V. BV Functions & Sets of Finite Perimeter

Definitions

1. $f \in L^1(U)$ has bounded variation in U if

$$\sup\left\{ \int_U f \operatorname{div} \varphi \, dx \;\middle|\; \varphi \in C_c^1(U;\mathbb{R}^n),\ |\varphi| \le 1 \right\} < \infty$$

$BV(U)$ — The space of functions of bdd variation

2. An $\mathcal{L}^n$-measurable subset $E \subset \mathbb{R}^n$ has finite perimeter in U if $\chi_E \in BV(U)$

3. $f \in L^1_{loc}(U)$ has locally bounded variation in U if, $\forall$ open set $V \subset\subset U$,

$$\sup\left\{ \int_V f \operatorname{div} \varphi \, dx \;\middle|\; \varphi \in C_c^1(V;\mathbb{R}^n),\ |\varphi| \le 1 \right\} < \infty$$

$BV_{loc}(U)$ — The space of functions of locally bdd variation

4. An $\mathcal{L}^n$-measurable subset $E \subset \mathbb{R}^n$ has locally finite perimeter in U if

$$\chi_E \in BV_{loc}(U)$$

Ex 1 $f \in W^{1,1}_{loc}(U)$

$\hookrightarrow$ $\forall\ V \subset\subset U$, $\varphi \in C^1_c(V;\mathbb{R}^n)$ with $|\varphi| \le 1$

$$\left|\int_U f \operatorname{div}\varphi\, dx\right| = \left|-\int_U Df\cdot\varphi\, dx\right| \le \int_V |Df|\,dx < \infty$$

$\hookrightarrow$ $f \in BV_{loc}(U)$

$\hookrightarrow$ $W^{1,1}_{loc}(U) \subset BV_{loc}(U)$

Similarly $\quad W^{1,1}(U) \subset BV(U)$.

In particular. $\quad W^{1,p}_{loc}(U) \subset BV_{loc}(U)$, $\forall 1 \le p \le \infty$

Ex 2 $\begin{cases} E \subset \mathbb{R}^n \text{ Smooth, open subset} \\ \mathcal{H}^{n-1}(\partial E \cap K) < \infty \qquad \forall K \subset\subset U \end{cases}$

$\hookrightarrow$ $\forall\ \varphi \in C^1_c(V;\mathbb{R}^n)$, $V \subset\subset U$

$$\int_E \operatorname{div}\varphi\, dx = \int_{\partial E\cap V} \varphi\cdot\nu\, d\mathcal{H}^{n-1} \quad \text{(Gauss-Green formula)}$$

The outward unit normal on ∂E

$\hookrightarrow$ $\left|\int_E \operatorname{div}\varphi\, dx\right| \le \mathcal{H}^{n-1}(\partial E\cap V) < \infty \Rightarrow \chi_E \in BV_{loc}(U)$.

$\hookrightarrow$ E has locally finite perimeter in U

But $\boxed{\chi_E \notin W^{1,1}_{loc}(U).}$

$\hookrightarrow$ $W^{1,1}_{loc}(U) \subsetneqq BV_{loc}(U)$. $\quad W^{1,1}(U) \subsetneqq BV(U)$.

Structure Thm for $BV_{loc}(U)$ functions

$f \in BV_{loc}(U)$

↳ $\exists \begin{cases} \text{a Radon measure } \mu \text{ on } U \\ \text{a } \mu\text{-measurable function } \sigma : U \to \mathbb{R}^n \end{cases}$

s.t.

$$\begin{cases} |\sigma(x)| = 1 & \mu\text{-a.e} \\ \int_U f \operatorname{div} \varphi \, dx = -\int_U \varphi \cdot \sigma \, d\mu, & \forall \varphi \in C_c^1(U; \mathbb{R}^n) \end{cases}$$

