_{\lambda(n)})$$

$\hookrightarrow$ For each $\lambda \in \Lambda(m,n)$, $\exists$ an n-D subspace

$$S_\lambda \triangleq \operatorname{Span}\{e_{\lambda(1)},\cdots,e_{\lambda(n)}\} \subset \mathbb{R}^m$$

s.t. P_λ is the projection of $\mathbb{R}^m$ onto S_λ.

Binet-Cauchy Formula

$$\begin{cases} n \le m \\ L : \mathbb{R}^n \to \mathbb{R}^m \text{ is linear} \end{cases}$$

$\hookrightarrow$
$$\llbracket L \rrbracket^2 = \sum_{\lambda \in \Lambda(m,n)} \Big(\underbrace{\det(P_\lambda \circ L)}\Big)^2$$

$$\Updownarrow$$

Determinants of each $(n\times n)$-submatrix of the $(m\times n)$-matrix representing L (w.r.t. the standard bases of $\mathbb{R}^n$ and $\mathbb{R}^m$).

Jacobians

$f=(f^1,\cdots,f^m):\mathbb{R}^n\longrightarrow\mathbb{R}^m$ Lipschitz

Rademacher's Thm

↳ f is differentiable $\mathcal{L}^n$-a.e.

↳ $Df(x)$ $\begin{cases}\text{exists}\\ \text{is a linear mapping: } \mathbb{R}^n\to\mathbb{R}^m\end{cases}$
for $\mathcal{L}^n$-a.e. $x\in\mathbb{R}^n$

$$Df=\begin{bmatrix}\frac{\partial f^1}{\partial x_1} & \cdots & \frac{\partial f^1}{\partial x_n}\\ \vdots & & \vdots\\ \frac{\partial f^m}{\partial x_1} & \cdots & \frac{\partial f^m}{\partial x_n}\end{bmatrix}_{m\times n}.$$

The Jacobian of f is

$$Jf(x)\triangleq[\![Df(x)]\!] \qquad \mathcal{L}^n\text{-a.e. } x$$

Exercise 20

$\boxed{\text{III}-3}$ The Area Formula $\boxed{n\le m}$

Let $\begin{cases} f:\mathbb{R}^n \longrightarrow \mathbb{R}^m \text{ be Lipschitz.} \\ n \le m \end{cases}$

$\Rightarrow$ $\forall$ $\mathcal{L}^n$-measurable subset $A\subset\mathbb{R}^n$:

$$\int_A Jf\,dx = \int_{\mathbb{R}^m} \mathcal{H}^0(A\cap f^{-1}\{y\})\,d\mathcal{H}^n(y)$$

Basic Facts

1. $L:\mathbb{R}^n \longrightarrow \mathbb{R}^m$ linear

   $\hookrightarrow$ $\mathcal{H}^n(L(A)) = [\![L]\!]\,\mathcal{L}^n(A)$

2. $\left[\begin{array}{ll} f:\mathbb{R}^n\longrightarrow\mathbb{R}^m & \text{Lipschitz} \\ A\subset\mathbb{R}^n & \mathcal{L}^n\text{-measurable} \end{array}\right.$

   $\hookrightarrow$ (i) $f(A)$ is $\mathcal{H}^n$-measurable

   (ii) The mapping $y \longmapsto \mathcal{H}^0(A\cap f^{-1}\{y\})$ is $\mathcal{H}^n$-measurable on $\mathbb{R}^n$.

   (iii) $\int_{\mathbb{R}^m} \mathcal{H}^0(A\cap f^{-1}\{y\})\,d\mathcal{H}^n \le (\mathrm{Lip}(f))^n\,\mathcal{L}^n(A)$

Rm: The mapping $y\longmapsto \mathcal{H}^0(A\cap f^{-1}\{y\})$ is called the multiplicity function

Basic Facts (Conti.)

3. $\begin{cases} B \triangleq \{x \mid Df(x) \text{ exists. } Jf(x) > 0\} \\ t > 1 \end{cases}$

$\Rightarrow$ $\exists \; \{E_k\}_{k=1}^{\infty} \subset \mathbb{R}^n$ Borel subsets. s.t.

