Nonlinear Conservation Laws and Divergence-Measure Fields

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Integration by Parts & Gauss-Green Theorem in Analysis

Integration by Parts (1675): Let $f(y), g(y) \in C^1(\mathbb{R})$. Then

$$\int_a^b f(y)g'(y)\,dy = \big(f(b)g(b) - f(a)g(a)\big) - \int_a^b f'(y)g(y)\,dy \qquad \text{for any } a \le b.$$

The rule is shown via the fundamental theorem of calculus (**Newton 1669**; ...) and the product rule for derivatives (**Leibniz**;...):

$$f(b)g(b) - f(a)g(a) = \int_a^b \frac{d}{dy}(f(y)g(y))\,dy = \int_a^b f'(y)g(y)\,dy + \int_a^b f(y)g'(y)\,dy.$$

Gauss-Green Theorem (Divergence Theorem): Let $\Omega \Subset \mathcal{D} \subset \mathbb{R}^N$ be compact and have a piecewise smooth boundary. If $\mathbf{F} \in C^1(\mathcal{D}; \mathbb{R}^N)$, then

$$\int_\Omega \varphi\, \mathrm{div}\mathbf{F}\, d\mathbf{y} = -\int_{\partial\Omega} \varphi\, \mathbf{F}\cdot\boldsymbol{\nu}\, dS - \int_\Omega \mathbf{F}\cdot\nabla\varphi\, d\mathbf{y}$$

for any $\varphi \in C^1(\mathbb{R}^N; \mathbb{R}^N)$, where $\boldsymbol{\nu}$ is the unit interior normal on $\partial\Omega$ to Ω and dS is the surface measure (**Carl Friedrich Gauss** in 1813, **George Green** in 1825).

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Achievements of 20th Century: Sobolev Spaces, BV Space, $\cdots$
Traces, Gauss-Green formula, $\cdots$

Bow Shock in Space generated by a Solar Explosion

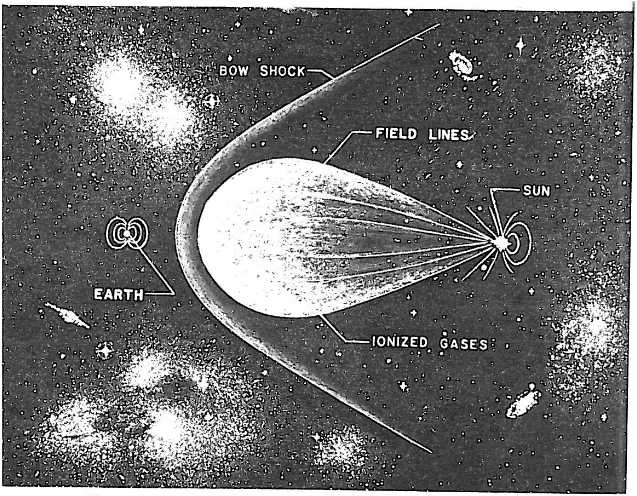
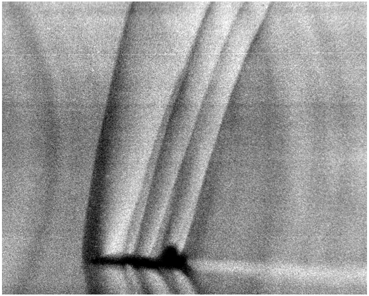
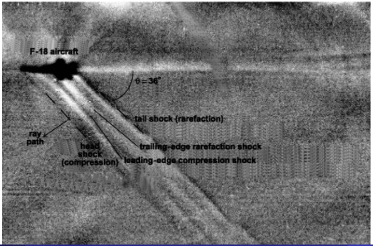


FIG. 50: SOLAR EXPLOSION

A shock wave in space generated by a solar eruption. The sketch shows the fully ionized nucleons attached to the solar magnetic field lines acting as the driving piston for the shock wave. (Courtesy: UTIAS, after Gold, 1962).

Shock Waves generated by Supersonic Aircrafts





Blast Wave from a TNT Surface Explosion

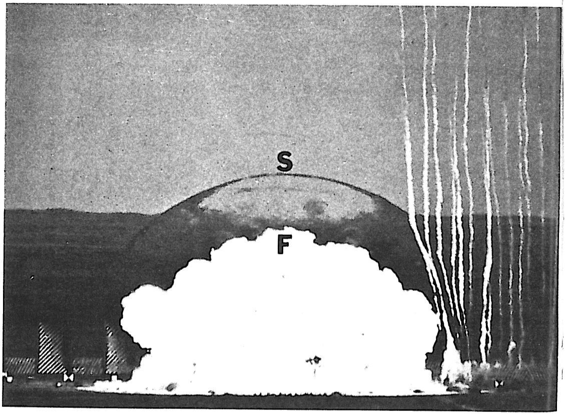
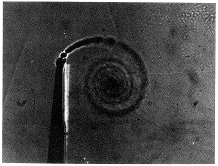


FIG. 22: EXPLOSION FROM A 20-TON HEMISPHERE OF TNT
The blast wave S, and fireball F, from a 20-ton TNT surface explosion are clearly shown. The backdrops are 50 feet by 30 feet and in conjunction with the rocket smoke trails, it is possible to distinguish shock waves and particle paths and to measure their velocities.