↳ The weak 1st partial derivatives of a BV function are Radon measures

Notation

(i) $f \in BV_{loc}(U)$. Write

$$\|Df\| \triangleq \mu$$

$$[Df] \triangleq \|Df\| \llcorner \sigma = \mu \llcorner \sigma$$

↳
$$\int_U f \operatorname{div} \varphi \, dx = -\int_U \varphi \cdot \sigma \, d\|Df\|$$
$$= -\int_U \varphi \cdot d[Df]$$
$$\forall \varphi \in C_c^1(U; \mathbb{R}^n)$$

We write

$$\mu^i := \|Df\| \llcorner \sigma^i \qquad i=1,2,\cdots,n$$
$$\sigma=(\sigma^1,\cdots,\sigma^n)$$

Lebesgue's Decomposition Thm

↳ $$\mu^i = \mu^i_{ac} + \mu^i_s \qquad \begin{cases} \mu^i_{ac} \ll \mathcal{L}^n \\ \mu^i_s \perp \mathcal{L}^n \end{cases}$$

↳ $\mu^i_{ac} = \mathcal{L}^n \llcorner g_i$ for some function $f_i \in L^1_{loc}(U)$
$i=1,2,\cdots,n.$

Write

$$Df :\triangleq \left(\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n}\right) \triangleq (g_1,\cdots,g_n)$$

$$[Df]_{ac} \triangleq (\mu^1_{ac},\cdots,\mu^n_{ac}) = \mathcal{L}^n \llcorner Df$$

$$[Df]_s \triangleq (\mu^1_s,\cdots,\mu^n_s)$$

↳ $$[Df] = [Df]_{ac} + [Df]_s = \mathcal{L}^n \llcorner Df + [Df]_s.$$

↳ $Df \in L^1_{loc}(U;\mathbb{R}^n)$ is the density of the absolutely continuous part of $[Df]$.

* $f \in BV_{loc}(U)$ belongs to $W^{1,p}_{loc}(U)$

$\Longleftrightarrow$ $f \in L^p_{loc}(U)$, $\underline{[Df]_s = 0}$, $Df \in L^p_{loc}(U)$

(iii) E is a set of locally finite perimeter in U

↳ $f = \chi_E \in BV$

↳ $\begin{cases} \|\partial E\| \stackrel{\Delta}{=} \mu \\ \nu_E \stackrel{\Delta}{=} -\sigma \end{cases}$

↳ $\int_E \operatorname{div}\varphi \, dx = \int_U \varphi \cdot \nu_E \, d\|\partial E\|, \quad \forall \varphi \in C_c^1(U; \mathbb{R}^n)$

(iv) $\|Df\|$ is the variation measure of f

$\|\partial E\|$ is the perimeter measure of E

$\|\partial E\|(U)$ is the perimeter of E in U

(v) $f \in BV_{loc}(U) \cap L^1(U)$

↳ $f \in BV(U) \iff \|Df\|(U) < \infty$

$$\|f\|_{BV(U)} \stackrel{\Delta}{=} \|f\|_{L^1(U)} + \|Df\|(U)$$

↳ $BV(U)$ is a Banach space

(vi) $\forall V \subset\subset U$

$$\|Df\|(V) = \sup\left\{ \int_V f \operatorname{div}\varphi \, dx \;\middle|\; \varphi \in C_c^1(V; \mathbb{R}^n), |\varphi| \le 1 \right\}$$

$$\|\partial E\|(V) = \sup\left\{ \int_E \operatorname{div}\varphi \, dx \;\middle|\; \varphi \in C_c^1(V; \mathbb{R}^n), |\varphi| \le 1 \right\}$$

Approximation and Compactness

1. Lower Semicontinuity of Variation Measure

$$\begin{cases} f_k \in BV(U), & k = 1, 2, \cdots \\ f_k \longrightarrow f & \text{in } L^1_{loc}(U) \end{cases}$$

