Elliptic-Hyperbolic Partial Differential Equations

Mathematics Taught Course Centre
Michaelmas Term
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In order to receive credits, you should write a miniproject (4-8 pages) after the end of the course on some (your favorite) topic which the course will cover.

Oxford grades: pass/fail, or distinction for particularly good work.

Course Homepage:
http://people.maths.ox.ac.uk/chengq/teach/tcc13/tcc-ehpde.html
Elliptic-Hyperbolic PDEs

arise naturally in Mechanics, Mathematical Physics, Analysis, Differential Geometry, Biology, Economics, ...

The solution of some fundamental issues in the areas greatly requires a deep understanding of nonlinear mixed elliptic-hyperbolic PDEs:

- Engineering, Physics, Biology, Economics, ...

• An introduction to some facets of techniques, approaches, and their applications to mixed PDEs.
• Prerequisite: Basic analysis and PDE theory.
• Some familiarity with nonlinear PDE theory, fluid mechanics, and differential geometry is desirable.
1. Introduction

2. Linear Degenerate Elliptic Equations
   [Elliptic Euler-Poisson-Darboux equations, …]

3. Nonlinear Degenerate Elliptic Equations

4. Fixed Point Theorems, Degree Theory, and Their Applications

5. Nonlinear Conservation Laws of Mixed Type and Shock Reflection-Diffraction Problems

6. Mathematics of Shock Reflection-Diffraction

7. Further Topics on PDEs of Mixed Elliptic-Hyperbolic Type
References:


Linear Partial Differential Equations

- Two of the Basic Types: Representatives

  Hyperbolic: \( \partial_{tt} u - \Delta_x u = 0 \) (Wave Equation)

  Elliptic: \( \Delta_x u = 0 \) (Laplace’s Equation)

  \( x = (x_1, \ldots, x_n), \quad \Delta_x = \sum_{j=1}^n \frac{\partial^2}{\partial^2 x_j} \)

- Mixed Hyperbolic-Elliptic Type

  Lavrentyev-Bitsadze Eq.: \( \partial_{xx} u + \text{sign}(x) \partial_{yy} u = 0 \)

  Tricomi Eq.: \( \partial_{xx} u + x \partial_{yy} u = 0 \) (hyperbolic degeneracy at \( x = 0 \))

  Keldysh Eq.: \( x \partial_{xx} u + \partial_{yy} u = 0 \) (parabolic degeneracy at \( x = 0 \))
Nonlinear PDEs of Mixed Elliptic-Hyperbolic Type I: Shock Reflection-Diffraction Problems in Fluid Mechanics
FIG. 50: SOLAR EXPLOSION
A shock wave in space generated by a solar eruption. The sketch shows the fully ionized nucleons attached to the solar magnetic field lines acting as the driving piston for the shock wave. (Courtesy: UTTAS, after Gold, 1962.)
FIG. 22: EXPLOSION FROM A 20-TON HEMISPHERE OF TNT

The blast wave S, and fireball F, from a 20-ton TNT surface explosion are clearly shown. The backdrops are 50 feet by 30 feet and in conjunction with the rocket smoke trails, it is possible to distinguish shock waves and particle paths and to measure their velocities.
Shock Waves generated by Supersonic Aircrafts
Shock Reflection-Diffraction

? Shock Wave Patterns around a Wedge (airfoils, inclined ramps, ...)
Complexity of Reflection-Diffraction Configurations:

Über den verlauf von funkenwellen in der ebene und im raume,


Supersonic Flow and Shock Waves,
Experimental Analysis: 1940s–

Walker Bleakney: Palmer Physical Laboratory
Princeton University, USA

Irvine Israel Glass: Institute for Aerospace Studies
University of Toronto, Canada

LeRoy Freame Henderson: School of Aerospace, Mechanical and Mechatronic Engineering, University of Sydney, Australia

Tatiana V. Bazhenova: Joint Institute of High Temperatures
Russian Academy of Sciences, Moscow, Russia