(i) $B = \bigcup_{k=1}^{\infty} E_k$

(ii) $f|_{E_k}$ is one-to-One, $k = 1, 2, \cdots$

(iii) $\forall \; k = 1, 2, \cdots,$ $\exists$ a symmetric automorphism $T_k : \mathbb{R}^n \longrightarrow \mathbb{R}^n$

s.t.

$$\begin{cases} Lip((f|_{E_k}) \circ T_k^{-1}) \leq t \\ Lip(T_k \circ (f|_{E_k})^{-1}) \leq t \\ t^{-n} |\det T_k| \leq Jf|_{E_k} \leq t^n |\det T_k| \end{cases}$$

Exercise 21

Area formula

$\begin{cases} f: \mathbb{R}^n \longrightarrow \mathbb{R}^m \quad \text{Lipschitz} \\ n \le m \end{cases}$

$\Rightarrow$ $\forall$ $\mathcal{L}^n$-measurable subset $A \subset \mathbb{R}^n$,

$$\int_A Jf \, dx = \int_{\mathbb{R}^m} \mathcal{H}^0(A \cap f^{-1}\{y\}) \, d\mathcal{H}^n(y).$$

Ideas of Proof:

1. Rademacher's Thm

↳ We may assume that $\begin{cases} Df(x) \\ Jf(x) \end{cases}$ exist for all $x \in A$.

We may also suppose $\mathcal{L}^n(A) < \infty$

2. Case 1 $A \subset \{Jf > 0\}$

$\begin{cases} \text{Fix } t > 1 \\ \text{Choose Borel sets } \{E_k\}_{k=1}^{\infty} \text{ as in Fact 3} \end{cases}$

↳ We may assume that $\begin{cases} \{E_k\}_{k=1}^{\infty} \text{ are disjoint} \\ \{Jf > 0\} = \bigcup_{k=1}^{\infty} E_k \end{cases}$

$$B_k \triangleq \left\{ Q \;\middle|\; \begin{matrix} Q = (a_1, b_1] \times \cdots \times (a_n, b_n], \; a_i = \frac{c_i}{k}, \; b_i = \frac{c_i + 1}{k}, \\ c_i \text{ integers}, \; i = 1, 2, \cdots n \end{matrix} \right\}$$

↳ $\mathbb{R}^n = \bigcup_{Q \in B_k} Q$

Set $F_j^i \triangleq E_j \cap Q_i \cap A$, $\quad Q_i \in B_k$

$i, j = 1, 2, \dots,$

↳ $\left\{\begin{array}{l} A \subset \bigcup_{i,j=1}^{\infty} F_j^i \\ F_j^i \text{ are disjoint} \end{array}\right.$

Claim $\lim_{k\to\infty} \underbrace{\sum_{i,j=1}^{\infty} \mathcal{H}^n(f(F_j^i))} = \int_{\mathbb{R}^m} \mathcal{H}^0(A \cap f^{-1}\{y\})\, d\mathcal{H}^n$

$$= \int_{\mathbb{R}^m} \boxed{g_k}\, d\mathcal{H}^n$$

$$= \boxed{\sum_{i,j=1}^{\infty} \chi_{f(F_j^i)}}$$

~ The number of the sets $\{F_j^i\}$ s.t. $F_j^i \cap f^{-1}\{y\} \neq \emptyset$

↓ Monotone increasing

$\mathcal{H}^0(A \cap f^{-1}\{y\})$ as $k \to \infty$

The Monotone Convergence Thm $\Rightarrow$ claim

Note

$$\mathcal{H}^n(f(F_j^i)) = \mathcal{H}^n(\underbrace{f|_{E_j}\circ T_j^{-1}}\circ T_j(F_j^i)) \overset{\substack{\text{Fact 3}\\ \downarrow}}{\leq} t^n\mathcal{L}^n(T_j(F_j^i))$$

$$\mathcal{L}^n(T_j(F_j^i)) = \mathcal{H}^n(T_j\circ(f|_{E_j})^{-1}\circ f(F_j^i)) \underset{\substack{\uparrow\\ \text{Fact 3}}}{\leq} t^n\mathcal{H}^n(f(F_j^i))$$

$$\xrightarrow{\text{Facts 1\&3}}$$

$$\begin{aligned} t^{-2n}\mathcal{H}^n(f(F_j^i)) &\leq t^{-n}\mathcal{L}^n(T_j(F_j^i)) = t^{-n}|\det T_j|\,\mathcal{L}^n(F_j^i) \\ &\leq \int_{F_j^i} Jf\,dx \\ &\leq t^n|\det T_i|\,\mathcal{L}^n(F_j^i) \\ &= t^n\mathcal{L}^n(T_j(F_j^i)) \\ &\leq t^{2n}\mathcal{H}^n(f(F_j^i)). \end{aligned}$$