↳ $$\|Df\|(U) \leq \liminf_{k\to\infty} \|Df_k\|(U)$$

$\forall \varphi \in C_c^1(U; \mathbb{R}^n)$, $|\varphi| \leq 1$

$$\int_U f \operatorname{div}\varphi \, dx = \lim_{k\to\infty} \int_U f_k \operatorname{div}\varphi \, dx = -\lim_{k\to\infty} \int_U \varphi \cdot \sigma_k \, d\|Df_k\|$$

$$\leq \liminf_{k\to\infty} \|Df_k\|(U)$$

↳ $$\|Df\|(U) = \sup\left\{ \int_U f \operatorname{div}\varphi \, dx \;\middle|\; \varphi \in C_c^1(U; \mathbb{R}^n), \ |\varphi| \leq 1 \right\}$$

$$\leq \liminf_{k\to\infty} \|Df_k\|(U).$$

2. Local Approximation by Smooth Functions

$f \in BV(U) \Rightarrow \exists \{f_k\}_{k=1}^\infty \subset BV(U) \cap C^\infty(U)$ s.t.

$$\begin{cases} f_k \to f & \text{in } L^1(U) \\ \|Df_k\|(U) \to \|Df\|(U), & k \to \infty \\ \mu_k \longrightarrow \mu & (\mathcal{M}) \end{cases}$$

* $\mu_k(B) = \int_{B\cap U} Df_k \, dx$. $\mu(B) = \int_{B\cap U} d[Df]$ $\forall B \subset \mathbb{R}^n$ Borel set

* $\|D(f_k - f)\|(U) \xrightarrow{?} 0$ $k \to \infty$

3. Compactness

Let $U \subset \mathbb{R}^n$ be open, bdd, with ∂U Lipschitz

Assume $\begin{cases} \{f_k\}_{k=1}^{\infty} \text{ is a sequence in } BV(U) \\ \sup_k \|f_k\|_{BV(U)} < \infty \end{cases}$

$\hookrightarrow \exists \begin{cases} \{f_{k_j}\}_{j=1}^{\infty} \subset \{f_k\}_{k=1}^{\infty} \\ f \in BV(U) \end{cases}$

Such that

$$f_{k_j} \longrightarrow f \quad \text{in } L^1(U),\ j \to \infty$$

Choose $g_k \in C^\infty(\Omega)$ s.t. $\xrightarrow{②} \begin{cases} \int_U |f_k - g_k|\,dx < \frac{1}{k} \\ \sup_k \int_U |Dg_k|\,dx < \infty \end{cases}$

$\hookrightarrow \|g_k\|_{W^{1,1}} \leq C \ \cancel{\forall}\ k$

$\hookrightarrow \exists \begin{cases} f \in L^1(U) \\ \{g_{k_j}\}_{j=1}^{\infty} \subset \{g_k\}_{k=1}^{\infty} \end{cases}$ s.t. $g_{k_j} \longrightarrow f$ in $L^1(U)$.

$\hookrightarrow \quad f_{k_j} \longrightarrow f \quad \text{in } L^1(U)$

$\xrightarrow{①} \quad f \in BV(U)$

Traces ~ The "boundary values" of f on ∂U

$\begin{cases} U \subset \mathbb{R}^n & \text{Open, bdd.} \\ \partial U & \text{Lipschitz} \end{cases}$

↳ The outer unit normal ν exists $\mathcal{H}^{n-1}$-a.e. on ∂U

↑ Rademacher's Thm

↳ $\exists$ a bdd linear mapping

$$T: BV(U) \longrightarrow L^1(\partial U; \mathcal{H}^{n-1})$$

s.t.

$$\int_U f \operatorname{div}\varphi \, dx = -\int_U \varphi \cdot d[Df] + \int_{\partial U} (\varphi \cdot \nu) Tf \, d\mathcal{H}^{n-1}$$

$$\forall f \in BV(U), \ \varphi \in C^1(\mathbb{R}^n; \mathbb{R}^n)$$

↳ Gauss-Green formula

* Tf is uniquely defined up to sets of $\mathcal{H}^{n-1} \llcorner \partial U$ measure zero

↳ Tf is called the trace of f on ∂U

Traces (Conti.)