Kazuyoshi Takayama: Institute of Fluid Science
Tohoku University, Japan

......
A New Mach Reflection-Diffraction Pattern:
A. M. Tesdall and J. K. Hunter: TSD, 2002
A. M. Tesdall, R. Sanders, and B. L. Keyfitz: NWE, 2006; Full Euler, 2008
B. Skews and J. Ashworth: J. Fluid Mech. 542 (2005), 105-114
Shock Reflection-Diffraction Patterns

- **Gabi Ben-Dor**  *Shock Wave Reflection Phenomena*
  Experimental results before 1991
  Various proposals for transition criteria

- **Peter O. K. Krehl**  *History of Shock Waves, Explosions and Impact*
  A Chronological and Biographical Reference
  2009, XXII, 1288 p. 1200 illus., 300 in color.

- **Milton Van Dyke**  *An Album of Fluid Motion*
  Various photographs of shock wave reflection phenomena
Scientific Issues

- Structure of the Shock Reflection-Diffraction Patterns
- Transition Criteria among the Patterns
- Dependence of the Patterns on the Parameters
  - wedge angle $\theta_w$
  - adiabatic exponent $\gamma \geq 1$
  - incident-shock-wave Mach number $M_s$
- ...

Interdisciplinary Approaches:

- Experimental Data and Photographs
- Large or Small Scale Computing
  - Colella, Berger, Deschambault, Glass, Glaz, Woodward, ...
  - Anderson, Hindman, Kutler, Schneyer, Shankar, ...
  - Yu. Dem'yanov, Panasenko, ...
- Asymptotic Analysis
  - Lighthill, Keller, Majda, Hunter, Rosales, Tabak, Gamba, Harabetian ...
  - Morawetz: CPAM 1994
- Rigorous Mathematical Analysis?? (Global Solutions)
  - Existence, Stability, Regularity, Bifurcation, ...
2-D Riemann Problem for Hyperbolic Conservation Laws

\[ \partial_t U + \nabla_x \cdot F(U) = 0, \quad x = (x_1, x_2) \in \mathbb{R}^2 \]

or

\[ \partial_t A(U, U_t, \nabla_x U) + \nabla_x \cdot B(U, U_t, \nabla_x U) = 0 \]

Books and Survey Articles:

Kurganov-Tadmor 2002, ⋯

Theoretical Roles:  Asymptotic States and Attractors
Local Structure and Building Blocks
FIG. 5.5A  
Density contour curves

FIG. 5.5B  
Self-Mach number contour curves

FIG. 5.5C  
Pressure contour curves
Riemann Solutions II

Fig. 5.6A
Density contour curves

Fig. 5.6B
Self-Mach number contour curves

Fig. 5.6C
Pressure contour curves
Full Euler Equations: \((t, x) \in \mathbb{R}_+^3 := (0, \infty) \times \mathbb{R}^2\)

\[
\begin{align*}
\partial_t \rho + \nabla_x \cdot (\rho \mathbf{v}) &= 0 \quad \text{(conservation of mass)} \\
\partial_t (\rho \mathbf{v}) + \nabla_x \cdot (\rho \mathbf{v} \otimes \mathbf{v}) + \nabla_x \rho &= 0 \quad \text{(conservation of momentum)} \\
\partial_t \left( \frac{1}{2} \rho |\mathbf{v}|^2 + \rho e \right) + \nabla_x \cdot \left( \left( \frac{1}{2} \rho |\mathbf{v}|^2 + \rho e + p \right) \mathbf{v} \right) &= 0 \quad \text{(conservation of energy)}
\end{align*}
\]

Constitutive Relations: \(p = p(\rho, e)\)

- \(\rho\)–density, \(\mathbf{v} = (v_1, v_2)^T\)–fluid velocity, \(p\)–pressure
- \(e\)–internal energy, \(\theta\)–temperature, \(S\)–entropy

For a polytropic gas: \(p = (\gamma - 1)\rho e\), \(e = c_v \theta\), \(\gamma = 1 + \frac{R}{c_v} > 1\)

\[
p = p(\rho, S) = \kappa \rho^{\gamma} e^{S/c_v}, \quad e = e(\rho, S) \frac{\kappa}{\gamma - 1} \rho^{\gamma - 1} e^{S/c_v},
\]