$\Rightarrow$

$$t^{-2n}\sum_{i,j=1}^{\infty}\mathcal{H}^n(f(F_j^i)) \leq \int_A Jf\,dx \leq t^{2n}\sum_{i,j=1}^{\infty}\mathcal{H}^n(f(F_j^i)).$$

$$\Big\downarrow\, k\to\infty \qquad\qquad\qquad\qquad \Big\downarrow\, k\to\infty$$

$$t^{-2n}\int_{\mathbb{R}^n}\mathcal{H}^0(A\cap f^{-1}\{y\})\,d\mathcal{H}^n \qquad t^{2n}\int_{\mathbb{R}^n}\mathcal{H}^0(A\cap f^{-1}\{y\})\,d\mathcal{H}^n$$

$\underline{t\to 1}$ $\longrightarrow$ $\square$

3. **Case 2** $A \subset \{Jf = 0\}$

Fix $\varepsilon > 0$

Factor $f = p \circ g : \begin{cases} g: \mathbb{R}^n \longrightarrow \mathbb{R}^m \times \mathbb{R}^n \\ \quad x \longrightarrow g(x) \overset{\triangle}{=} (f(x), \varepsilon x) \\ p: \mathbb{R}^m \times \mathbb{R}^n \longrightarrow \mathbb{R}^m \\ \quad (y, z) \longrightarrow p(y,z) = y \end{cases}$

Claim $\exists\ C > 0$ s.t. $0 < Jg(x) \leq C\varepsilon,\ \forall x \in A$

↳ $\because$ $p: \mathbb{R}^m \times \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is a projection

$$\mathcal{H}^n(f(A)) \leq \mathcal{H}^n(g(A)) \leq \int_{\mathbb{R}^{n+m}} \mathcal{H}^0(A \cap g^{-1}\{y,z\})\, d\mathcal{H}^n(y,z)$$

$$\rotatebox{60}{$=$}\ \text{Case 1}$$

$$\int_A Jg(x)\, dx$$

$$\rotatebox{-90}{$\leq$}$$

$$\varepsilon C \mathcal{L}^n(A)$$

$$\downarrow \varepsilon \to 0$$

$$0.$$

↳

$$\mathcal{H}^n(f(A)) = 0.$$

$$\xrightarrow{\operatorname{spt}\mathcal{H}^0(A \cap f^{-1}\{y\}) \subset f(A)} \int_{\mathbb{R}^n} \mathcal{H}^0(A \cap f^{-1}\{y\})\, d\mathcal{H}^n = 0$$

□

4. For general A.

Write

$$A = A_1 \cup A_2$$
$$A_1 \subset \{Jf > 0\}$$
$$A_2 \subset \{Jf = 0\}$$

Apply Cases 1 & 2 above

$\hookrightarrow$ Result

Exercise 22

Applications of the Area Formula

1. Change of Variables Formula

$\begin{cases} f: \mathbb{R}^n \to \mathbb{R}^m \text{ Lipschitz} \\ n \le m \end{cases}$

↳ $\forall$ $\mathcal{L}^n$- summable function $g: \mathbb{R}^n \to \mathbb{R}$

$$\int_{\mathbb{R}^n} g(x) Jf(x) dx = \int_{\mathbb{R}^m} \Big(\underline{\sum_{x \in f^{-1}\{y\}}} g(x) \Big) d\mathcal{H}^n(y)$$

⇑

By the Area formula, $f^{-1}\{y\}$ is at most countable for $\mathcal{H}^n$-a.e. $y \in \mathbb{R}^m$

2. Length of a Curve ($n=1$, $m \ge 1$)

$f: \mathbb{R} \to \mathbb{R}^m$ Lipschitz, One-to-One

↳ $f = (f^1, \cdots, f^m)$. $Df = (\dot{f}^1, \cdots, \dot{f}^m)$

$$= |\dot{f}| \left(\frac{\dot{f}^1}{|\dot{f}|}, \cdots \frac{\dot{f}^m}{|\dot{f}|} \right)$$

$$Jf = |Df| = |\dot{f}|$$

For $-\infty < a < b < \infty$. $C \triangleq f([a,b]) \subset \mathbb{R}^m$. a curve

$$\mathcal{H}^1(C) = \text{"length" of } C = \int_a^b |\dot{f}| dt$$

3. Surface Area of a Graph ($n \geq 1$, $m = n+1$)

$g: \mathbb{R}^n \longrightarrow \mathbb{R}$ Lipschitz

$f: \mathbb{R}^n \longrightarrow \mathbb{R}^{n+1}$

$\quad x \longrightarrow f(x) \triangleq (x, g(x))$

↳ $$Df = \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \\ \frac{\partial g}{\partial x_1} & \cdots & \frac{\partial g}{\partial x_n} \end{bmatrix}_{(n+1)\times n}$$

