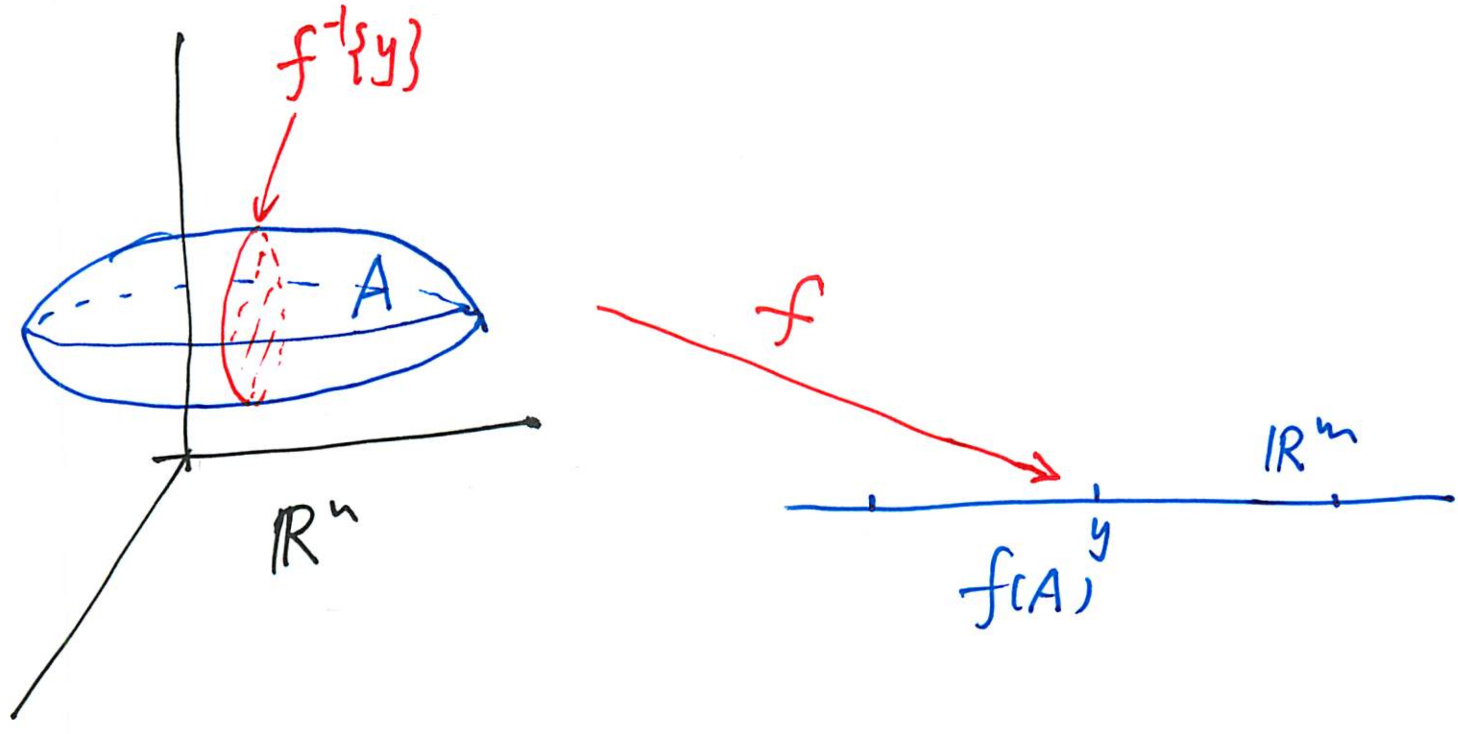
III-4 The Coarea Formula ($n \geq m$)

Let $\begin{cases} f : \mathbb{R}^n \to \mathbb{R}^m \text{ be Lipschitz.} \\ n \geq m \end{cases}$

$\Rightarrow$ $\forall$ $\mathcal{L}^n$-measurable set $A \subset \mathbb{R}^n$:

$$\int_A Jf\, dx = \int_{\mathbb{R}^m} \mathcal{H}^{n-m}(A \cap f^{-1}\{y\})\, dy$$

* The Coarea formula is a kind of "curvilinear" generalization of Fubini's Thm.



