Linear Degenerate Elliptic Equations.

Linear PDEs of Mixed Type:

① Tricomi type: $Lu = K(y)u_{xx} + u_{yy}$ + Lower-order-terms.

where $K(0)=0$ and $yK(y)>0$ for $y \neq 0$.

② Keldysh type: $Lu = K(x)u_{xx} + u_{yy}$ + Lower-order-terms.

where $K(0)=0$ and $xK(x)>0$ for $x \neq 0$.

Remark: Any linear elliptic-hyperbolic PDE of second order is locally of either Tricomi or Keldysh type, except countable singular points.

Remark: The two types are fundamental different.

Analytic Difference:

① Characteristics:

Keldysh: degenerate at the parabolic transition,
intersect the sonic line tagentially.

Tricomi: intersect the sonic line perpendicularly.

Boundary data, characteristic boundary ...

② Symbol:

Tricomi: of real principal type.

Keldysh: not. (lower order terms cannot be controleed by higher...)

For example: $K(x)u_{xx} + \alpha K'(x) u_x + u_{yy} = 0$

Properties for $\alpha = 1$ are much better than the one for $\alpha = \frac{1}{2}$.

③ Self-adjoint:

Tricomi: Yes

Keldysh: depends on lower order terms.
($\alpha = 1$ self-adjoint; $\alpha = \frac{1}{2}$ is not).

Typical difference (Seleted):

① Existence and regularity to Dirichlet problem:

Tricomi equation: Unique existence of H^1_{loc} weak solution by Lupo-Morawetz-Payne.

Keldysh equation: No answer to H^1_{loc} weak solution.

② Maximum principle:

Tricomi equation: done by Lupo-Payne (solutions prescribed on the elliptic part of the boundary and a characteristic).

Keldysh equation: No such result.

Remark: Tricomi equation is closer to elliptic equations.

Euler-Poisson-Darboux Equations:

$$\frac{\partial^2 u}{\partial x^2} \pm \frac{\partial^2 u}{\partial y^2} + \frac{2\beta}{y}\frac{\partial u}{\partial y} = 0 \quad , \qquad \text{where } \beta \text{ is constant.}$$

Relationship between PDE of Mixed Type and EPD Equations:

① Tricomi: $\quad u_{xx} + x\, u_{yy} = 0$

if $x > 0$: $\tau = \frac{2}{3} x^{\frac{3}{2}} \Rightarrow \quad u_{\tau\tau} + u_{yy} + \frac{\frac{1}{3}}{\tau} u_\tau = 0 \qquad$ (Elliptic EPD equation)

if $x < 0$: $\tau = \frac{2}{3} (-x)^{\frac{3}{2}} \Rightarrow \quad u_{\tau\tau} - u_{yy} + \frac{\frac{1}{3}}{\tau} u_\tau = 0 \qquad$ (Hyperbolic EPD equation)

② Keldysh: $\quad x\, u_{xx} + u_{yy} = 0$

if $x > 0$: $\tau = \frac{1}{2} x^{\frac{1}{2}} \Rightarrow \quad u_{\tau\tau} + u_{yy} - \frac{\frac{1}{4}}{\tau} u_\tau = 0 \qquad$ (Elliptic)

if $x < 0$: $\tau = \frac{1}{2} (-x)^{\frac{1}{2}} \Rightarrow \quad u_{\tau\tau} - u_{yy} - \frac{\frac{1}{4}}{\tau} u_\tau = 0 \qquad$ (Hyperbolic).

Remark: for general Tricomi: $u_{xx} + x^m u_{yy} = 0 \Rightarrow \tau = \frac{2}{2+m} (\pm x)^{\frac{2+m}{2}}$.

for general Keldysh: $x^m u_{xx} + u_{yy} = 0 \Rightarrow \tau = \frac{2}{2-m} (\pm x)^{\frac{2-m}{2}}$.

Elliptic Euler-Poisson-Darboux Equations:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{2\beta}{y}\frac{\partial u}{\partial y} = 0, \quad \text{where } \beta \text{ is constant.}$$

Basic properties:

- **Inviarant w.r.t. $\frac{1}{y}\frac{\partial}{\partial y}$:**

  - take $\partial_y \Rightarrow (u_y)_{xx} + (u_y)_{yy} + \frac{2\beta}{y}(u_y)_y - \frac{2\beta}{y^2}u_y = 0$
  - Let $v = \frac{1}{y}u_y \Rightarrow v_{xx} + v_{yy} + \frac{2(\beta+1)}{y}v_y = 0.$

  $\therefore \boxed{\frac{1}{y}\frac{\partial}{\partial y}u(x,y,\beta) = u(x,y,\beta+1)}$

- **Inviarant w.r.t. translation:**

  If ~~$u(x,y,\beta)$~~ $u(x,y)$ is a solution, then $u(x+\mu,y)$ is also a solution.