$f \in BV(U)$

↳ For $\mathcal{H}^{n-1}$-a.e. $x \in \partial U$

$$\lim_{r \to 0} \fint_{B(x,r) \cap U} |f - Tf(x)| \, dy = 0$$

↳

$$\boxed{Tf(x) = \lim_{r \to 0} \fint_{B(x,r) \cap U} f(y) \, dy}$$

* If $f \in BV(U) \cap C(\bar{U})$

↳ $Tf = f|_{\partial U}$ $\mathcal{H}^{n-1}$-a.e.

Extensions

$$\begin{cases} U \subset \mathbb{R}^n & \text{open, bdd} \\ \partial U & \text{Lipschitz} \end{cases}$$

$$\begin{cases} f_1 \in BV(U) \\ f_2 \in BV(\mathbb{R}^n - \bar{U}) \end{cases}$$

Define

$$\bar{f}(x) \triangleq \begin{cases} f_1(x) & x \in U \\ f_2(x) & x \in \mathbb{R}^n - \bar{U} \end{cases}$$

$$\Rightarrow \begin{cases} \bar{f} \in BV(\mathbb{R}^n) \\ \|D\bar{f}\|(\mathbb{R}^n) = \|Df_1\|(U) + \|Df_2\|(\mathbb{R}^n - \bar{U}) \\ \qquad + \int_{\partial U} |Tf_1 - Tf_2| \, d\mathcal{H}^{n-1} \end{cases}$$

$\hookrightarrow$ (i) The extension

$$Ef \triangleq \begin{cases} f & \text{on } U \\ 0 & \text{on } \mathbb{R}^n - U \end{cases} \in BV(\mathbb{R}^n).$$

provided $f \in BV(U)$

(ii) The set U has finite perimeter

& $\|\partial U\|(\mathbb{R}^n) = \mathcal{H}^{n-1}(\partial U)$

Coarea formula for BV functions

$$\begin{cases} f \in BV(U) \\ E_t \triangleq \{ x \in U \mid f(x) > t \} \end{cases}$$

$\hookrightarrow$ (i) E_t has finite perimeter for $\mathcal{L}^1$-a.e. $t \in \mathbb{R}$

(ii) $\|Df\|(U) = \int_{-\infty}^{\infty} \|\partial E_t\|(U)\, dt$

Conversely, if $\begin{cases} f \in L^1(U) \\ \int_{-\infty}^{\infty} \|\partial E_t\|(U)\, dt < \infty \end{cases}$

$\hookrightarrow$ $f \in BV(U)$

Sobolev's and Poincaré's Inequalities for BV

(i) $\exists\, C_1 > 0$ s.t.

$$\|f\|_{L^{\frac{n}{n-1}}(\mathbb{R}^n)} \leq C_1 \|Df\|(\mathbb{R}^n), \quad \forall f \in BV(\mathbb{R}^n)$$

(ii) $\exists\, C_2 > 0$ s.t.

$$\|f - \underbrace{(f)_{x,r}}_{= ⨍_{B(x,r)} f\,dy}\|_{L^{\frac{n}{n-1}}(B(x,r))} \leq C_2 \|Df\|(B(x,r))$$

$$\forall f \in BV_{loc}(\mathbb{R}^n), \ \forall B(x,r) \subset \mathbb{R}^n$$

(iii) $\forall\, 0 < \alpha \leq 1$, $\exists\, C_3 = C_3(\alpha)$ s.t.

$$\|f\|_{L^{\frac{n}{n-1}}(B(x,r))} \leq C_3 \|Df\|(B(x,r))$$

$\forall B(x,r) \subset \mathbb{R}^n$.

