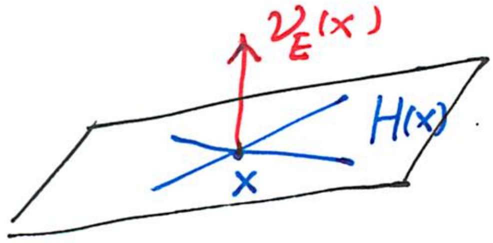
Blow-up

Notation Fix $x \in \partial^* E$

Hyperplane $H(x) := \{ y \in \mathbb{R}^n \mid \nu_E(x) \cdot (y-x) = 0 \}$

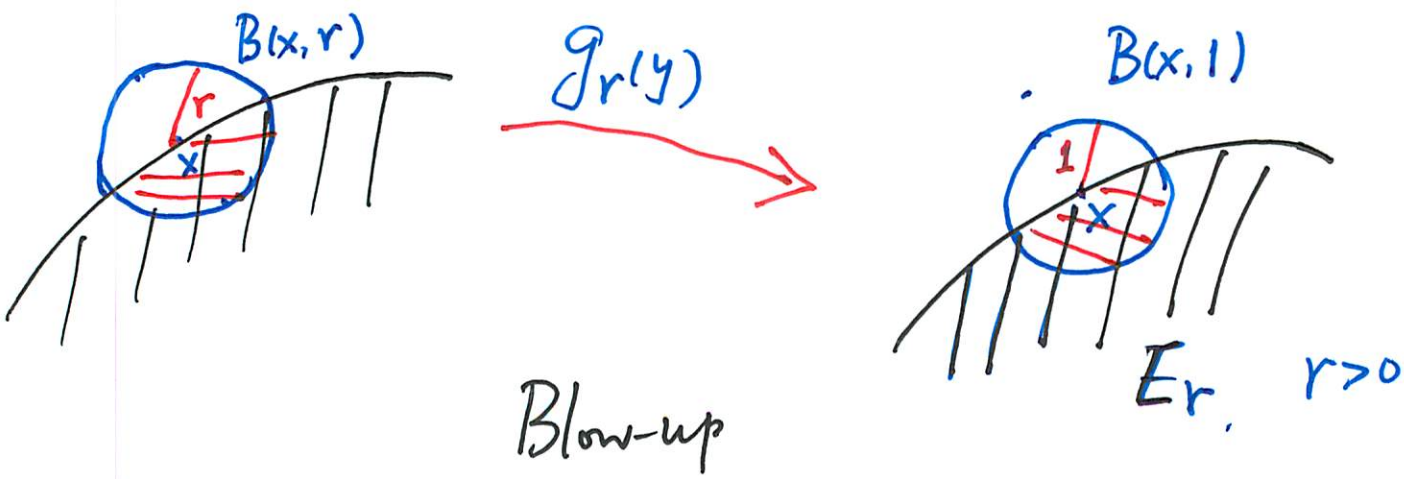


Half-Spaces

$$H^{\pm}(x) := \{ y \in \mathbb{R}^n \mid \pm \nu_E(x) \cdot (y-x) \geq 0 \}$$

$$E_r := \{ y \in \mathbb{R}^n \mid r(y-x) + x \in E \}, \quad r > 0.$$

$\hookrightarrow$ $y \in E \cap B(x,r) \iff g_r(y) := x + \frac{y-x}{r} \in E_r \cap B(x,1)$



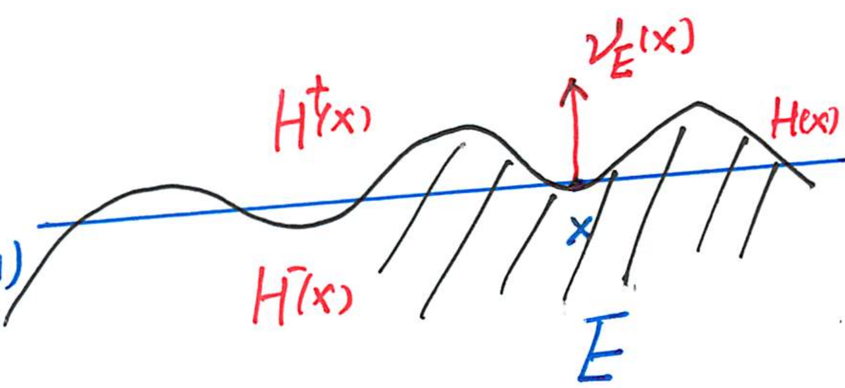
Thm Blow-up of Reduced Boundary

$x \in \partial^* E$

↳ $\chi_{E_r} \longrightarrow \chi_{H^-(x)}$ in $L^1_{loc}(\mathbb{R}^n)$ as $r \to 0$.