Vortex from a Wedge

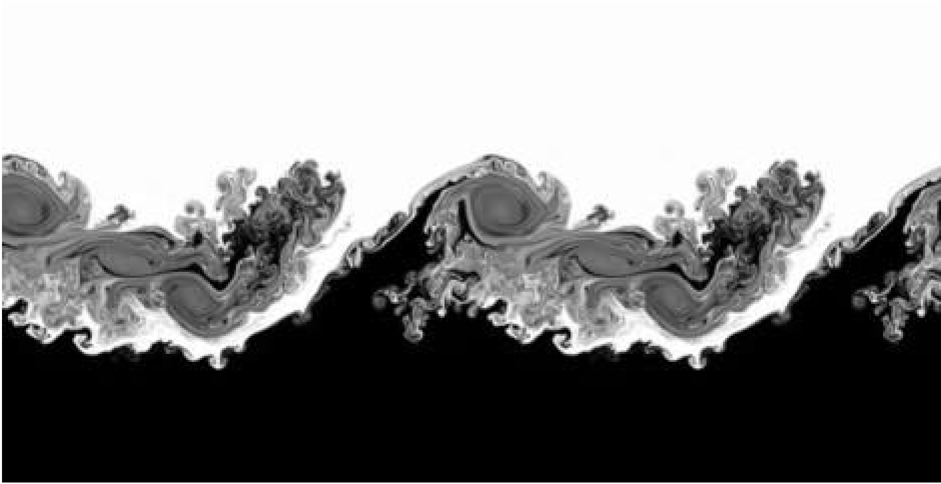


82. Vortex from a wedge in a shock tube. This schlieren photograph shows the vortex that spirals from the tip of a thin wedge after the air is set in motion normal to it by the passage of a weak plane shock wave, which is out of sight to the right. Other photographs show that the flow pattern is "conical" or "pseudo-stationary," remaining always similar to itself but growing in size in proportion to the time. *Photograph by Walker Bleakney*

83. Density in a vortex from a wedge. A quite different view of the phenomenon above is given by this infinite-fringe interferogram, which shows lines of constant density. A striking feature is the almost perfectly circular density distribution about the center of the vortex, extending nearly to the wedge. *Photograph by Walker Bleakney*

Kelvin-Helmholtz Instability I: Clouds over Colorado

Kelvin-Helmholtz Instability II



$$\partial_t \mathbf{u} + \nabla \cdot \mathbf{f}(\mathbf{u}) = 0$$

$\mathbf{u} = (u_1, \cdots, u_m)^\top$, $\mathbf{x} = (x_1, \cdots, x_d)$, $\nabla = (\partial_{x_1}, \cdots, \partial_{x_d})$

$\mathbf{f} = (\mathbf{f}_1, \cdots, \mathbf{f}_d) : \mathbb{R}^m \to (\mathbb{R}^m)^d$ is a nonlinear mapping

$\mathbf{f}_i : \mathbb{R}^m \to \mathbb{R}^m$ for $i = 1, \cdots, d$

$$\partial_t \mathbf{A}(\mathbf{u}, \mathbf{u}_t, \nabla \mathbf{u}) + \nabla \cdot \mathbf{B}(\mathbf{u}, \mathbf{u}_t, \nabla \mathbf{u}) = 0$$

$\mathbf{A}, \mathbf{B} : \mathbb{R}^m \times \mathbb{R}^m \times (\mathbb{R}^m)^d \to \mathbb{R}^m$ are nonlinear mappings

Connections and Applications:

- Fluid Mechanics and Related: Euler Equations and Related Equations
  Gas, shallow water, elastic body, reacting gas, plasma, ...
- Special Relativity: Relativistic Euler Equations and Related Equations
  General Relativity: Einstein Equations and Related Equations
- Differential Geometry: Isometric Embeddings, Nonsmooth Manifolds..
- ······

Hyperbolicity

$$\partial_t \mathbf{u} + \nabla \cdot \mathbf{f}(\mathbf{u}) = 0, \quad \mathbf{u} \in \mathbb{R}^m, \, \mathbf{x} \in \mathbb{R}^d$$

Plane Wave Solutions: $\mathbf{u}(t, \mathbf{x}) = \mathbf{w}(t, \, \omega \cdot \mathbf{x})$

$\mathbf{w}(t, \xi)$ is determined by: $\partial_t \mathbf{w} + (\nabla_{\mathbf{w}} \mathbf{f}(\mathbf{w}) \cdot \omega) \, \partial_\xi \mathbf{w} = 0$

?? Existence of stable plane wave solutions ??

Hyperbolicity in D: For any $\omega \in S^{d-1}$, $\mathbf{u} \in D$,

$$(\nabla_{\mathbf{u}} \mathbf{f}(\mathbf{u}) \cdot \omega)_{m \times m} \, \mathbf{r}_j(\mathbf{u}, \omega) = \lambda_j(\mathbf{u}, \omega) \, \mathbf{r}_j(\mathbf{u}, \omega), \; 1 \le j \le m$$

$\lambda_j(\mathbf{u}, \omega)$ **are real**

Main Features/Challenges:

Finiteness of Propagation Speeds;

Formation of Singularities: Discontinuous/Singular Solutions, $\cdots$

Well-Posedness: Existence, Uniqueness, Stability, $\cdots$

Convex Entropy and Hyperbolicity

$$\partial_t \mathbf{u} + \nabla \cdot \mathbf{f}(\mathbf{u}) = 0 \qquad (*)$$

Entropy: $\eta : \mathbb{R}^m \to \mathbb{R}$ if $\exists$ $\mathbf{q} = (q_1, \cdots, q_d) : \mathbb{R}^m \to \mathbb{R}^d$,

satisfying $\nabla q_i(\mathbf{u}) = \nabla \eta(\mathbf{u}) \nabla \mathbf{f}_i(\mathbf{u}), \; i = 1, \cdots, d$

(Strictly) convex entropy $\eta(\mathbf{u})$: $(\nabla^2 \eta(\mathbf{u}) > 0)$ $\nabla^2 \eta(\mathbf{u}) \geq 0$

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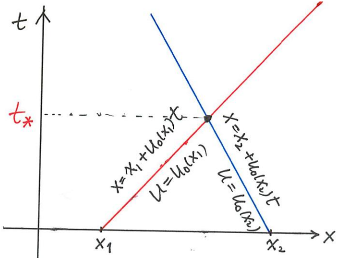
Theorem. If system (*) is endowed with a strictly convex entropy η in a state domain D, then system (*) must be hyperbolic and symmetrizable in D.

Lax-Friedrichs 1971; Godunov 1961, 1978, 1987; Boillat 1965; $\cdots$

$\Longrightarrow$ Local Existence of Smooth Solutions in $C([0,T]; H^s) \cap C^1([0,T]; H^{s-1}), \ s > \frac{d}{2} + 1.$

Singularities ⟹ Discontinuous/Singular Solutions

- Shock Waves, Vortex Sheets, Vorticity Waves, ...
- Focusing and Breaking of Waves, ...
- Concentration, Cavitation, ...
- ......