$\forall f \in BV_{loc}(\mathbb{R}^n)$ satisfying $\dfrac{|B(x,r) \cap \{f=0\}|}{|B(x,r)|} \geq \alpha$

Isoperimetric Inequality

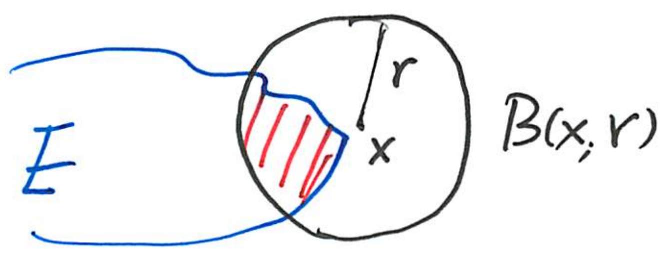
$$\begin{cases} E \subset \mathbb{R}^n \\ \|\partial E\|(\mathbb{R}^n) < \infty \end{cases}$$ Set of finite perimeter

↳ (i) $|E|^{1-\frac{1}{n}} \leq C_1 \|\partial E\|(\mathbb{R}^n)$ (Isoperimetric Ineq.)

(ii) $\forall B(x,r) \subset \mathbb{R}^n$:

$$\min\{|B(x,r) \cap E|, |B(x,r) - E|\}^{1-\frac{1}{n}} \leq C_2 \|\partial E\|(B(x,r))$$

(Relative Isoperimetric Ineq.)



$\mathcal{H}^{n-1} \longleftrightarrow \mathrm{Cap}_1$

$A \subset\subset \mathbb{R}^n$

↳ $\mathrm{Cap}_1(A) = 0 \iff \mathcal{H}^{n-1}(A) = 0.$

Ideas of Proof

"$\Longleftarrow$" $\mathrm{Cap}_1(A) \le C\,\mathcal{H}^{n-1}(A).$ (proved)

"$\Longrightarrow$" $\because$ $\mathrm{Cap}_1(A) = 0$

↳ $\forall \varepsilon > 0,\ \exists f \in K^1$ s.t.

$$\begin{cases} A \subset \{f \ge 1\}^\circ \\ f : \mathbb{R}^n \to \mathbb{R},\quad f \ge 0,\quad f \in L^1(\mathbb{R}^n),\ Df \in L^1(\mathbb{R}^n;\mathbb{R}^n) \\ \int_{\mathbb{R}^n} |Df|\,dx \le \varepsilon \end{cases}$$

$\xrightarrow[\text{Formula}]{\text{The Coarea}}$ $\displaystyle\int_0^1 \|\partial E_t\|(\mathbb{R}^n)\,dt \le \int_{\mathbb{R}^n} |Df|\,dx$

for $E_t \triangleq \{f > t\}$

↳ $\exists\, t \in (0,1)$ s.t.

$$\|\partial E_t\|(\mathbb{R}^n) \le \int_{\mathbb{R}^n} |Df|\,dx$$

Clearly $\begin{cases} A \subset E_t^{\circ} \\ \mathcal{L}^n(E_t) < \infty \end{cases}$ $\Leftarrow$ Isoperimetric Ineq.

$\hookrightarrow$ $\forall\, x \in A$, $\exists\, r > 0$. s.t.

$$\frac{\mathcal{L}^n(E_t \cap B(x,r))}{\alpha(n)\, r^n} \geq \frac{1}{4}$$

$\hookrightarrow$

$$\left[\frac{1}{4}\alpha(n) r^n\right]^{\frac{n-1}{n}} \leq \left(\mathcal{L}^n(E_t \cap B(x,r))\right)^{\frac{n-1}{n}}$$

$$\leq C \|\partial E_t\|(B(x,r)).$$

Relative Isoperimetric Ineq.