- **Inviarant w.r.t. scaling:**

  If $u(x,y)$ is a solution, then $u(\lambda x,\lambda y)$ is also a solution.

Basic properties (Continued):

- Inviarant w.r.t. $r'=\frac{1}{r}$, where $r=\sqrt{x^2+y^2}$, $r'=\sqrt{x'^2+y'^2}$.

Notice that $u_x = u_r\frac{x}{r} - u_\theta\frac{y}{r^2}$, $u_y = u_r\frac{y}{r} + u_\theta\frac{x}{r^2}$,

$$u_{xx} = u_{\theta\theta}\frac{y^2}{r^4} - 2u_{r\theta}\frac{xy}{r^3} + 2u_\theta\frac{xy}{r^4} + u_r\frac{y^2}{r^3} + u_{rr}\frac{x^2}{r^2},$$

$$u_{yy} = u_{\theta\theta}\frac{x^2}{r^4} + 2u_{r\theta}\frac{xy}{r^3} - 2u_\theta\frac{xy}{r^4} + u_r\frac{x^2}{r^3} + u_{rr}\frac{y^2}{r^2}.$$

$\therefore$ in (r,θ)-coordinates:

$$u_{rr} + \frac{1+2\beta}{r}u_r + \frac{2\beta}{r^2}\operatorname{ctg}\theta\cdot u_\theta + \frac{1}{r^2}u_{\theta\theta} = 0.$$

Let $r'=\frac{1}{r} \Rightarrow u_{r'r'} + \frac{1-2\beta}{r'}u_{r'} + \frac{2\beta}{r'^2}\operatorname{ctg}\theta\cdot u_\theta + \frac{1}{r'^2}u_{\theta\theta} = 0$.

$\therefore$ If $u(r,\theta)$ is a solution, then $r^{-2\beta}u(\frac{1}{r},\theta)$ is also a solution.

- Inviarant w.r.t. coordinates transformations:

$$x' = \frac{(ax+b)(cx+d)+acy^2}{(cx+d)^2+c^2y^2}, \quad y' = \frac{(bc-ad)y}{(cx+d)^2+c^2y^2}, \text{ where } bc-ad\neq 0.$$

All together, if $u(x,y)$ is a solution, then

$$[(cx+d)^2+c^2y^2]^{-\beta}\cdot u\left(\frac{(ax+b)(cx+d)+acy^2}{(cx+d)^2+c^2y^2}, \frac{(bc-ad)y}{(cx+d)^2+c^2y^2}\right)$$ is also a solution.

A special solution:

In (r,θ)- coordinate, $\quad u_{rr} + \frac{1+2\beta}{r} u_r + \frac{2\beta}{r^2} ctg\theta \cdot u_\theta + \frac{1}{r^2} u_{\theta\theta} = 0$

Let $u(r,\theta) = R(r) T(\theta)$, then

$$r^2 \frac{R''(r)}{R(r)} + (1+2\beta) r \frac{R'(r)}{R(r)} + 2\beta ctg\theta \cdot \frac{T'(\theta)}{T(\theta)} + \frac{T''(\theta)}{T(\theta)} = 0 .$$

$\therefore$ R(r) satisfies: $\quad r^2 R'' + (1+2\beta) R' \cdot r + \alpha^2 R = 0$, $\quad$ where α is constant.

$\Rightarrow R = r^k \quad$ with $-\alpha^2 = k(k+2\beta)$.

$\therefore$ $T(\theta)$ satisfies: $\quad T'' + 2\beta\, ctg\theta \cdot T' + k(k+2\beta) T = 0$.

Let $t = \cos^2\frac{\theta}{2}$, then $T(t)$ satisfies:

$$t(1-t) T_{tt} + [\tfrac{1}{2} + \beta - (1+2\beta) t] T_t + k(k+2\beta) T = 0 .$$

Let $c = \frac{1}{2} + \beta$, $b = k + 2\beta$, $a = -k$, then $T(t)$ satisfies

$t(1-t) T_{tt} + c(1-2t) T_t - abT = 0$, $\quad$ the well-known hypergeometric equation,

then $\quad T(t) = F(-k, k+2\beta, \frac{1}{2}+\beta, t)$

A special solution (Continued):

Notice that when $c = \frac{a+b+1}{2}$, Kummer's identity:

$$F\left(a, b, \frac{a+b+1}{2}, t\right) = F\left(\frac{a}{2}, \frac{b}{2}, \frac{a+b+1}{2}, 4t(1-t)\right).$$