$u_0(x_1) > u_0(x_2)$

1-D Example:

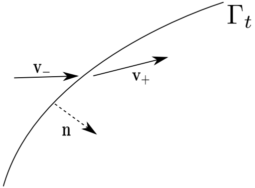
$$\begin{cases} u_t + \left(\frac{u^2}{2}\right)_x = 0 \\ u|_{t=0} = u_0(x) \end{cases}$$

$$u_t + u u_x = 0$$

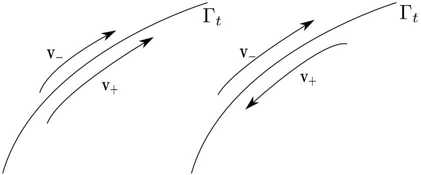
$$\begin{cases} \frac{dx}{dt} = u, \longrightarrow x = x_j + u_0(x_j)t \\ \frac{du}{dt} = 0 \longrightarrow u = u_0(x_j), \ j=1,2 \end{cases}$$

When $t \to t_* = \frac{x_2 - x_1}{u_0(x_1) - u_0(x_2)} > 0$, $u(t,x)$ is Multi-Valued

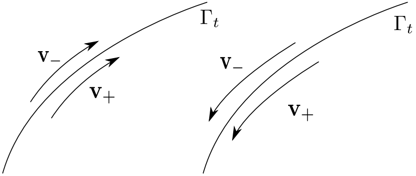
Noncharacteristic Discontinuities: Shock Waves:



Characteristic Discontinuities: Vortex Sheets/Entropy Waves

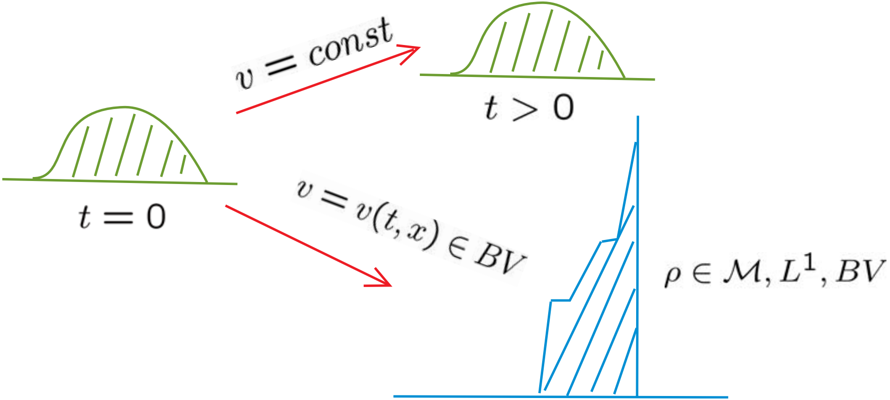


(i) $(p_+, \rho_+) = (p_-, \rho_-)$, $\mathbf{v}_+ \neq \mathbf{v}_-$



(ii) $(p_+, \mathbf{v}_+) = (p_-, \mathbf{v}_-)$, $\rho_+ \neq \rho_-$

Transport Equation: $\partial_t \rho + \partial_x(v\rho) = 0$

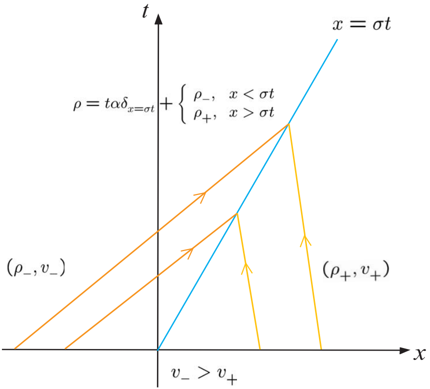


$$\int \rho(t,x)dx = \int \rho(0,x)dx$$

* ρ-density, v-velocity

Pressureless Euler Equations

$$\partial_t \rho + \partial_x(\rho v) = 0, \qquad \partial_t(\rho v) + \partial_x(\rho v^2) = 0$$



$$\alpha = \frac{1}{\sqrt{1+\sigma^2}}(\sigma[\rho] - [\rho v]) > 0, \qquad \sigma = \frac{\sqrt{\rho_+}v_+ + \sqrt{\rho_-}v_-}{\sqrt{\rho_+} + \sqrt{\rho_-}} \in (v_+, v_-)$$

$$\partial_t \mathbf{u} + \nabla \cdot \mathbf{f}(\mathbf{u}) = 0$$

Challenges: Singularity →Discontinuous/Singular Solutions

- Shock Waves, Vortex Sheets, Vorticity Waves, Entropy Waves, ...
- Focusing and Breaking of Waves, ...
- Concentration, Cavitation, ...
- ......

Entropy Solutions:

(i) $\mathbf{u}(t, \mathbf{x}) \in BV, L^\infty, L^p, \mathcal{M}$;

(ii) For any convex entropy pair $(\eta, \mathbf{q})$, $\partial_t \eta(\mathbf{u}) + \nabla_{\mathbf{x}} \cdot \mathbf{q}(\mathbf{u}) \leq 0 \;\; \mathcal{D}'$

as long as $(\eta(\mathbf{u}(t, \mathbf{x})), \mathbf{q}(\mathbf{u}(t, \mathbf{x}))) \in \mathcal{D}'$ (i.e. $(\eta, \mathbf{q}) := (\eta, q_1, \dots, q_d)$ is a solution of $\nabla q_k(\mathbf{u}) = \nabla \eta(\mathbf{u}) \nabla \mathbf{f}_k(\mathbf{u}), 1 \leq k \leq d$).

Posed Spaces for Entropy Solutions ??

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Posed Spaces for Entropy Solutions ??

Candidates: BV, L^∞, L^p, $\mathcal{M}$, ...

BV Space: Well-Posedness Space for Entropy Solutions?