$\hookrightarrow$

$$r^{n-1} \leq C \|\partial E_t\|(B(x,r)).$$

$\xrightarrow[\text{Covering}]{\text{Vitali's}}$ $\exists\, \{B(x_j, r_j)\}_{j=1}^{\infty}$ s.t. $\begin{cases} B(x_i, r_i) \cap B(x_j, r_j) = \phi \quad i \neq j \\ x_j \in A \\ A \subset \bigcup_{j=1}^{\infty} B(x_j, 5r_j) \end{cases}$

$\hookrightarrow$ $$\sum_{j=1}^{\infty} (5r_j)^{n-1} \leq C\|\partial E_t\|(\mathbb{R}^n) \leq C\int_{\mathbb{R}^n} |Df|\, dx \leq \varepsilon.$$

$\hookrightarrow$ $$r_j \leq \left(C\|\partial E_t\|(\mathbb{R}^n)\right)^{\frac{1}{n-1}} \leq C\,\varepsilon^{\frac{1}{n-1}}$$

$\hookrightarrow$ $$\mathcal{H}^{n-1}(A) = 0$$

The Reduced Boundary

$$\begin{cases} E \subset \mathbb{R}^n \\ \|\partial E\|(K) < \infty \qquad \forall\, K \subset\subset \mathbb{R}^n \end{cases}$$

Set of locally finite perimeter

$$\hookrightarrow \int_E \operatorname{div} \varphi \, dx = \int_U \varphi \cdot \nu_E \, d\|\partial E\|$$

__Definition__ $x \in \mathbb{R}^n$.

We say $x \in \partial^* E$, __the reduced bdry of E__ if

(i) $\|\partial E\|(B(x,r)) > 0 \qquad \forall\, r > 0$

(ii) $\lim_{r \to 0} ⨍_{B(x,r)} \nu_E \, d\|\partial E\| = \nu_E(x)$

(iii) $|\nu_E(x)| = 1$

__Lebesgue-Besicovitch Differentiation Thm__

$$\hookrightarrow \qquad \|\partial E\|(\mathbb{R}^n - \partial^* E) = 0$$

The Reduced Boundary (Conti.).

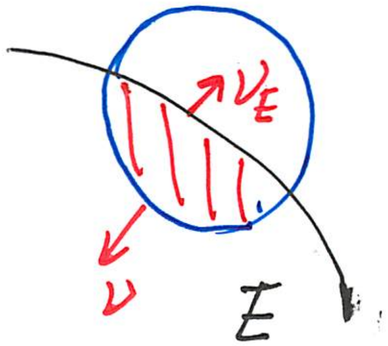
↳ 1. $\varphi \in C_c^1(\mathbb{R}^n; \mathbb{R}^n)$

$\forall\ x \in \mathbb{R}^n$

$$\int_{E\cap B(x,r)} \operatorname{div}\varphi \, dy = \int_{B(x,r)} \varphi\cdot\nu_E \, d\|\partial E\| + \int_{E\cap\partial B(x,r)} \varphi\cdot\nu \, d\mathcal{H}^{n-1}$$

for $\mathcal{L}^1$-a.e. $r>0$.

where ν is the outward unit normal to $\partial B(x,r)$



2. $\exists\ A_j > 0$, $j = 1, 2, \cdots, 5$, s.t. only n

$\forall\ x \in \partial^* E$

$$\begin{cases} \liminf\limits_{r\to 0} \dfrac{|B(x,r)\cap E|}{r^n} > A_1 > 0 \\ \liminf\limits_{r\to 0} \dfrac{|B(x,r) - E|}{r^n} > A_2 > 0 \\ \liminf\limits_{r\to 0} \dfrac{\|\partial E\|(B(x,r))}{r^{n-1}} > A_3 > 0 \\ \limsup\limits_{r\to 0} \dfrac{\|\partial E\|(B(x,r))}{r^{n-1}} \le A_4 \\ \limsup\limits_{r\to 0} \dfrac{\|\partial(E\cap B(x,r))\|(\mathbb{R}^n)}{r^{n-1}} \le A_5 \end{cases}$$