Exercise 23

* Apply the Coarea Formula to $A = \{Jf = 0\}$.

$\hookrightarrow$ $\mathcal{H}^{n-m}\{\{Jf=0\} \cap f^{-1}\{y\}\} = 0$. $\mathcal{L}^m$-a.e. $y \in \mathbb{R}^m$

f is required to be Lipschitz

This is a weak version/variant of the Morse-Sard Theorem:

$\{Jf = 0\} \cap f^{-1}\{y\} = \phi$ $\quad \mathcal{L}^m$-a.e. y

Provided $f \in C^k(\mathbb{R}^n; \mathbb{R}^m)$ for $k = 1+n-m$.

Applications of the Coarea Formula

1. Change of Variables Formula:

$$\begin{cases} f : \mathbb{R}^n \longrightarrow \mathbb{R}^m \quad \text{Lipschitz} \\ n \geq m \end{cases}$$

$\Rightarrow$ $\forall$ $\mathcal{L}^n$-summable function $g : \mathbb{R}^n \to \mathbb{R}$

$$\begin{cases} g|_{f^{-1}\{y\}} \text{ is } \mathcal{H}^{n-m}\text{-summable for } \mathcal{L}^m\text{-a.e. } y \\ \int_{\mathbb{R}^n} g(x) Jf(x)\,dx = \int_{\mathbb{R}^m} \left[\int_{f^{-1}\{y\}} g \, d\mathcal{H}^{n-m} \right] dy \end{cases}$$

* for each $y \in \mathbb{R}^m$, $f^{-1}\{y\}$ is closed

$\hookrightarrow$ $\mathcal{H}^{n-m}$-measurable

- $g = g^+ - g^-$.
- W.O.L.G. we assume $g \geq 0$.

$g = \sum_{i=1}^{\infty} \frac{1}{i} \chi_{A_i}$ appropriate $\mathcal{L}^n$-measurable set $\{A_i\}_{i=1}^{\infty}$.

$$\xrightarrow[\text{Convergence Thm}]{\text{Monotone}} \int_{\mathbb{R}^n} g\, Jf\, dx = \sum_{i=1}^{\infty} \frac{1}{i} \int_{A_i} Jf\, dx$$

$$= \sum_{i=1}^{\infty} \frac{1}{i} \int_{\mathbb{R}^m} \mathcal{H}^{n-m}(A_i \cap f^{-1}\{y\})\, dy$$

$$= \int_{\mathbb{R}^m} \sum_{i=1}^{\infty} \frac{1}{i} \mathcal{H}^{n-m}(A_i \cap f^{-1}\{y\})\, dy$$

$$= \int_{\mathbb{R}^m} \left[\int_{f^{-1}\{y\}} g \, d\mathcal{H}^{n-m} \right] dy$$

2. Polar Coordinates

$g : \mathbb{R}^n \longrightarrow \mathbb{R}$ $\mathcal{L}^n$-summable

$\hookrightarrow$ $$\int_{\mathbb{R}^n} g\,dx = \int_0^\infty \left(\int_{\partial B(0,r)} g\,d\mathcal{H}^{n-1} \right) dr$$

$\hookrightarrow$ $$\frac{d}{dr}\left(\int_{B(0,r)} g\,dx \right) = \int_{\partial B(0,r)} g\,d\mathcal{H}^{n-1}$$

$\mathcal{L}^1$-a.e. $r > 0$

$$\boxed{f(x) = |x|, \quad Df(x) = \frac{x}{|x|} ; \quad Jf(x) = 1. \quad x \neq 0}$$

3. Level sets.

① $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ Lipschitz

$\hookrightarrow$ $$\boxed{\int_{\mathbb{R}^n} \underbrace{|Df|}_{= Jf}\,dx = \int_{-\infty}^{\infty} \mathcal{H}^{n-1}(\{f = t\})\,dt}$$