1-D: Glimm's Theorem (1965): $\|\mathbf{u}(t,\cdot)\|_{BV} \leq C\,\|\mathbf{u}_0\|_{BV}$

L^1-**Stability**: $\|\mathbf{u}(t,\cdot)-\mathbf{v}(t,\cdot)\|_{L^1(\mathbb{R})} \leq C\,\|\mathbf{u}(0,\cdot)-\mathbf{v}(0,\cdot)\|_{L^1(\mathbb{R})}$

Bressan et al, Liu-Yang, Bianchini-Bressan, LeFloch, ...

Strictly hyperbolic systems, initial data of small BV: Works!

Large initial data, nonstrictly hyper. systems: Fails in general!!

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Multi-D: **(?)** $\|\mathbf{u}(t,\cdot)\|_{BV} \le C\,\|\mathbf{u}_0\|_{BV}$ (*)

Rauch (1986): A necessary condition for (*) is

$$\nabla\mathbf{f}_k(\mathbf{u})\nabla\mathbf{f}_l(\mathbf{u}) = \nabla\mathbf{f}_l(\mathbf{u})\nabla\mathbf{f}_k(\mathbf{u}) \qquad \text{for all } k,l = 1,2,\dots,d$$

Special cases: $m = 1$ or $d = 1$

$\mathbf{f}_k(\mathbf{u}) = \phi_k(|\mathbf{u}|^2)\mathbf{u}, \qquad k = 1,2,\dots,d$

2003: Ambrosio-De Lellis, Bressan: (*) fails

2005: De Lellis: Blowup of $\|\mathbf{u}(t,\cdot)\|_{BV}$ in finite time

Frameworks for Studying Entropy Solutions of Multidimensional Conservation Laws?

A general mathematical framework may be derived from the theory of divergence-measure vector fields, which is based on the following class of Entropy Solutions:

(i) $\mathbf{u}(t, \mathbf{x}) \in \mathcal{M}$, L^∞, L^p;

(ii) For any convex entropy pair $(\eta, \mathbf{q})$,

$$\partial_t \eta(\mathbf{u}) + \nabla_{\mathbf{x}} \cdot \mathbf{q}(\mathbf{u}) \le 0 \qquad \mathcal{D}'$$

as long as $(\eta(\mathbf{u}(t, \mathbf{x})), \mathbf{q}(\mathbf{u}(t, \mathbf{x}))) \in \mathcal{D}'$

Existence: **Isentropic Euler Equations** via Compensated Compactness
DiPerna, Chen, Ding-Chen-Luo, Lions-Perthame-Souganidis-Tadmor, Chen-LeFloch

Schwartz's lemma $\Longrightarrow$

$$\mathrm{div}_{(t,\mathbf{x})}(\eta(\mathbf{u}(t, \mathbf{x})), \mathbf{q}(\mathbf{u}(t, \mathbf{x}))) \in \mathcal{M}$$

$\Longrightarrow$ The field $(\eta(\mathbf{u}(t, \mathbf{x})), \mathbf{q}(\mathbf{u}(t, \mathbf{x})))$ is a divergence-measure vector field

Divergence-Measure Fields over an open set $\mathcal{D} \subset \mathbb{R}^N$

- For $1 \le p \le \infty$, $\mathbf{F}$ is called a $\mathcal{DM}^p(\mathcal{D})$-field if $\mathbf{F} \in L^p(\mathcal{D})$ and

$$\|\mathbf{F}\|_{\mathcal{DM}^p(\mathcal{D})} := \|\mathbf{F}\|_{L^p(\mathcal{D};\mathbb{R}^N)} + \|\operatorname{div}\mathbf{F}\|_{\mathcal{M}(\mathcal{D})} < \infty; \qquad (1)$$

- The field $\mathbf{F}$ is called a $\mathcal{DM}^{ext}(\mathcal{D})$-field if $\mathbf{F} \in \mathcal{M}(\mathcal{D})$ and

$$\|\mathbf{F}\|_{\mathcal{DM}^{ext}(\mathcal{D})} := \|(\mathbf{F}, \operatorname{div}\mathbf{F})\|_{\mathcal{M}(\mathcal{D})} < \infty. \qquad (2)$$

- $\mathbf{F}$ is called a $\mathcal{DM}^p_{loc}(\mathcal{D})$ field if $\mathbf{F} \in \mathcal{DM}^p(\Omega)$ and $\mathbf{F}$ called a $\mathcal{DM}^{ext}_{loc}(\mathcal{D})$ if $\mathbf{F} \in \mathcal{DM}^{ext}(\Omega)$, for any open set $\Omega \Subset \mathcal{D}$.

$\mathcal{DM}^p(\mathcal{D})$ and $\mathcal{DM}^{ext}(\mathcal{D})$ are **Banach spaces**, which are LARGER than the space of BV fields (they coincide when $N = 1$).

BV theory (esp. the Gauss-Green Formula and Traces) has significantly advanced our understanding of solutions of nonlinear PDEs and related problems in the calculus of variations, differential geometry,...

Goal:

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Goal: Develop a $\mathcal{DM}$ theory to deal with entropy solutions without bounded variation for nonlinear conservation laws and related problems

Examples

1: $\mathbf{F}(y_1, y_2) = (\sin\left(\frac{1}{y_1 - y_2}\right), -\sin\left(\frac{1}{y_1 - y_2}\right))$.

(i) $\mathbf{F} \in \mathcal{DM}^\infty(\mathbb{R}^2)$, while $F_j \notin BV(\mathbb{R}^2)$ for $j = 1, 2$;

(ii) $\mathbf{F}$ has an essential singularity at each point of $L = \{y_1 = y_2\}$, therefore, $\mathbf{F}$ has no trace on L in the classical sense.

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2: $\mathbf{F}(y_1, y_2) = \left(\frac{-y_2}{y_1^2 + y_2^2}, \frac{y_1}{y_1^2 + y_2^2}\right) \in \mathcal{DM}^1_{loc}(\mathbb{R}^2)$.