↳ For small enough $r>0$. $E \cap B(x,r)$ approximately equals the half ball $H^-(x) \cap B(x,r)$.

Ideas of Proof



W.O.L.G. we may assume

$$\begin{cases} x=0, \ \nu_E(0) = e_n = (0,\cdots,0,1) \\ H(0) = \{ y \in \mathbb{R}^n \mid y_n = 0 \} \\ H^+(0) = \{ y \in \mathbb{R}^n \mid y_n \geq 0 \} \\ H^-(0) = \{ y \in \mathbb{R}^n \mid y_n \leq 0 \} \end{cases}$$

1. Choose any sequence $r_k \to 0$. It suffices to show that $\exists \ \{s_j\}_{j=1}^{\infty} \subset \{r_k\}_{k=1}^{\infty}$ for which

$$\chi_{E_{s_j}} \longrightarrow \chi_{H^-(0)} \quad \text{in } L^1_{loc}(\mathbb{R}^n)$$

2. Fix $L>0$

Let $D_r := E_r \cap B(0,L)$, $\quad g_r(y) = \frac{y}{r} \in E_r$

$y \in E$

$\forall\, \varphi \in C_c^1(\mathbb{R}^n;\mathbb{R}^n)$, $\quad |\varphi|<1$,

$$\int_{D_r} \operatorname{div}\varphi\, dz = \frac{1}{r^{n-1}} \int_{E\cap B(0,rL)} \operatorname{div}(\varphi\circ g_r)\, dy$$

$$= \frac{1}{r^{n-1}} \int_{\mathbb{R}^n} (\varphi\circ g_r)\cdot \nu_{E\cap B(0,rL)}\, d\|\partial(E\cap B(0,rL))\|$$

$$\leq \frac{\|\partial(E\cap B(0,rL))\|(\mathbb{R}^n)}{r^{n-1}} \leq C < \infty \quad \forall r\in(0,1]$$

Property of the Reduced Bdry

$\hookrightarrow$

$$\|\partial D_r\|(\mathbb{R}^n) \leq C<\infty, \qquad 0<r\leq 1.$$

Also $\quad \|\chi_{D_r}\|_{L^1(\mathbb{R}^n)} = \mathcal{L}^n(D_r) \leq \mathcal{L}^n(B(0,L)) < \infty, \quad r>0.$

$\hookrightarrow$

$$\|\chi_{D_r}\|_{BV(\mathbb{R}^n)} \leq C<\infty \qquad \forall\, 0<r\leq 1$$

$\xrightarrow[\text{Thm}]{\text{Compactness}}$ $\exists \begin{cases} \{s_j\}_{j=1}^{\infty} \subset \{r_k\}_{k=1}^{\infty} \\ f\in BV_{loc}(\mathbb{R}^n) \end{cases}$ s.t.

$$\chi_{E_j} \longrightarrow f \quad \text{in} \begin{cases} L^1_{loc}(\mathbb{R}^n) \\ \mathcal{L}^n\text{-a.e.} \end{cases}$$

$E_j = E_{s_j}$

$\hookrightarrow$ $f(x) \in \{0, 1\}$ for $\mathcal{L}^n$-a.e. x.

$\hookrightarrow$ $\begin{cases} f = \chi_F & \mathcal{L}^n\text{-a.e.} \\ F \subset \mathbb{R}^n & \text{has locally finite perimeter} \end{cases}$

$\Longrightarrow$ $\forall \varphi \in C_c^1(\mathbb{R}^n; \mathbb{R}^n)$

$$(*) \quad \int_F \operatorname{div} \varphi \, dy = \int_{\mathbb{R}^n} \varphi \cdot \nu_E \, d\|\partial F\|$$

Some $\|\partial F\|$-measurable function with $|\nu_F| = 1$, $\|\partial F\|$-a.e.