3. Level sets (Conti)

② $\begin{cases} f: \mathbb{R}^n \longrightarrow \mathbb{R} & \text{Lipschitz} \\ \operatorname{ess\,inf} |Df| > 0 & \\ g: \mathbb{R}^n \longrightarrow \mathbb{R} & \mathcal{L}^n\text{-summable} \end{cases}$

$$\Longrightarrow \int_{\{f>t\}} g\, dx = \int_t^{\infty} \left(\int_{\{f=s\}} \frac{g}{|Df|}\, d\mathcal{H}^{n-1} \right) ds$$

$$\hookrightarrow \frac{d}{dt}\left(\int_{\{f>t\}} g\, dx \right) = -\int_{\{f=t\}} \frac{g}{|Df|}\, d\mathcal{H}^{n-1}$$

$\mathcal{L}^1$-a.e. t.

$Jf = |Df|$

$E_t \triangleq \{f>t\}$

$$\int_{\{f>t\}} g\, dx = \int_{\mathbb{R}^n} \underbrace{\chi_{E_t} \frac{g}{|Df|}} Jf\, dx$$

$$= \int_{-\infty}^{\infty} \left(\int_{\partial E_s} \frac{g}{|Df|} \chi_{E_t}\, d\mathcal{H}^{n-1} \right) ds$$

$$= \int_t^{\infty} \left(\int_{\partial E_s} \frac{g}{|Df|}\, d\mathcal{H}^{n-1} \right) ds$$

IV Capacity

* As a way to study certain "small" subsets of $\mathbb{R}^n$
* Suited for characterizing the fine properties of Sobolev functions

Fix $1 \le p < n$. $p^* = \frac{np}{n-p} > p$ Sobolev conjugate of p.

$\left(\frac{1}{p^*} + \frac{1}{n} = \frac{1}{p}\right)$.

Definitions

- $K^p \triangleq \{ f : \mathbb{R}^n \to \mathbb{R} \mid f \ge 0, \ f \in L^{p^*}(\mathbb{R}^n), \ Df \in L^p(\mathbb{R}^n; \mathbb{R}^n) \}$
- If $A \subset \mathbb{R}^n$, set

$$\mathrm{Cap}_p(A) = \inf\left\{ \int_{\mathbb{R}^n} |Df|^p dx \;\middle|\; f \in K^p, \ A \subset \{f \ge 1\}^{\circ} \right\}$$

Interior

$\boxed{p\text{-Capacity of } A}$

↳ (i) Using regularization, we see

$$\mathrm{Cap}_p(K) = \inf\left\{ \int_{\mathbb{R}^n} |Df|^p dx \;\middle|\; f \in C_c^\infty(\mathbb{R}^n), \ f \ge \chi_K \right\}$$

for each $K \subset\subset \mathbb{R}^n$

(ii) $A \subset B \Rightarrow \mathrm{Cap}_p(A) \le \mathrm{Cap}_p(B)$

Properties of K^p

1. If $f \in K^p$ for some $1 \le p < n$

$\hookrightarrow \exists \{f_k\}_{k=1}^{\infty} \subset W^{1,p}(\mathbb{R}^n)$ s.t.

$$\begin{cases} \|f_k - f\|_{L^{p^*}(\mathbb{R}^n)} \longrightarrow 0 \\ \|Df_k - Df\|_{L^p(\mathbb{R}^n)} \longrightarrow 0 \end{cases} \quad \text{as } k \to \infty$$

2. If $f \in K^p$, then

$$\|f\|_{L^{p^*}(\mathbb{R}^n)} \le \underbrace{C}_{\text{only } p,\, n.} \|Df\|_{L^p(\mathbb{R}^n)}$$

G-N-S Inequality

(Gagliardo-Nirenberg-Sobolev)

3. $f, g \in K^p$

$\hookrightarrow \quad h \triangleq \overbrace{\max\{f, g\}}^{=f+(g-f)^+} \in K^p$

$$Dh = \begin{cases} Df & \mathcal{L}^n\text{-a.e. on } \{f \ge g\} \\ Dg & \mathcal{L}^n\text{-a.e. on } \{f \le g\} \end{cases}$$

$+ \quad h \triangleq \min\{f, g\} \in K^p.$

4. If $f \in K^p$, $t \ge 0 \Rightarrow h \triangleq \min\{f, t\} \in K^p.$

5. Given a sequence $\{f_k\}_{k=1}^{\infty} \subset K^p$, define

$$g \triangleq \sup_{1 \le k < \infty} f_k, \qquad h \triangleq \sup_{1 \le k < \infty} |Df_k|.$$

If $h \in L^p(\mathbb{R}^n) \Rightarrow g \in K^p$, $|Dg| \le h$. $\mathcal{L}^n$-a.e.