However, for $\Omega = \{\mathbf{y} : |\mathbf{y}| < 1, y_2 > 0\}$,

$$\int_\Omega \operatorname{div} \mathbf{F} = 0 \neq -\int_{\partial\Omega} \mathbf{F} \cdot \boldsymbol{\nu}\, d\mathcal{H}^1 = \pi \quad \text{(in the classical sense)},$$

where $\boldsymbol{\nu}$ is the interior unit normal on $\partial\Omega$ to Ω

$\Rightarrow$ The classical Gauss-Green theorem fails for a $\mathcal{DM}$-field.

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$\Rightarrow$ The classical Gauss-Green theorem fails for a $\mathcal{DM}$-field.

3: For any $\mu_i \in \mathcal{M}(\mathbb{R}), i = 1, 2$, with finite total variation,

$$\mathbf{F}(y_1, y_2) = (\mu_1(y_2), \mu_2(y_1)) \in \mathcal{DM}^{ext}(\mathbb{R}^2).$$

Cauchy Flux (Cauchy 1823-27): Derivation of Conservation Laws

Balance Laws: Cauchy Flux & Production

$\Leftrightarrow$ **Gauss-Green Formula and Normal Trace for $\mathcal{DM}$ Fields**

References:

C. M. Dafermos: Hyperbolic Conservation Laws in Continuum Physics, 3rd Edition, Springer-Verlag, 2010.

Chen-Torres-Ziemer: Gauss-Green Theorem for Weakly Differentiable Fields, Sets of Finite Perimeter, and Balance Laws, Comm. Pure Appl. Math. **62** (2009), 242–304.

Divergence-Measure Fields: $\mathbf{F} \in \mathcal{DM}^\infty(\mathcal{D})$

- $\exists$ C^∞ vector fields $\mathbf{F}^j, j = 1, 2, \ldots,$ such that
  $$|\mathrm{div}\,\mathbf{F}^j|(\mathcal{D}) \to |\mathrm{div}\,\mathbf{F}|(\mathcal{D});$$

Divergence-Measure Fields: $\mathbf{F} \in \mathcal{DM}^\infty(\mathcal{D})$

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$$|\mathrm{div}\, \mathbf{F}^j|(\mathcal{D}) \to |\mathrm{div}\, \mathbf{F}|(\mathcal{D});$$
- If $g \in BV \cap L^\infty(\mathcal{D})$, then $g\mathbf{F} \in \mathcal{DM}^\infty(\mathcal{D})$ and the product rule holds:
$$\mathrm{div}\,(g\mathbf{F}) = g^* \mathrm{div}\, \mathbf{F} + \overline{\mathbf{F} \cdot \nabla g},$$
where $g^*(= g, a.e.)$ is the limit of the mollifiers of g and
$$|\overline{\mathbf{F} \cdot \nabla g}| \le \|F\|_{L^\infty} |\nabla g|, \qquad (\overline{\mathbf{F} \cdot \nabla g})_{ac} = \mathbf{F} \cdot (\nabla g)_{ac}.$$

Divergence-Measure Fields: $\mathbf{F} \in \mathcal{DM}^\infty(\mathcal{D})$

- $\exists$ C^∞ vector fields $\mathbf{F}^j, j = 1, 2, \ldots,$ such that
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- If $E \Subset \mathcal{D}$ is a set of finite perimeter with $|\mathrm{div}\,\mathbf{F}|(\partial E) = 0$, then the Gauss-Green formula holds on E:
$$\int_E \mathrm{div}\,\mathbf{F} = -\int_{\partial^* E} \mathbf{F}(\mathbf{y}) \cdot \boldsymbol{\nu}(\mathbf{y})\, d\mathcal{H}^{N-1}(\mathbf{y}).$$

See: Chen-Frid: ARMA 1999

Related: Fuglede, Anzellotti, Ziemer, Silhavy, $\cdots\cdots$

Deformable Lipschitz Boundary $\partial\Omega$: $\Omega \subset \mathbb{R}^N$ open, bdd

- For any $\mathbf{x} \in \partial\Omega$, there exist $r > 0$ and a Lipschitz map $\gamma : \mathbb{R}^{N-1} \to \mathbb{R}$ such that, after rotating and relabeling coordinates if necessary,

$$\Omega \cap Q(\mathbf{x}, r) = \{\mathbf{y} \in \mathbb{R}^N \ : \ \gamma(y_1, \dots, y_{N-1}) < y_N \} \cap Q(\mathbf{x}, r),$$

where $Q(\mathbf{x}, r) = \{\mathbf{y} \in \mathbb{R}^N \ : \ |y_i - x_i| \le r, \ i = 1, \dots, N \}$;

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- There exists $\Psi : \partial\Omega \times [0,1] \to \overline{\Omega}$ such that Ψ is a homeomorphism, bi-Lipschitz over its image, and $\Psi(\omega, 0) = \omega$ for any $\omega \in \partial\Omega$. The map Ψ is called a Lipschitz deformation of the boundary $\partial\Omega$.

Notations: For $s \in [0,1]$, $\partial\Omega_s := \Psi(\partial\Omega \times \{s\})$

Ω_s=the open subset of Ω bounded by $\partial\Omega_s$.

* The domains with deformable Lipschitz boundaries clearly include star-shaped domains and domains whose boundaries satisfy the cone property. It is also clear that, if Ω is the image through a bi-Lipschitz map of a domain $\bar{\Omega}$ with a Lipschitz deformable boundary, then Ω itself possesses a Lipschitz deformable boundary.