$\overset{?}{\hookrightarrow}$ $F = H^{-}(0)$.

3. <u>Claim</u>: $\nu_F = e_n \quad \|\partial F\|$-a.e.

$\forall \varphi \in C_c^1(\mathbb{R}^n; \mathbb{R}^n)$.

$$\int_{\mathbb{R}^n} \varphi \cdot \underset{\underset{\nu_{E_j}}{\parallel}}{\nu_j} \, d\|\partial E_j\| = \int_{E_j} \operatorname{div} \varphi \, dy \qquad j = 1, 2, \cdots$$

$\chi_{E_j} \to \chi_F$ in L^1_{loc}

$$\int_F \operatorname{div} \varphi \, dy \overset{(*)}{=} \int_{\mathbb{R}^n} \varphi \cdot \nu_E \, \|\partial F\|.$$

$\hookrightarrow$ $\boxed{\nu_j \|\partial E_j\| \longrightarrow \nu_F \|\partial F\|. \quad (\mathcal{M})}$

$\hookrightarrow$ For each $L>0$ for which $\|\partial F\|(\partial B(0,L))=0$

i.e. for all but at most countably many $L>0$,

$$(**) \quad \int_{B(0,L)} \nu_j \, d\|\partial E_j\| \longrightarrow \int_{B(0,L)} \nu_F \, d\|\partial F\|.$$

<u>On the other hand.</u> $\forall\, \varphi \in C_c^1(\mathbb{R}^n;\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} \varphi\cdot\nu_j \, d\|\partial E_j\| = \frac{1}{s_j^{n-1}} \int_{\mathbb{R}^n} (\varphi\circ g_{s_j})\cdot \nu_E \, d\|\partial E\|.$$

$\hookrightarrow$

$$(***) \quad \begin{cases} \|\partial E_j\|(B(0,L)) = \dfrac{1}{s_j^{n-1}} \|\partial E\|(B(0,s_jL)) \\ \displaystyle\int_{B(0,L)} \nu_j \, d\|\partial E_j\| = \frac{1}{s_j^{n-1}} \int_{B(0,s_jL)} \nu_E \, d\|\partial E\| \end{cases}$$

$\hookrightarrow$

$$(****) \quad \lim_{j\to\infty} ⨍_{B(0,L)} \nu_j \, d\|\partial E_j\| = \lim_{j\to\infty} ⨍_{B(0,s_jL)} \nu_E \, d\|\partial E\|$$

$$= \nu_E(0) \qquad (0 \in \partial^* E)$$

$$= e_n$$

If $\|\partial F\|(\partial B(0,L))=0$

$$\hookrightarrow \|\partial F\|(B(0,L)) \overset{\text{L.S.C.}}{\leq} \liminf_{j\to\infty} \|\partial E_j\|(B(0,L))$$

$$\overset{(****)}{=\!=} \lim_{j\to\infty} \int_{B(0,L)} e_n\cdot\nu_j \, d\|\partial E_j\|$$

$$\overset{(**)}{=\!=} \int_{B(0,L)} e_n\cdot\nu_F \, d\|\partial F\|$$

$\hookrightarrow \quad \|\partial F\|(B(0,L)) \leq \int_{B(0,L)} e_n \cdot \nu_F \, d\|\partial F\|$

$\hookrightarrow$ $|\nu_F|=1 \quad \|\partial F\|$-a.e.

$$\begin{cases} \nu_F = e_n & \|\partial F\|\text{-a.e.} \\ \|\partial F\|(B(0,L)) = \lim_{j\to\infty} \|\partial E_j\|(B(0,L)) & \\ \qquad \forall \ \|\partial F\|(\partial B(0,L)) = 0 & \end{cases}$$

4. Claim: F is a half-space

Step 3 $\longrightarrow$ $\forall \ \varphi \in C_c^1(\mathbb{R}^n;\mathbb{R}^n)$

$$\boxed{\int_F \operatorname{div} \varphi \, dz = \int_{\mathbb{R}^n} \varphi \cdot e_n \, d\|\partial F\|}$$