Thm Cap_p is a measure on $\mathbb{R}^n$.

(an outer measure)

* Cap_p is not a Borel measure.

In fact, if $\begin{cases} A \subset \mathbb{R}^n \\ 0 < Cap_p(A) < \infty \end{cases}$

↳ A is not Cap_p-measurable in general.

$\boxed{\text{Exercise 24}}$

Properties of Capacity $A, B \subset \mathbb{R}^n$

1. $Cap_p(A) = \inf\{Cap_p(U) \mid U \text{ open}, A \subset U\}$

2. $Cap_p(\lambda A) = \lambda^{n-p} Cap_p(A)$, $\lambda > 0$

3. $Cap_p(L(A)) = Cap_p(A)$ for each affine isometry $L: \mathbb{R}^n \to \mathbb{R}^n$

4. $Cap_p(B(x,r)) = r^{n-p} Cap_p(B(0,1))$

5. $Cap_p(A) \le C \mathcal{H}^{n-p}(A)$ for some constant $\underbrace{C}_{p \text{ and } n}$

Fix $\delta > 0$,

Suppose $A \subset \bigcup_{k=1}^{\infty} B(x_k, r_k)$, $2r_k < \delta$, $k = 1, 2 \cdots$

$\hookrightarrow Cap_p(A) \le \sum_{k=1}^{\infty} Cap_p(B(x_k, r_k)) = Cap_p(B(0,1)) \sum_{i=1}^{\infty} r_k^{n-p}$

$\hookrightarrow Cap_p(A) \le C \mathcal{H}^{n-p}(A)$

6. $\mathcal{L}^n(A) \le C\, Cap_p(A)^{\frac{n}{n-p}}$ $C \overset{\text{only}}{\sim} p, n$.

$\forall \varepsilon > 0$, $\exists f \in K^p$ s.t. $\begin{cases} A \subset \{f \ge 1\}^\circ \triangleq U \\ \int_{\mathbb{R}^n} |Df|^p dx \le Cap_p(A) + \varepsilon \end{cases}$

$$\mathcal{L}^n(A)^{\frac{1}{p^*}} \le \left(\int_{\mathbb{R}^n} f^{p^*} dx\right)^{\frac{1}{p^*}} \overset{G\text{-}N\text{-}S}{\le} C\left(\int_{\mathbb{R}^n} |Df|^p dx\right)^{\frac{1}{p}}$$
$$\le C\left(Cap_p(A) + \varepsilon\right)^{\frac{1}{p}}$$

7. $\mathrm{Cap}_p(A\cup B)+\mathrm{Cap}_p(A\cap B)\le \mathrm{Cap}_p(A)+\mathrm{Cap}_p(B)$

8. If $A_1\subset\cdots\subset A_k\subset A_{k+1}\subset\cdots$

then
$$\lim_{k\to\infty}\mathrm{Cap}_p(A_k)=\mathrm{Cap}_p\Big(\bigcup_{k=1}^{\infty}A_k\Big)$$

- $1<p<n$ easier
- $p=1$

[Federer-Ziemer: Indiana U. Math. J. 22 (1972), 139-158

9. If $A_1\supset\cdots\supset A_k\supset A_{k+1}\supset\cdots$ are compact.

↳ $$\lim_{k\to\infty}\mathrm{Cap}_p(A)=\mathrm{Cap}_p\Big(\bigcap_{k=1}^{\infty}A_k\Big)$$