Theorem (Chen-Frid): $\partial\Omega$ Lip. Deformable; $\mathbf{F} \in \mathcal{DM}^\infty(\mathcal{D})$

There exists $\mathbf{F}\cdot\boldsymbol{\nu} \in L^\infty(\partial\Omega)$ such that, for any $\varphi \in C_0^1(\mathbb{R}^N)$,

- $\int_\Omega \varphi \,\mathrm{div}\, \mathbf{F} = -\int_{\partial\Omega} \varphi\,(\mathbf{F}\cdot\boldsymbol{\nu})\, d\mathcal{H}^{N-1} - \int_\Omega \mathbf{F}\cdot\nabla\varphi\, d\mathbf{y}$;

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- $\|\mathbf{F} \cdot \boldsymbol{\nu}\|_{L^\infty(\partial\Omega)} \leq \|\mathbf{F}\|_{L^\infty(\Omega)}$;
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  $\langle \mathbf{F} \cdot \boldsymbol{\nu}, \, \psi \rangle_{\partial\Omega} = \operatorname{ess\,lim}_{s\to 0} \int_{\partial\Omega_s} (\mathbf{F} \cdot \boldsymbol{\nu}) \, (\psi \circ \Psi_s^{-1}) \, d\mathcal{H}^{N-1}$;

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- Let $h : \mathbb{R}^N \to \mathbb{R}$ be the level set function of $\partial\Omega_s$:

$$h(\mathbf{y}) := \begin{cases} 0 & \text{for } \ \mathbf{y} \in \mathbb{R}^N - \bar{\Omega}, \\ 1 & \text{for } \ \mathbf{y} \in \Omega - \Psi(\partial\Omega \times [0,1]), \\ s & \text{for } \ \mathbf{y} \in \partial\Omega_s, 0 \le s \le 1. \end{cases}$$

Then, for any $\psi \in \mathrm{Lip}(\partial\Omega)$,

$$\langle \mathbf{F} \cdot \boldsymbol{\nu},\, \psi \rangle_{\partial\Omega} = \lim_{s\to 0} \frac{1}{s} \int_{\Psi(\partial\Omega\times(0,s))} \mathcal{E}(\psi)\, \nabla h \cdot \mathbf{F} \, d\mathbf{y},$$

where $\mathcal{E}(\psi)$ is any Lipschitz extension of ψ to the whole space $\mathbb{R}^N$.

Sets of Finite Perimeter $E \subset \mathbb{R}^N$: $\chi_E \in BV(\mathbb{R}^N)$

For every $\alpha \in [0, 1]$, define the set of all points with density α:

$$E^\alpha := \{\mathbf{y} \in \mathbb{R}^N \,:\, \lim_{r\to 0} \frac{|E\cap B(\mathbf{y},r)|}{|B(\mathbf{y},r)|} = \alpha\}.$$

E^0—Measure-theoretic Exterior, E^1—Measure-theoretic Interior

$\partial^m E := \mathbb{R}^N \setminus (E^0 \cup E^1)$—Measure-theoretic Boundary

Sets of Finite Perimeter $\Longleftrightarrow \mathcal{H}^{N-1}(\partial^m E) < \infty$

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Reduced Boundary $\partial^* E$: Set of all points $\mathbf{y} \in \Omega$ such that

(i) $\|\nabla\chi_E\|\,(B(\mathbf{y},r)) > 0$ for all $r > 0$;

(ii) The limit $\boldsymbol{\nu}_E(\mathbf{y}) := \lim_{r\to 0} \frac{\nabla\chi_E(B(\mathbf{y},r))}{\|\nabla\chi_E\|(B(\mathbf{y},r))}$ exists.

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$\Rightarrow$ (i) For $\mathcal{H}^{N-1}$-a.e. $\mathbf{y} \in \partial^* E$, $\lim_{r\to 0} \frac{\|\nabla\chi_E\|(B(\mathbf{y},r))}{\alpha(N-1)r^{N-1}} = 1$;

(ii) $\|\nabla \chi_E\| = \mathcal{H}^{N-1} \llcorner \partial^* E$.

$\boldsymbol{\nu}_E(\mathbf{y})$ (unit vector)–measure-theoretic interior unit normal to E at $\mathbf{y}$

- $\partial^* E \subset E^{\frac{1}{2}} \subset \partial^m E$; $\mathcal{H}^{N-1}(\partial^m E \setminus \partial^* E) = 0$.

Consult: **H. Federer (1969), L. Evans and R. Gariepy (1992)**, $\cdots$

Approximation Theorem: Almost One-Sided Smooth Approximation of a Set of Finite Perimeter, E

The set E cannot be approximated by smooth sets that lie completely in the interior of E.

For example, let U be the open unit disk with a single radius removed. Then $\mathcal{H}^1(\partial U) = 2\pi + 1$, while $\mathcal{H}^1(\partial^* U) = 2\pi$. Thus, if U_k is an approximating open subset of U, then its boundary will be close to that of boundary U and so $\mathcal{H}^1(\partial U_k)$ will be close to $2\pi + 1$. Adding more radii, say m of them, will force the approximating set to have boundaries whose Hausdorff measure close to 2π plus m.

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Theorem (Chen-Torres-Ziemer)

Let μ be a Radon measure on $\mathbb{R}^N$ satisfying $|\mu| \ll \mathcal{H}^{N-1}$. Then there exists a family of smooth sets A_k such that

- $|\mu|\big((A_k \setminus E^1) \cup (E^1 \setminus A_k)\big) \to 0$;
- $\lim_{k\to\infty} \mathcal{H}^{N-1}(\partial A_k \cap (E^0 \cup \partial^* E)) = 0.$

Theorem (Chen-Torres-Ziemer): $\mathbf{F} \in \mathcal{DM}^\infty(\mathcal{D})$
$E := E^1 \cup \partial^* E \Subset \mathcal{D}$ be a bounded set of finite perimeter

there exists a signed measure σ_{in} and a family of sets A_k with smooth boundaries such that