- \(R > 0\) may be taken to be the universal gas constant divided by the effective molecular weight of the particular gas
- \(c_v > 0\) is the specific heat at constant volume
- \(\gamma > 1\) is the adiabatic exponent, \(\kappa > 0\) is any constant under scaling
Euler Equations for Potential Flow: $v = \nabla \Phi$, $\gamma > 1$

\[
\begin{align*}
\partial_t \rho + \nabla_x \cdot (\rho \nabla_x \Phi) &= 0, & \text{(Conservation of mass)} \\
\partial_t \Phi + \frac{1}{2} |\nabla_x \Phi|^2 + \frac{\rho \gamma^{-1}}{\gamma - 1} &= \frac{\rho_0^\gamma}{\gamma - 1}, & \text{(Bernoulli's law)}
\end{align*}
\]

or, equivalently,

\[
\partial_t \rho (\partial_t \Phi, \nabla_x \Phi, \rho_0) + \nabla_x \cdot (\rho (\partial_t \Phi, \nabla_x \Phi, \rho_0) \nabla_x \Phi) = 0,
\]

with

\[
\rho (\partial_t \Phi, \nabla_x \Phi, \rho_0) = (\rho_0^{\gamma - 1} - (\gamma - 1)(\partial_t \Phi + \frac{1}{2} |\nabla_x \Phi|^2)) \frac{1}{\gamma - 1}.
\]

- The potential flow equations and the full Euler equations coincide in important regions of the solution and are very close each other in the other regions in the configuration of regular shock reflection-diffraction.

- Aerodynamics/Gas Dynamics: Fundamental PDE

- Morawetz: *CPAM* 1994,  
- Majda-Thomann: *CPDE* 1987,  
- ………
Initial-Boundary Value Problem: \( 0 < \rho_0 < \rho_1, \ u_1 > 0 \)

Initial condition: \((\rho, \Phi)|_{t=0} = \begin{cases} (\rho_0, 0), & |x_2| > x_1 \tan \theta_w, x_1 > 0, \\ (\rho_1, u_1x_1), & x_1 < 0; \end{cases}\)

Slip boundary condition on the wedge boundary: \( \nabla \Phi \cdot \nu = 0 \)

Invariant under the Self-Similar Scaling:
\( (t, \mathbf{x}) \rightarrow (\alpha t, \alpha \mathbf{x}), \quad (\rho, \Phi) \rightarrow (\rho, \Phi/\alpha) \quad \alpha \neq 0 \)
Seek Self-Similar Solutions: \( (\xi, \eta) = (\frac{x_1}{t}, \frac{x_2}{t}) \)

\[
\rho(t, x) = \rho(\xi, \eta), \quad \Phi(t, x) = t(\varphi(\xi, \eta) + \frac{1}{2}(\xi^2 + \eta^2))
\]

\[
\nabla \cdot \left( \rho(\nabla \varphi, \varphi, \rho_0) \nabla \varphi \right) + 2\rho(\nabla \varphi, \varphi, \rho_0) = 0
\]

- Incompressible: \( \rho = \text{const.} \implies \Delta \varphi + 2 = 0 \)
- Elliptic: \( |\nabla \varphi| < c_*(\varphi, \rho_0) := \sqrt{\frac{2}{\gamma+1}(\rho_0^{\gamma-1} - (\gamma - 1)\varphi)} \)
- Hyperbolic: \( |\nabla \varphi| > c_*(\varphi, \rho_0) := \sqrt{\frac{2}{\gamma+1}(\rho_0^{\gamma-1} - (\gamma - 1)\varphi)} \)

Second-order nonlinear equations of mixed hyperbolic-elliptic type
Near the Sonic Circle with rescaled coordinates \((x,y)\):

\[
(2x - (\gamma + 1)\psi_x)\psi_{xx} + \frac{1}{c^2}\psi_{yy} - \psi_x \sim o(x^2)
\]

**Ellipticity:**

\[
\psi_x \leq \frac{2x}{\gamma + 1}
\]