↳ $(Jf)^2 =$ sum of squares of $(n\times n)$-subdeterminant

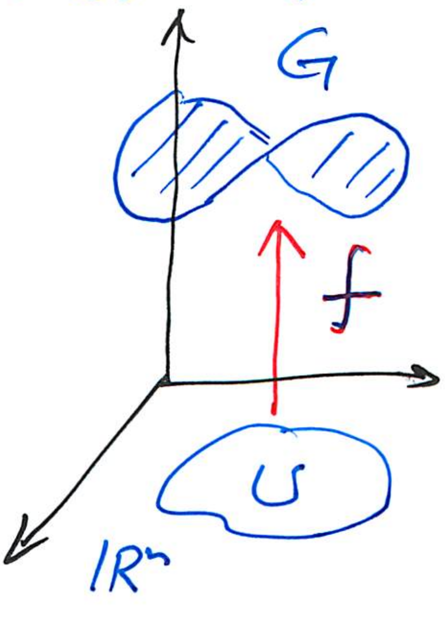
$\quad = 1 + |Dg|^2$

$\forall$ open set $U \subset \mathbb{R}^n$, define the graph of g over U:

$$G = G(g; U) \triangleq \{ (x, g(x)) \mid x \in U \} \subset \mathbb{R}^{n+1}$$

↳ $\mathcal{H}^n(G) =$ "Surface area" of G

$$= \int_U (1 + |Dg|^2)^{1/2} dx$$



4. Surface Area of a Parametric Hypersurface ($n \geq 1$, $m = n+1$)

$f: \mathbb{R}^n \longrightarrow \mathbb{R}^{n+1}$ Lipschitz

One-to-one

$f = (f^1, \cdots, f^{n+1})$

$$Df = \begin{bmatrix} \frac{\partial f^1}{\partial x_1} & \cdots & \frac{\partial f^1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f^{n+1}}{\partial x_1} & \cdots & \frac{\partial f^{n+1}}{\partial x_n} \end{bmatrix}_{(n+1)\times n}.$$

↳ $(Jf)^2$ = sum of squares of $(n\times n)$-subdeterminants

$$= \sum_{k=1}^{n+1} \left[\frac{\partial(f^1, \cdots, f^{k-1}, f^{k+1}, \cdots f^{n+1})}{\partial(x_1, \cdots, x_n)} \right]^2$$

$\forall$ open set $U \subset \mathbb{R}^n$, write

$$S \triangleq f(U) \subset \mathbb{R}^{n+1}$$

↳ $\mathcal{H}^n(S)$ = "surface area" of S

$$= \int_U \left(\sum_{k=1}^{n+1} \left[\frac{\partial(f^1, \cdots, f^{k-1}, f^{k+1}, \cdots, f^{n+1})}{\partial(x_1, \cdots, x_n)} \right]^2 \right)^{\frac{1}{2}} dx$$

5. Submanifolds

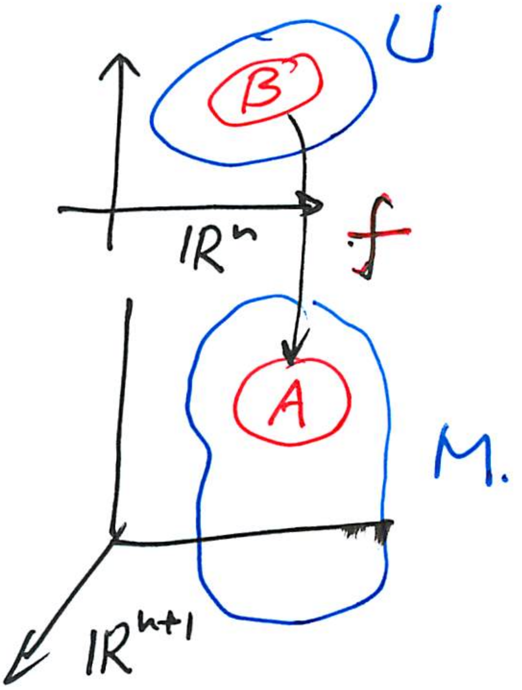
$M \subset \mathbb{R}^m$ Lipschitz. n-D embedded submanifold

$U \subset \mathbb{R}^n$

$f: U \to M$ is a chart for M

$A \subset f(U)$ Borel

$B = f^{-1}(A)$



$$g_{ij} \overset{\Delta}{=} \frac{\partial f}{\partial x_i} \cdot \frac{\partial f}{\partial x_j}, \quad 1 \leq i, j \leq n$$

$$g = \det(g_{ij}).$$

$$(g_{ij}) = (Df)^* \circ Df$$

↳ $Jf = g^{1/2}$

↳

$$\mathcal{H}^n(A) = \text{"Volume" of } A \text{ in } M$$

$$= \int_B g^{1/2}\, dx$$