- $\|\mathrm{div}\,\mathbf{F}\|\big((A_k \setminus E^1) \cup (E^1 \setminus A_k)\big) \to 0$;
- $\lim_{k\to\infty} \mathcal{H}^{N-1}(\partial A_k \cap (E^0 \cup \partial^* E)) = 0$;
- $\lim_{k\to\infty} \|\sigma_k\|(E^0 \cup \partial^* E) = 0$, where the measures σ_k are defined by $\sigma_k(B) = \int_{B\cap\partial A_k} \mathbf{F}\cdot\boldsymbol{\nu}\, d\mathcal{H}^{N-1}$ for any Borel set $B \subset \mathcal{D}$ with $\mathbf{F}\cdot\boldsymbol{\nu}$ being the normal trace over the smooth boundary ∂A_k;
- $\sigma_{in} := w^* - \lim_{k\to\infty} \sigma_k$ in $\mathcal{M}(\mathcal{D})$ is carried by $\partial^* E$ with $\|\sigma_{in}\|\,(\mathcal{D} \setminus \partial^* E) = 0$ and $\|\sigma_{in}\| \ll \mathcal{H}^{N-1} \llcorner \partial^* E$;
- The density of σ_{in}, $(\mathbf{F}\cdot\boldsymbol{\nu})_{in}$, is called the interior normal trace relative to E of $\mathbf{F}$ on $\partial^* E$ and satisfies the Gauss-Green formula:

  $$\int_{E^1} \mathrm{div}\,\mathbf{F} = (\mathrm{div}\,\mathbf{F})(E^1) = -\sigma_{in}(\partial^* E) = -\int_{\partial^* E} (\mathbf{F}\cdot\boldsymbol{\nu})_{in}(\mathbf{y})\, d\mathcal{H}^{N-1}(\mathbf{y});$$
- $\|\sigma_{in}\| = \|(\mathbf{F}\cdot\boldsymbol{\nu})_{in}\|_{L^\infty(\partial^* E, \mathcal{H}^{N-1})} \le \|\mathbf{F}\|_{L^\infty(\mathcal{D})}$.

Remarks

- Exterior Normal Traces $(\mathbf{F} \cdot \boldsymbol{\nu})_{ex}$: Similar Theorem
- Let $E \Subset \mathcal{D}$ be a set of finite perimeter. Then, for any $\varphi \in C_0^1(\mathcal{D})$,

$$\int_{E^1} \varphi \,\mathrm{div}\, \mathbf{F} = -\int_{\partial^* E} \varphi \,(\mathbf{F} \cdot \boldsymbol{\nu})_{in}\, d\mathcal{H}^{N-1} - \int_{E^1} \mathbf{F} \cdot \nabla\varphi \, d\mathbf{y}$$

Chen-Torres: ARMA 2005

$\mathbf{F} \in \mathcal{DM}^p$ for $1 < p < \infty$: Silhavy's Example (2009)

$\mathbf{F} : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}^2$ with $1 < \alpha < 3$:

$$\mathbf{F}(\mathbf{x}) = \frac{1}{|\mathbf{x}|^\alpha}(x_2, -x_1) \qquad \text{for every } \mathbf{x} = (x_1, x_2) \in \mathbb{R}^2 \setminus \{0\}$$

$\Omega = \{\mathbf{x} = (x_1, x_2) : |\mathbf{x}| < 1, x_2 < 0\}$

$\Longrightarrow \mathbf{F} \in \mathcal{DM}^p(\Omega) \qquad (1 \le p \le \infty$ for $\alpha = 1$; $1 \le p\frac{2}{\alpha - 1}$ for $1 < \alpha < 3)$:

$$\mathbf{F} \in L^p(\Omega; \mathbb{R}^2); \qquad \operatorname{div} \mathbf{F} = 0 \qquad \text{in } \mathbb{R}^2 \setminus \{0\}$$

For any $g \in Lip(\partial\Omega)$, we have

$$\mathbf{F} \cdot \boldsymbol{\nu}(g) = \begin{cases} \int_{-1}^{1} g(t,0)\operatorname{sgn}(t)|t|^{1-\alpha} dt, & 1 \le \alpha < 2, \\ \lim_{\varepsilon \to 0} \int_{\{\varepsilon < |t| < 1\}} g(t,0)\operatorname{sgn}(t)|t|^{1-\alpha} dt, & 2 \le \alpha < 3, \end{cases}$$

- For $1 \le \alpha \le 2$, $\mathbf{F} \cdot \boldsymbol{\nu}$ is a measure;
- For $2 \le \alpha < 3$, $\mathbf{F} \cdot \boldsymbol{\nu}$ is NOT a measure.
  Notice that the principal value (in the case $2 \le \alpha < 3$) exists for each $g : \partial U \to \mathbb{R}$ Hölder continuous of exponent $\beta > \alpha - 2$.

$\mathbf{F} \in \mathcal{DM}(\mathcal{D})$: Gauss-Green Theorem & Normal Trace

Theorem (Chen-Frid, Silhavy)

If $\mathbf{F} \in \mathcal{DM}(\mathcal{D})$, then, for any $\Omega \subset \mathcal{D}$, there exists a linear functional $\mathbf{F} \cdot \boldsymbol{\nu} : Lip(\partial\Omega) \to \mathbb{R}$ such that, for any $g \in Lip(\mathbb{R}^N)$,

$$\mathbf{F} \cdot \boldsymbol{\nu}(g|\partial\Omega) = \int_\Omega \overline{\nabla g \cdot \mathbf{F}} + \int_\Omega g \operatorname{div} \mathbf{F},$$

and $\quad |\mathbf{F} \cdot \boldsymbol{\nu}(h)| \le \|F\|_{\mathcal{DM}} \|h\|_{Lip(\partial\Omega)}.$

Furthermore, if

- $m : \mathbb{R}^N \to \mathbb{R}^+$ *is Lipschitz,* $\operatorname{supp} m \subset \bar{\Omega}$, *and* $m(\mathbf{x}) > 0, \mathbf{x} \in \Omega$;
- *For* $\varepsilon > 0$, $L^\varepsilon := \{\mathbf{x} \in \Omega \,:\, 0 < m(\mathbf{x}) < \varepsilon\}$.

then

- *For* $g \in Lip(\mathbb{R}^N)$, $\quad \mathbf{F} \cdot \boldsymbol{\nu}(g|\partial\Omega) = -\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{L^\varepsilon} g \overline{\nabla m \cdot \mathbf{F}}$
- *If* $\quad \liminf_{\varepsilon \to 0} \frac{1}{\varepsilon} |\overline{\nabla m \cdot \mathbf{F}}|(L^\varepsilon) < \infty$, *then* $\mathbf{F} \cdot \boldsymbol{\nu}$ *is a measure over* $\partial\Omega$.