$\left[\begin{array}{l} \forall \varepsilon > 0 \\ f^\varepsilon := \eta_\varepsilon * \chi_F \end{array} \right. \Longrightarrow \quad f^\varepsilon \in C^\infty(\mathbb{R}^n)$

the mollifier

$$\int_{\mathbb{R}^n} f^\varepsilon \operatorname{div} \varphi \, dz = \int_F \operatorname{div}(f^\varepsilon * \varphi) \, dz$$

$$= \int_{\mathbb{R}^n} \eta_\varepsilon * (\varphi \cdot e_n) \, d\|\partial F\|$$

$$\parallel$$

$$-\int_{\mathbb{R}^n} Df^\varepsilon \cdot \varphi \, dz$$

$\hookrightarrow$

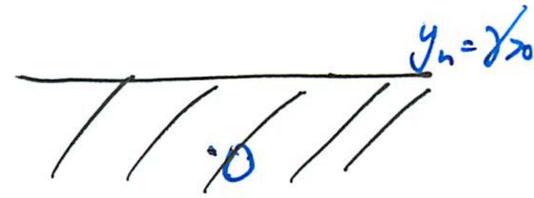
$$\begin{cases} \dfrac{\partial f^\varepsilon}{\partial z_i} = 0, & i = 1, 2, \cdots, n-1 \\ \dfrac{\partial f^\varepsilon}{\partial z_n} \leq 0 & \end{cases}$$

$\because$ $f_\varepsilon \longrightarrow \chi_F$ $\mathcal{L}^n$-a.e. as $\varepsilon \to 0$.

$\hookrightarrow$ Up to a set of $\mathcal{L}^n$-measure zero.

$F = \{ y \in \mathbb{R}^n \mid y_n \le \gamma \}$ for some $\gamma \in \mathbb{R}$

5 $\quad \boxed{F = H^-(0)} \Longleftrightarrow \boxed{\gamma = 0}$



If $\gamma > 0$,

$$\boxed{\chi_{E_j} \longrightarrow \chi_F \quad L^1_{loc}(\mathbb{R}^n)}$$

$$\begin{aligned}
\alpha(n)\gamma^n &= \mathcal{L}^n(B(0,\gamma) \cap F) \\
&= \lim_{j\to\infty} \mathcal{L}^n(B(0,\gamma) \cap E_j) \\
&= \lim_{j\to\infty} \frac{\mathcal{L}^n(B(0,\gamma s_j) \cap E)}{s_j^n} \\
&= \boxed{\lim_{r_j \to 0} \frac{\mathcal{L}^n(B(0,r_j) \cap E)}{\alpha(n) r_j^n}} \alpha(n)\gamma^n.
\end{aligned}$$

$< 1.$ $\quad \boxed{\lim_{r_j\to 0} \frac{|B(0,r_j) - E|}{\alpha(n) r_j} > A_2 > 0}$

If $\gamma < 0$

$$\begin{aligned}
\alpha(n)|\gamma|^n &= \mathcal{L}^n(B(0,|\gamma|) - F) = \lim_{j\to\infty} \mathcal{L}^n(B(0,|\gamma|) - E_j) \\
&= \boxed{\lim_{j\to\infty} \frac{|B(0,|\gamma| s_j) - E|}{\alpha(n)(|\gamma| s_j)^n}} \alpha(n)\gamma^n
\end{aligned}$$

$< 1.$

$\boxed{\text{Contradiction}}$

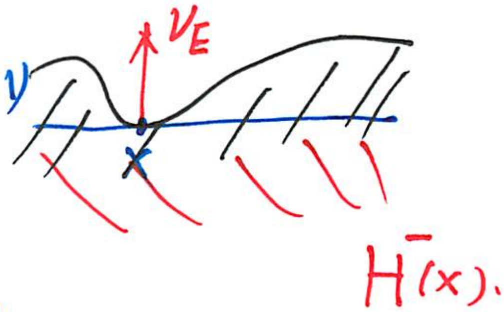
Thm $x \in \partial^* E$

$\hookrightarrow$ $\chi_{E_r} \xrightarrow[r \to 0]{} \chi_{H^-(x)}$ in $L^1_{loc}(\mathbb{R}^n)$

$E_r = \{ y \in \mathbb{R}^n \mid x + r(y-x) \in E \}$ $\qquad$ $H^-(x) = \{ y \in \mathbb{R}^n \mid \nu_E(x) \cdot (y-x) \le 0 \}$