* False if $\{A_k\}_{k=1}^{\infty}$ are not compact

10. (i) $1<p<n$: If $\mathcal{H}^{n-p}(A)<\infty$

↳ $\mathrm{Cap}_p(A)=0$

(ii) $\left\{\begin{array}{l} A\subset\mathbb{R}^n,\ 1\le p<\infty \\ \mathrm{Cap}_p(A)=0\end{array}\right. \Longrightarrow \mathcal{H}^s(A)=0 \quad \forall\, s>n-p$

(iii) $\mathrm{Cap}_1(A)=0 \iff \mathcal{H}^{n-1}(A)=0$

Ideas of Proof

(i) $\begin{cases} 1 < p < n \\ \mathcal{H}^{n-p}(A) < \infty \end{cases} \Longrightarrow \mathrm{Cap}_p(A) = 0$

We may assume that A is compact.

$\boxed{\text{Claim}}$ $\exists$ a constant C $\underset{\text{only}}{\sim}$ n, A.

s.t. $\forall$ open set $V \supset A$.

$\hookrightarrow$ $\exists \begin{cases} \text{an open set } W \\ f \in K^p \end{cases}$ s.t. $\begin{cases} A \subset W \subset \{f = 1\} \\ \mathrm{spt}(f) \subset V \\ \int_{\mathbb{R}^n} |Df|^p dx \le C \end{cases}$

$\xrightarrow{\text{Inductively}}$ $\exists \begin{cases} \text{Open sets } \{V_k\}_{k=1}^{\infty} \\ \text{functions } f_k \in K^p \end{cases}$

s.t.

$$\begin{cases} A \subset V_{k+1} \subset V_k \\ \overline{V_{k+1}} \subset \{f_k = 1\}^{\circ} \\ \mathrm{spt}(f_k) \subset V_k \\ \int_{\mathbb{R}^n} |Df_k|^p dx \le C \end{cases}$$

Set
$$\begin{cases} S_j = \sum_{k=1}^{j} \frac{1}{k} \\ g_j = \frac{1}{S_j} \sum_{k=1}^{j} \frac{f_k}{k} \end{cases}$$

$\hookrightarrow$
$$\begin{cases} g_j \in K^p \\ g_j \geq 1 \quad \text{on } V_{j+1} \end{cases}$$

$\because \quad \operatorname{spt}(|Df_k|) \subset V_k - \overline{V_{k+1}}$

$\hookrightarrow$
$$\begin{aligned} \operatorname{Cap}_p(A) &\leq \int_{\mathbb{R}^n} |Dg_j|^p dx \\ &= \frac{1}{S_j^p} \sum_{k=1}^{j} \frac{1}{k^p} \int_{\mathbb{R}^n} |Df_k|^p dx \\ &\leq \frac{C}{S_j^k} \sum_{k=1}^{j} \frac{1}{k^p} \longrightarrow 0 \quad \text{as } j \to \infty \end{aligned}$$

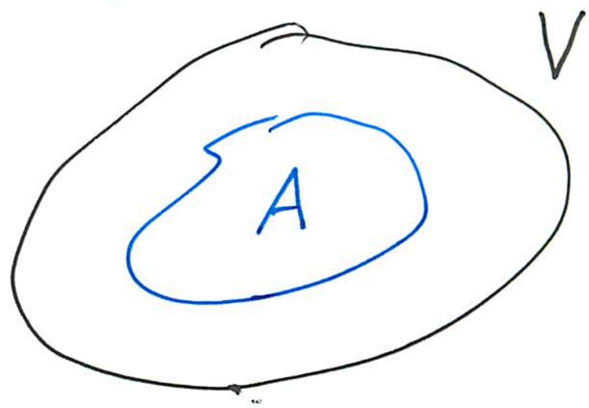
for $\underline{p > 1}$

<u>Proof of Claim</u> $\exists\ C \overset{\text{only}}{\sim} n, A$

s.t $\forall$ open set $V \supset A$.

$$\exists \begin{cases} \text{open set } W \\ f \in K^p \end{cases} \text{ s.t. } \begin{cases} A \subset W \subset \{f=1\} \\ spt(f) \subset V \\ \int_{\mathbb{R}^n} |Df|^p dx \le C \end{cases}$$