Example: $\quad m(\mathbf{x}) = dist(\mathbf{x}, \partial\Omega)$ *if* $\mathbf{x} \in \Omega$; $m(\mathbf{x}) = 0$ *if* $\mathbf{x} \in \mathbb{R}^N \setminus \Omega$.

Product Rules

Theorem (Chen-Frid 1999)

Let $\mathbf{F} \in \mathcal{DM}^\infty(\mathcal{D})$. *If* $g \in BV \cap L^\infty(\mathcal{D})$, *then* $g\mathbf{F} \in \mathcal{DM}^\infty(\mathcal{D})$ *and*

$$\mathrm{div}\,(g\mathbf{F}) = g^*\mathrm{div}\,\mathbf{F} + \overline{\mathbf{F} \cdot \nabla g},$$

where $g^*(= g, \text{a.e.})$ *is the limit of the mollifiers of* g *and*

$$|\overline{\mathbf{F} \cdot \nabla g}| \le \|F\|_{L^\infty} |\nabla g|, \qquad (\overline{\mathbf{F} \cdot \nabla g})_{ac} = \mathbf{F} \cdot (\nabla g)_{ac}.$$

Theorem (Silhavy 2009)

Let $\mathbf{F} \in \mathcal{DM}^{ext}(\mathcal{D})$. *If* $g \in W^{1,\infty}(\mathcal{D})$, *then* $g\mathbf{F} \in \mathcal{DM}^{ext}(\mathcal{D})$ *and*

$$\mathrm{div}\,(g\mathbf{F}) = g\mathrm{div}\,\mathbf{F} + \overline{\mathbf{F} \cdot \nabla g} \qquad \text{as Radon measures in } \mathcal{D}.$$

- $|\overline{\mathbf{F} \cdot \nabla g}|(\Omega) \le \|\nabla g\|_\infty |\mathbf{F}|(\Omega)$; $\qquad (\overline{\mathbf{F} \cdot \nabla g})_{ac} = (\mathbf{F})_{ac} \cdot \nabla g$;
- *If* $h \in W^{1,\infty}(\Omega)$,
  $\overline{\mathbf{F} \cdot \nabla(gh)} = \overline{\mathbf{F} \cdot \nabla g}\, h + \overline{\mathbf{F} \cdot \nabla h} g = \overline{h\mathbf{F} \cdot \nabla g} + \overline{g\mathbf{F} \cdot \nabla h}$;

A Different Point of View

Question: ?? Find a continuous or L^p vector field that solves

$$\operatorname{div} \mathbf{F} = \mu \qquad \text{in } \Omega,$$

for a given Radon measure μ.

Continuous $\mathcal{DM}$–Fields:

Bourgain-Brezis (JAMS 2003): $d\mu = f\,dx$ with $f \in L^n_{loc}(\Omega)$

De Pauw-Pfeffer (CPAM 2008): Necessary/sufficient Condition is: μ is a strong charge; i.e., given $\varepsilon > 0$ and a compact set $K \subset \Omega$, there is $\theta > 0$ such that

$$\int_\Omega \phi\, d\mu \le \varepsilon \|\nabla \phi\|_{L^1} + \theta \|\phi\|_{L^1} \qquad \text{for any } \phi \in C_0^\infty(K).$$

$\mathcal{DM}$–Fields in L^p: Phuc-Torres (IUMJ 2008)

L^∞–Fields: For any open set U with smooth boundary, $|\mu(U)| \le C\mathcal{H}^{n-1}(U)$

L^p–Fields, $\frac{n}{n-1} < p < \infty$: $\mu \in \mathcal{M}_+$ satisfies the finite $(1,p)$–energy:

$$\int_{\mathbb{R}^n} \big(I_1\mu(x)\big)^p dx < \infty \quad \text{with} \quad I_1\mu(x) = \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-1}} d\mu(y).$$

L^p–Fields, $1 \le p \le \frac{n}{n-1}$: $\mu \in \mathcal{M}_+$ must be $\mu \equiv 0$.

Entropy Solutions

- $\mathbf{u}(t,\mathbf{x}) \in \mathcal{M}(\mathbb{R}^{d+1}_+)$ **or** $L^p(\mathbb{R}^{d+1}_+), 1 \le p \le \infty$;
- **For any convex entropy-entropy flux pair** $(\eta, \mathbf{q})$ **so that** $(\eta(\mathbf{u}), \mathbf{q}(\mathbf{u}))(t,\mathbf{x})$ **is a distributional field,**

$$\mu_\eta := \partial_t \eta(\mathbf{u}) + \nabla_{\mathbf{x}} \cdot \mathbf{q}(\mathbf{u}) \le 0$$

in the sense of distributions (i.e. $(\eta, \mathbf{q}) := (\eta, q_1, \dots, q_d)$ **is a solution of** $\nabla q_k(\mathbf{u}) = \nabla\eta(\mathbf{u})\nabla\mathbf{f}_k(\mathbf{u}), 1 \le k \le d$**).**

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$\Rightarrow$ $(\eta(\mathbf{u}(t,\mathbf{x})), \mathbf{q}(\mathbf{u}(t,\mathbf{x}))) \in \mathcal{DM}^\infty(\mathbb{R}_+ \times \mathbb{R}^n)$.

$\Rightarrow$ Integration by parts, traces, ...

$\Rightarrow$ Properties of entropy solutions, ...

Some Fundamental Problems in Multidimensional Hyperbolic Conservation Laws and Related Nonlinear PDEs

- **Construction and Existence of Entropy Solutions for the Cauchy Problem and Initial-Boundary Value Problems**
- **Traces of Entropy Solutions on Shock Waves, Vortex Sheets, Entropy Waves, ...??**
- BV**-like Structure of Entropy Solutions in** L^∞ **or** L^p **??**
- **Generalized Characteristics ??**
- **Free Boundary Problems**
- **Uniqueness and Stability of Entropy Solutions ??**
- **Asymptotic Behavour of Entropy Solutions: Decay of Periodic Solutions, Stability of Riemann Solutions,....**
- **Efficient Numerical Algorithms and Methods,....**
- **New Mathematical Ideas, Frameworks, Methods, Approaches,...**
- ......