**Apriori Estimate:**

\[
|\psi_x| \leq \frac{4x}{3(\gamma + 1)}
\]

**Asymptotics:**

\[
\psi \sim \frac{x^2}{2(\gamma + 1)} + h.o.t. \quad \text{when } x \approx 0
\]
Important Nonlinear Models and Equations of Mixed Type

- **Transonic Small Disturbance Equation:**
  \[
  ((u - x)u_x + \frac{u}{2})_x + u_{yy} = 0
  \]
  or, for \( v = u - x \),
  \[
  v v_{xx} + v_{yy} + \text{l.o.t.} = 0
  \]
  Morawetz, Hunter, Canic-Keyfitz-Lieberman-Kim, ⋯

- **Nonlinear Wave System:** Canic-Keyfitz-Kim, ⋯
- **Pressure Gradient System:** Y. Zheng, ⋯

- **Steady Potential Flow Equation of Aerodynamics**
  \[
  \nabla \cdot (\rho(\nabla \varphi, \rho_0)\nabla \varphi) = 0
  \]

- **Elliptic:**
  \[
  |\nabla \varphi| < c_*(\rho_0) := \sqrt{\frac{2}{\gamma+1} \rho_0^{\gamma-1}}
  \]

- **Hyperbolic:**
  \[
  |\nabla \varphi| > c_*(\rho_0)
  \]
Steady Potential Flow Equation of Aerodynamics

$$\nabla \cdot \left( \rho \left( \nabla \varphi, \rho_0 \right) \nabla \varphi \right) = 0$$

- **Pure Elliptic Case:** Subsonic Flow past an Obstacle
  Shiffman, L. Bers, Finn-Gilbarg, G. Dong, …

- **Degenerate Elliptic Case:** Subsonic-Sonic Flows
  Shiffman, Chen-Dafermos-Slemrod-Wang, Elling-Liu, Xin, …

- **Pure Hyperbolic Case (even Full Euler Eqs.):**
  Gu, Li, Schaeffer, S. Chen, Xin-Yin, Y. Zheng, …
  T.-P. Liu-Lien, S. Chen-Zhang-Wang, Chen-Zhang-Zhu, …

- **Elliptic-Hyperbolic Mixed Case**
  Transonic Nozzles: Chen-Feldman, S. Chen, J. Chen, Yuan, Xin-Yin,...
  Wedge or Conical Body: S. Chen, B. Fang, Chen-Fang, …
  Transonic Flow past an Obstacle: Morawetz, Chen-Slemrod-Wang,...
Mach Reflection: Self-Similar Solutions for the Full Euler Equations \((u, v, p, \rho)(t, x) = (u, v, p, \rho)(\xi, \eta), \ (\xi, \eta) = \left(\frac{x_1}{t}, \frac{x_2}{t}\right)\)

\[
\begin{align*}
(pU)_\xi + (pV)_\eta + 2\rho &= 0, \\
(pU^2 + p)\xi + (pUV)_\eta + 3\rho U &= 0, \\
(pUV)\xi + (pV^2 + p)_\eta + 3\rho V &= 0, \\
(U\left(\frac{1}{2}\rho q^2 + \frac{\gamma p}{\gamma - 1}\right))\xi + (V\left(\frac{1}{2}\rho q^2 + \frac{\gamma p}{\gamma - 1}\right))_\eta + 2\left(\frac{1}{2}\rho q^2 + \frac{\gamma p}{\gamma - 1}\right) &= 0,
\end{align*}
\]

where \(q = \sqrt{U^2 + V^2}\) and \((U, V) = (u - \xi, v - \eta)\) is the pseudo-velocity.

**Eigenvalues:** 
\[\lambda_0 = \frac{V}{U}\text{ (repeated)}, \quad \lambda_{\pm} = \frac{UV \pm c\sqrt{q^2 - c^2}}{U^2 - c^2},\]

where \(c = \sqrt{\gamma p/\rho}\) is the sonic speed.