$\Downarrow$

$y \in E \cap B(x,r) \Leftrightarrow x + \frac{y-x}{r} \in E_r \cap B(x,1)$



$\hookrightarrow$ $x \in \partial^* E$

$$\begin{cases} \lim_{r \to 0} \dfrac{|B(x,r) \cap E \cap H^+(x)|}{r^n} = 0 \quad (\text{✲}) \\[2ex] \lim_{r \to 0} \dfrac{|(B(x,r) - E) \cap H^-(x)|}{r^n} = 0 \\[2ex] \lim_{r \to 0} \dfrac{\|\partial E\|(B(x,r))}{\alpha(n-1)\, r^{n-1}} = 1 \end{cases}$$

* The unit vector $\nu_E(x)$ for which (✲) holds is called the <u>measure-theoretic unit outer normal to E at x</u>

Structure Thm for Sets of Finite Perimeter

$E \subset \mathbb{R}^n$ has locally finite perimeter

$\hookrightarrow$

(i) $\|\partial E\| = \mathcal{H}^{n-1} \llcorner \partial^* E$

(ii) $\partial^* E = \bigcup_{k=1}^{\infty} K_k \cup N$

$$\begin{cases} K_k \subset\subset S_k \text{ — a } C^1\text{-hypersurface} \\ \|\partial E\|(N) = 0 \end{cases}$$

(iii) $\nu_E|_{S_k}$ is the normal to S_k, $k = 1, 2, \ldots$

$\hookrightarrow$ A set of locally finite perimeter has "measure-theoretically a C^1-boundary

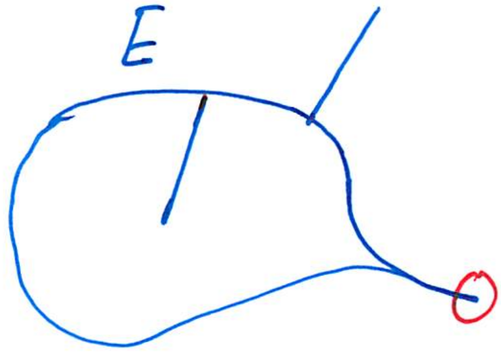
The Measure-theoretic Boundary

$E \subset \mathbb{R}^n$ a set of locally finite perimeter

$x \in \partial_* E$ ~ the measure-theoretic boundary of E

$\Updownarrow$

$$\begin{cases} \limsup_{r\to 0} \dfrac{|B(x,r)\cap E|}{r^n} > 0 \\ \limsup_{r\to 0} \dfrac{|B(x,r)-E|}{r^n} > 0 \end{cases}$$



Thm

(i) $\partial^* E \subset \partial_* E$

(ii) $\mathcal{H}^{n-1}(\partial_* E - \partial^* E) = 0$

* This is a refinement of the classical theorem

$E \subset \mathbb{R}^n$ is $\mathcal{L}^n$-measurable

$\hookrightarrow$

$$\begin{cases} \lim_{r\to 0} \dfrac{|B(x,r)\cap E|}{|B(x,r)|} = 1 & \mathcal{L}^n\text{-a.e. } x \in E \\ \lim_{r\to 0} \dfrac{|B(x,r)\cap E|}{|B(x,r)|} = 0 & \mathcal{L}^n\text{-a.e. } x \in \mathbb{R}^n - E \end{cases}$$

Gauss-Green Theorem

$E \subset \mathbb{R}^n$ locally finite perimeter

$\hookrightarrow$

(i) $\mathcal{H}^{n-1}(\partial_* E \cap \overline{K}) < \infty \qquad \forall\, K \subset\subset \mathbb{R}^n$

(ii) For $\mathcal{H}^{n-1}$-a.e. $x \in \partial_* E$,

$\hookrightarrow$ $\exists\, 1$ measure-theoretic unit outer normal $\nu_E(x)$ such that