Here $4t(1-t) = \frac{(r+x)(r-x)}{r^2} = \sin^2\theta = \frac{y^2}{r^2}$,

$\therefore$ $u(r,\theta)$ has a special solution:

$$r^k F\left(-\frac{k}{2}, \frac{k}{2}+\beta, \frac{1}{2}+\beta, \frac{y^2}{r^2}\right).$$

- Consider the special hypergeometric function at singular point ∞, (by transformation $z = \frac{1}{z_1}$):

$$u(r,\theta) = y^{-\beta-s} F\left(\frac{\beta+s}{2}, \frac{1-\beta+s}{2}, 1+s, \frac{r^2}{y^2}\right),$$

where $k = -s-\beta$.

Fundamental Solutions:

· Transformations:

Let $r_A' = \sqrt{(x'+R)^2+y'^2}$, $\quad r_A = \sqrt{(x+R)^2+y^2}$, $\quad r_A' = \frac{1}{r_A}$, $\quad x_1' = -x'$,

$r^2 = x^2+y^2$, $\quad r'^2 = x_1'^2 + y'^2$.

then: $y' = 2R^2 \frac{y}{r_A^2}$,

$$x_1' = -2R^2 \frac{x+R}{r_A^2} + R = \frac{R}{r_A^2}(r^2+2xR+R^2-2xR-2R^2) = \frac{r^2-R^2}{r_A^2}R.$$

$$r'^2 = x_1'^2+y'^2 = \frac{R^2}{r_A^4}[(r^2-R^2)^2+4y^2R^2] = \frac{R^2}{r_A^4}[(r^2+R^2)^2-(2xR)^2] = \frac{R^2 r_B^2}{r_A^2},$$

where $r_B^2 = r^2+R^2-2xR = (x-R)^2+y^2$.

$\therefore$ The special solution become:

$$y^{-\beta-s}(2R)^{-\beta-s} r_A^{2s} F\left(\frac{s+\beta}{2}, \frac{1+s-\beta}{2}, 1+s, \frac{r_A^2 r_B^2}{4R^2y^2}\right).$$

Let $R^2 = -y_0^2$, then

$$r_A^2 r_B^2 = (r^2-R^2)^2+4R^2y^2 = (r^2+y_0^2)^2 - 4y^2y_0^2 = r^2 r_1^2 := [x^2+(y-y_0)^2][x^2+(y+y_0)^2].$$

Let $s=0$, then:

$$y^{-\beta} y_0^{-\beta} F\left(\frac{\beta}{2}, \frac{1-\beta}{2}, 1, -\frac{r^2 r_1^2}{4y^2y_0^2}\right),$$

Fundamental Solutions (Continued):

- Notice that if $t = \frac{r_1^2}{4yy_0}$, then

$$4t(1-t) = 4\,\frac{x^2+(y+y_0)^2}{4yy_0}\cdot\left[\frac{4yy_0 - x^2 - (y+y_0)^2}{4yy_0}\right],$$

by Kummer identity:

$$y^{-\beta}y_0^{-\beta}F\left(\beta, 1-\beta, 1, \frac{r_1^2}{4yy_0}\right).$$

- For supergeometric equation, let $y = \frac{1}{y_1}$, (change singularity from ∞ to 0):

by the properties of supergeometric functions, we have.

$$r_1^{-2\beta}F\left(\beta, \beta, 2\beta, \frac{4yy_0}{r_1^2}\right),$$

$$y^{1-2\beta}y_0^{1-2\beta}r_1^{2\beta-2}F\left(1-\beta, 1-\beta, 2-2\beta, \frac{4yy_0}{r_1^2}\right).$$

- Let $x' = x - x_0$, then the fundamental solution is:

$$\begin{cases} q_1(x,y;x_0,y_0) = k_1 y^{2\beta} r_1^{-2\beta} F\left(\beta, \beta, 2\beta, \frac{4yy_0}{r_1^2}\right) \\ q_2(x,y;x_0,y_0) = k_2\, y y_0^{1-2\beta} r_1^{2\beta-2} F\left(1-\beta, 1-\beta, 2-2\beta, \frac{4yy_0}{r_1^2}\right). \end{cases}$$

where k_1, k_2 are constants, $r^2 = (x-x_0)^2 + (y-y_0)^2$, $r_1^2 = (x-x_0)^2 + (y+y_0)^2$.