**When the flow is pseudo-subsonic:** \(q < c\), the system consists of

- 2-transport equations: Compressible vortex sheets
- 2-nonlinear equations of mixed hyperbolic-elliptic type: Two kinds of transonic flow: Transonic shocks and sonic curves
Nonlinear PDEs of Mixed Elliptic-Hyperbolic Type II: Isometric Embedding Problems in Differential Geometry
Given a metric $g_{ij}$ and certain curvatures $h_{ij}$:

**Inverse Problem:** CAN we find a surface in our real world with this metric $g_{ij}$ and corresponding curvatures?

**Realization Question?**
Question: **CAN we produce even more sophisticated surfaces or thin sheets?**

**Fundamental**

- **Mathematics: Differential Geometry, Topology,** ……
- Understanding evolution of sophisticated shapes of surfaces or thin sheets in nature, including
  --Elasticity, Materials Science, ……
  --Biology and Algorithmic Origami: Protein Folding, ……
  *US DARPA’s 10th question of the 23 Challenge Questions in the Sciences*
  [US Defense Advanced Research Project Agency]:
  Build a stronger mathematical theory for isometric and rigid embedding that can give insight into protein folding.
- **Design, Visual Arts,** ……

Nash Isometric Embedding Theorem

\((C^k\text{ embedding theorem, } k \geq 3)\)

Every \(n\)-Dimensional Riemannian manifold (analytic or \(C^k\), \(k \geq 3\)) can be \(C^k\) isometrically imbedded in the Euclidean space \(\mathbb{R}^N\):

**Compact Case:** \(N = 3s_n + 4n\)

**Noncompact Case:** \(N = (n + 1)(3s_n + 4n)\)

Gromov (1986): \(N = s_n + 2n + 3\)

Günther (1989): \(N = \max\{s_n + 2n, s_n + n + 5\}\)

**Open Problems**

Important for Applications

Lowest Target Dimension? \(N = s_n = \frac{n(n+1)}{2}\)?

Optimal or Assigned Regularity?

\(C^{1,1}\) Isometric Embedding? What about \(BV(C^1)\)?

Current Research Activities, …..

Efimov’s Example (1966): No \(C^2\) Isometric Embedding when \(n = 2, s_n = 3\).
Gauss-Codazzi System: Compatibility/Constraint

Fundamental Theorem in Differential Geometry:
There exists a surface in $\mathbb{R}^3$ with 1st and 2nd fundamental form coefficients $\{g_{ij}\}$ and $\{h_{ij}\}$, $\{g_{ij}\}$ being positive definite, provided that the coefficients satisfy the Gauss-Codazzi system.

*This theorem holds even when $h_{ij} \in L^p$ (Mardare 2003–05)

Given $\{g_{ij}\}$, $\{h_{ij}\}$ is determined by the Codazzi Eqs. (Compatibility):

$$\begin{align*}
\partial_x M - \partial_y L &= L \Gamma^{(2)}_{22} - 2M \Gamma^{(2)}_{12} + N \Gamma^{(2)}_{11}, \\
\partial_x N - \partial_y M &= -L \Gamma^{(1)}_{22} + 2M \Gamma^{(1)}_{12} - N \Gamma^{(1)}_{11},
\end{align*}$$

satisfying the Gauss Equation (Constraint):

$$LN - M^2 = K,$$

where $L = \frac{h_{11}}{\sqrt{|g|}}$, $M = \frac{h_{12}}{\sqrt{|g|}}$, $N = \frac{h_{22}}{\sqrt{|g|}}$, $|g| = g_{11}g_{22} - g_{12}^2$

$\Gamma^{(k)}_{ij}$—Christoffel symbols, depending on $g_{ij}$ up to their 1st derivatives

$K(x, y)$—Gauss curvature, determined by $g_{ij}$ up to their 2nd derivatives

*Nonlinear PDEs of Mixed Elliptic-Hyperbolic Equations: Sign of $K$
Surfaces with Gauss Curvature of Changing Sign

Gauss Curvature $K$ on a Torus: Toroidal Shell or Doughnut Surface
Fluid Dynamics Formalism for Isometric Embedding

Set \( L = \rho v^2 + p, \ M = -\rho uv, \ N = \rho u^2 + p, \ q^2 = u^2 + v^2. \)