(*) $$\boxed{\int_E \operatorname{div} \varphi \, dx = \int_{\partial_* E} \varphi \cdot \nu_E \, d\mathcal{H}^{n-1} \qquad \forall\, \varphi \in C_c^1(\mathbb{R}^n; \mathbb{R}^n)}$$

$$\boxed{\|\partial E\| = \mathcal{H}^{n-1} \llcorner \partial_* E}$$

* (*) holds for $E = U$, an open set with Lipschitz boundary

Pointwise Properties of BV functions

$f \in BV(\mathbb{R}^n)$

Define

$$\mu(x) := \underset{y\to x}{\operatorname{ap\,lim\,sup}}\, f(y) = \inf\left\{ t \;\middle|\; \lim_{r\to 0} \frac{|B(x,r)\cap\{f>t\}|}{r^n} = 0 \right\}$$

$$\lambda(x) := \underset{y\to x}{\operatorname{ap\,lim\,inf}}\, f(y) = \sup\left\{ t \;\middle|\; \lim_{r\to 0} \frac{|B(x,r)\cap\{f<t\}|}{r^n} = 0 \right\}.$$

↳ $\begin{cases} -\infty \le \lambda(x) \le \mu(x) \le \infty & \forall x \in \mathbb{R}^n. \\ \lambda(x),\ \mu(x) \text{ are Borel measurable} \end{cases}$

$$J := \{ x \in \mathbb{R}^n \mid \lambda(x) < \mu(x) \}$$

the set of points at which the approximate limit of f does not exist.

↳ $\mathcal{L}^n(J) = 0$

$\Rightarrow$ (i) $\exists$ countably many C^1 hypersurfaces $\{S_k\}_{k=1}^{\infty}$ s.t.

$$\mathcal{H}^{n-1}\left(J - \bigcup_{k=1}^{\infty} S_k\right) = 0$$

* A BV function is "measure theoretically piecewise continuous" with "jumps along a measure theoretically C^1-surface"

(ii) $-\infty < \lambda(x) \le \mu(x) < \infty \qquad \mathcal{H}^{n-1}$-a.e. $x \in \mathbb{R}^n$

Notation

$$\begin{cases} F(x) := \dfrac{\lambda(x)+\mu(x)}{2} \\ \nu \in \mathbb{R}^n \text{ unit vector} \\ H_\nu := \{ y \in \mathbb{R}^n \mid \nu \cdot (y-x) = 0 \} \text{ hyperplane} \\ H_\nu^\pm := \{ y \in \mathbb{R}^n \mid \pm\nu \cdot (y-x) \geq 0 \} \text{ half-spaces} \end{cases}$$

Fine Properties of BV functions $\quad f \in BV(\mathbb{R}^n)$

(i) $\lim\limits_{r\to 0} \fint_{B(x,r)} |f - F(x)|^{\frac{n}{n-1}} dy = 0 \quad \mathcal{H}^{n-1}\text{-a.e. } x \in \mathbb{R}^n - J$

(ii) For $\mathcal{H}^{n-1}$-a.e. $x \in J$, $\exists$ a unit vector $\nu = \nu(x)$ s.t.

$$\begin{cases} \lim\limits_{r\to 0} \fint_{B(x,r)\cap H_\nu^-} |f-\mu(x)|^{\frac{n}{n-1}} dy = 0 \\ \lim\limits_{r\to 0} \fint_{B(x,r)\cap H_\nu^+} |f-\lambda(x)|^{\frac{n}{n-1}} dy = 0 \end{cases}$$

↳ $\mu(x) = \operatorname{ap}\lim\limits_{\substack{y\to x \\ y\in H_\nu^+}} f(y), \quad \lambda(x) = \operatorname{ap}\lim\limits_{\substack{y\to x \\ y \in H_\nu^-}} f(y).$

↳ For $\mathcal{H}^{n-1}$-a.e. $x \in J$, f has a "measure-theoretic jump" across the hyperplane $H_{\nu(x)}$.

↳

(i) $f^*(x) = \lim\limits_{r\to 0} (f)_{x,r} = F(x)$ exists $\mathcal{H}^{n-1}$-a.e. $x \in \mathbb{R}^n$

(ii) If η_ε is the standard mollifier and $f^\varepsilon = \eta_\varepsilon * f$,

$$f^*(x) = \lim_{\varepsilon\to 0} (\eta_\varepsilon * f)(x) \quad \mathcal{H}^{n-1}\text{-a.e. } x \in \mathbb{R}^n$$

Criterion for Finite Perimeter

$E \subset \mathbb{R}^n$ is $\mathcal{L}^n$-measurable

E has locally finite perimeter

$$\Updownarrow$$

$$\mathcal{H}^{n-1}(\partial_* E \cap K) < \infty$$

for each $K \subset\subset \mathbb{R}^n$