$\delta \triangleq \frac{1}{2} dist(A; \mathbb{R}^n \setminus V)$



$$\begin{cases} \mathcal{H}^{n-p}(A) < \infty \\ A \subset\subset \mathbb{R}^n \end{cases}$$

$\hookrightarrow \exists$ a finite collection

$\{U(x_i, r_i)\}_{i=1}^m$ of open balls

s.t.
$$\begin{cases} 2r_i < \delta \\ U(x_i, r_i) \cap A \ne \phi \\ A \subset \bigcup_{i=1}^m U(x_i, r_i) \\ \sum_{i=1}^m \alpha(n-p)\, r_i^{n-p} \le C\mathcal{H}^{n-p}(A) + 1 \end{cases} \quad \text{Some } C>0$$

Set $W = \bigcup_{i=1}^m U(x_i, r_i)$. Define $f_i(x) = \begin{cases} 1 & |x-x_i| \le r_i \\ 2 - \frac{|x-x_i|}{r_i} & r_i \le |x-x_i| \le 2r_i \\ 0 & 2r_i \le |x-x_i| \end{cases}$

$\hookrightarrow \int_{\mathbb{R}^n} |Df_i|^p dx \le C r_i^{n-p}$

$$\boxed{f = \max_{1\le i\le m} f_i} \Rightarrow \begin{cases} f \in K^p, \quad W \subset \{f=1\} \\ spt(f) \subset V \\ \int_{\mathbb{R}^n} |Df|^p dx \le \sum_{i=1}^m \int_{\mathbb{R}^n} |Df_i|^p dx \\ \quad \le C \sum_{i=1}^m r_i^{n-p} \le C(\mathcal{H}^{n-p}(A) + 1) \end{cases}$$

(ii) $\begin{cases} A \subset \mathbb{R}^n \\ 1 \le p < \infty \\ \mathrm{Cap}_p(A) = 0 \end{cases} \Rightarrow \mathcal{H}^s(A) = 0 \quad \forall\, s > n-p$

$\mathrm{Cap}_p(A) = 0 \qquad n-p < s < \infty$

$\hookrightarrow \forall\, i \ge 1, \quad \exists\, f_i \in K^p$ s.t. $\begin{cases} A \subset \{f_i \ge 1\}^\circ \\ \int_{\mathbb{R}^n} |Df_i|^p dx < \frac{1}{2^i} \end{cases}$

Let $g = \sum_{i=1}^{\infty} f_i$

$\hookrightarrow$ $$\left(\int_{\mathbb{R}^n} |Dg|^p dx\right)^{\frac{1}{p}} \le \sum_{i=1}^{\infty} \left(\int_{\mathbb{R}^n} |Df_i|^p dx\right)^{\frac{1}{p}} < \infty$$

$$\vee\!/$$

$$\left(\int_{\mathbb{R}^n} |g|^{p^*} dx\right)^{\frac{1}{p^*}}$$

$\hookrightarrow \quad g \in K^p$

Note $\quad A \subset \{g \ge m\}^\circ \qquad \forall\, m \ge 1$

$\forall\, a \in A \Rightarrow \forall\, r \ll 1$, s.t. $B(a,r) \subset \{g \ge m\}^\circ$

$\hookrightarrow (g)_{a,r} \ge m$

$\hookrightarrow (g)_{a,r} \longrightarrow \infty$ as $r \to 0$

claim $\forall a \in A$

$$\limsup_{r\to 0} \frac{1}{r^s}\int_{B(a,r)} |Dg|^p dx = \infty$$

$\hookrightarrow$
$$A \subset \left\{ a\in\mathbb{R}^n \,\middle|\, \limsup_{r\to 0} \frac{1}{r^s}\int_{B(a,r)} |Dg|^p dx = \infty \right\}$$
$$\subset \left\{ a\in\mathbb{R}^n \,\middle|\, \limsup_{r\to 0} \frac{1}{r^s}\int_{B(a,r)} |Dg|^p dx > 0 \right\}$$
$$\parallel\!\mid$$
$$\Lambda_s$$