Choose \( p \) as the Chaplygin type gas: \( p = -1/\rho. \)

The Codazzi Equations become the Momentum Equations:

\[
\begin{align*}
\partial_x (\rho uv) + \partial_y (\rho v^2 + p) &= - (\rho v^2 + p) \Gamma^{(2)}_{22} - 2 \rho uv \Gamma^{(2)}_{12} - (\rho u^2 + p) \Gamma^{(2)}_{11}, \\
\partial_x (\rho u^2 + p) + \partial_y (\rho uv) &= - (\rho v^2 + p) \Gamma^{(1)}_{22} - 2 \rho uv \Gamma^{(1)}_{12} - (\rho u^2 + p) \Gamma^{(1)}_{11},
\end{align*}
\]

and the Gauss Equation becomes the Bernoulli Relation:

\[ p = -\sqrt{q^2 + K}. \]

Define the sound speed: \( c^2 = p'(\rho). \) Then \( c^2 = 1/\rho^2 = q^2 + K. \)

- \( c^2 > q^2 \) and the “flow” is subsonic when \( K > 0, \)
- \( c^2 < q^2 \) and the “flow” is supersonic when \( K < 0, \)
- \( c^2 = q^2 \) and the “flow” is sonic when \( K = 0. \)

\[ \text{?? Existence/Continuity of Isometric Embedding} \]

\[ \Leftarrow \text{Compensated Compactness and Entropy Analysis} \]

Gauss-Codazzi-Ricci System for Isometric Embedding of $d$-D Riemannian Manifolds into $\mathbb{R}^N$: $d \geq 3$

**Gauss equations:**

\[ h_{ji}^a h_{kl}^a - h_{ki}^a h_{jl}^a = R_{ijkl} \]

**Codazzi equations:**

\[ \frac{\partial h_{ij}^a}{\partial x^k} - \frac{\partial h_{kj}^a}{\partial x^i} + \Gamma_{lj}^m h_{km}^a - \Gamma_{kj}^m h_{lm}^a + \kappa_{kb}^a h_{lj}^b - \kappa_{lb}^a h_{kj}^b = 0 \]

**Ricci equations:**

\[ \frac{\partial \kappa_{lb}^a}{\partial x^k} - \frac{\partial \kappa_{kb}^a}{\partial x^l} - g^{mn} \left( h_{ml}^a h_{kn}^b - h_{mk}^a h_{ln}^b \right) + \kappa_{kc}^a \kappa_{lb}^c - \kappa_{lc}^a \kappa_{kb}^c = 0 \]

$R_{ijkl}$ is the Riemann curvature tensor, $\kappa_{kb}^a = -\kappa_{ka}^b$ is the coefficients of the connection form (torsion coefficients) on the normal bundle; the indices $a, b, c$ run from 1 to $N$, and $i, j, k, l, m, n$ run from 1 to $d \geq 3$.

*The Gauss-Codazzi-Ricci system has no type, neither purely hyperbolic nor purely elliptic for general Riemann curvature tensor $R_{ijkl}$.

\[ \partial_t u + \nabla \cdot f(u) = 0, \quad u \in \mathbb{R}^m, \quad x \in \mathbb{R}^d \]

Plane Wave Solutions: \[ u(t, x) = w(t, \omega \cdot x) \]

\[ w(t, \xi) \] is determined by:
\[ \partial_t w + (\nabla_w f(w) \cdot \omega) \partial_\xi w = 0 \]

?? Existence of stable plane wave solutions ??

Hyperbolicity in \( D \): For any \( \omega \in S^{d-1}, u \in D \),
\[ (\nabla_u f(u) \cdot \omega)_{m \times m} r_j(u, \omega) = \lambda_j(u, \omega) r_j(u, \omega), \quad 1 \leq j \leq m \]
\[ \lambda_j(u, \omega) \] are real

Main Features:
Finiteness of Propagation Speeds;
Discontinuities of Solutions, \cdots\cdots \]
Well-Posedness: Existence, Uniqueness, Stability, \cdots