$|Dg|^p$ is $\mathcal{L}^n$-summable $\Longrightarrow \mathcal{H}^s(\Lambda_s) = 0$

If $\exists\, a\in A$ s.t. $\displaystyle\limsup_{r\to 0} \frac{1}{r^s}\int_{B(a,r)} |Dg|^p dx < \infty$

$\hookrightarrow \exists\, M < \infty$ s.t. $\displaystyle\frac{1}{r^s}\int_{B(a,r)} |Dg|^p dx \le M,\ \forall 0<r\le 1$

$\hookrightarrow \forall\, 0<r\le 1$

$$\fint_{B(a,r)} |g-(g)_{a,r}|^p dx \le C r^p \fint_{B(a,r)} |Dg|^p dx$$
$$\le C r^{s-(n-p)} > 0$$

$\hookrightarrow$
$$|(g)_{a,\frac{r}{2}} - (g)_{a,r}| = \frac{1}{|B(a,\frac{r}{2})|}\left|\int_{B(a,\frac{r}{2})} (g-(g)_{a,r})dx\right|$$
$$\le 2^n \fint_{B(a,r)} |g-(g)_{a,r}|\, dx$$
$$\le 2^n \left(\fint_{B(a,r)} |g-(g)_{a,r}|^p dx\right)^{\frac{1}{p}}$$
$$= C r^{\frac{\theta}{p}} \qquad \theta = s-(n-p)$$

$\hookrightarrow \forall\, k>j$

$$\left|(g)_{a,\frac{1}{2^k}}-(g)_{a,\frac{1}{2^j}}\right| \le \sum_{l=j+1}^{k}\left|(g)_{a,\frac{1}{2^l}}-(g)_{a,\frac{1}{2^{l-1}}}\right|$$

$$\le C\sum_{l=j+1}^{k}\left(\frac{1}{2^{l-1}}\right)^{\frac{\theta}{p}} \quad >0$$

The tail of a geometric series

$\hookrightarrow$ $\left\{(g)_{a,\frac{1}{2^k}}\right\}_{k=1}^{\infty}$ is a Cauchy sequence

$\hookrightarrow$ $(g)_{a,\frac{1}{2^k}} \not\longrightarrow \infty.$

Contradiction

11. $\begin{cases} f\in K^p \\ \varepsilon>0 \\ A\equiv\{x\in\mathbb{R}^n \mid (f)_{x,r}>\varepsilon \text{ for some } r>0\} \end{cases}$

$\hookrightarrow$ $$\boxed{Cap_p(A)\le \frac{C}{\varepsilon^p}\int_{\mathbb{R}^n}|Df|^p\,dx}$$

where $C \underset{\sim}{\text{only}}\ n, p$

$*$. $$|\{x\in\mathbb{R}^n \mid f(x)>\varepsilon\}| \le \frac{1}{\varepsilon^p}\int_{\mathbb{R}^n}|f|^p\,dx$$

Exercise 25

<u>p-quasicontinuity</u>. A function f is p-quasicontinuous if $\forall \varepsilon > 0$, $\exists$ an open set V such that

$$\begin{cases} Cap_p(V) \le \varepsilon \\ f|_{\mathbb{R}^n - V} \text{ is continuous} \end{cases}$$

<u>Fine Properties of Sobolev Functions</u> $f \in W^{1,p}(\mathbb{R}^n)$
$1 \le p < n$.

(i) $\exists$ a Borel set $E \subset \mathbb{R}^n$ s.t.

$$\begin{cases} Cap_p(E) = 0 \\ \lim_{r \to 0} (f)_{x,r} \equiv f^*(x) \text{ exists for each } x \in \mathbb{R}^n - E \end{cases}$$

(ii) $\lim_{r \to 0} \fint_{B(x,r)} |f - f^*(x)|^{p^*} dy = 0 \quad \forall x \in \mathbb{R}^n - E$

(iii) The precise representative f^* is p-quasicontinuous

$$f^* = \lim_{r \to 0} \fint_{B(x,r)} f \, dy$$

$\boxed{\text{Exercise 26}}$