Properties of the fundamental solution:

① $q_i(x,y;x_0,y_0)$ satisfies equation $u_{xx}+u_{yy}+\frac{2\beta}{y}u_y=0$ for (x_0,y_0).
while the adjoint equation $u_{xx}+u_{yy}-\frac{2\beta}{y}u_y+\frac{2\beta}{y^2}u=0$ for (x,y).

② $q_i(x,y;x_0,y_0)$ has singularity ~~also~~ when $(x,y)\to(x_0,y_0)$.

③ To be constant via integrating along some curves:

For $q_1(x,y;x_0,y_0)$: Let Γ be a differientiable curve in upper half-plane $y>0$, ended at $y=0$. Let D be the domain bounded by Γ and x-aixs, then for some constant k_1,

$$\int_\Gamma\left(q_{1y}-\frac{2\beta}{y}q_1\right)dx-q_{1x}dy=\begin{cases}1, & \text{when } (x_0,y_0)\in\bar{D}\setminus\Gamma\\ \frac{1}{2}, & \text{when } (x_0,y_0)\in\Gamma\\ 0, & \text{when } (x_0,y_0)\bar{\in}\bar{D}.\end{cases}$$

For $q_2(x,y;x_0,y_0)$: For the same Γ and D, then for some constant k_2,

$$\int_\Gamma\left(q_{2y}-\frac{2\beta}{y}q_2\right)dx-q_{2x}dy=\begin{cases}i(x,y)+1, & \text{when } (x_0,y_0)\in\bar{D}\setminus\Gamma\\ i(x,y)+\frac{1}{2}, & \text{when } (x_0,y_0)\in\Gamma\\ i(x,y), & \text{when } (x_0,y_0)\bar{\in}\bar{D}.\end{cases}$$

where $i(x,y)=-\int_a^b\left(q_{2y}-\frac{2\beta}{y}q_2\right)\Big|_{y=0}dx$, a,b are the end points of Γ.

Green Functions:

Similar to the procedure from fundamental solutions to Green functions for Laplace equation,

Let $\bar{x}_0 = \frac{x_0}{r_0^2}$, $\bar{y}_0 = \frac{y_0}{r_0^2}$, $r_0^2 = x_0^2 + y_0^2$,

then $G(x,y;x_0,y_0) = q_2(x,y;x_0,y_0) - r_0^{-2\beta} q_2(x,y;\bar{x}_0,\bar{y}_0)$

$$= k_2 y y_0^{1-2\beta} r_1^{2\beta-2} F\left(1-\beta, 1-\beta, 2-2\beta, \frac{4yy_0}{r_1^2}\right)$$

$$- k_2 y y_0^{1-2\beta} r_0^{2\beta-2} \bar{r}_1^{2\beta-2} F\left(1-\beta, 1-\beta, 2-2\beta, \frac{4y\bar{y}_0}{\bar{r}_1^2}\right).$$

or $G(x,y;x_0,y_0) = q_1(x,y;x_0,y_0) - r_0^{-2\beta} q_1(x,y;\bar{x}_0,\bar{y}_0)$

$$= k_1 y^{2\beta} r_1^{-2\beta} F\left(\beta,\beta,2\beta,\frac{4yy_0}{r_1^2}\right) - k_1 y^{2\beta} r_0^{-2\beta} \bar{r}_1^{-2\beta} F\left(\beta,\beta,2\beta,\frac{4y\bar{y}_0}{\bar{r}_1^2}\right).$$

where $k_2 = \frac{2^{-2\beta}\Gamma^{2}(1-\beta)}{\pi\Gamma(2-2\beta)}$, $\bar{r}_1^2 = (x-\bar{x}_0)^2 + (y+\bar{y}_0)^2$.

Singular boundary problems in semi-disc:

$$\begin{cases} u_{xx} + u_{yy} + \frac{2\beta}{y} u_y = 0, & \beta < \frac{1}{2}, \quad \text{when } x^2+y^2<1 \text{ and } y>0. \\ u = \varphi & \text{on } x^2+y^2=1 \text{ and } y>0. \\ u = t(x) & \text{on } y=0. \end{cases}$$

Green formula:

for any smooth function v in domain D,

$$\iint_D \left[v \left(u_{xx}+u_{yy}+\frac{2\beta}{y}u_y\right) - u\left(v_{xx}+v_{yy}-\frac{2\beta}{y}v_y+\frac{2\beta}{y^2}v\right)\right] dxdy$$

$$= \int_{\partial D} u\left[\left(v_y-\frac{2\beta}{y}v\right)dx - v_x dy\right] - v\left(u_y dx - u_x dy\right)\Big].$$

Let $D = \{(x,y) \mid x^2+y^2 \le 1, y \ge 0\} \setminus C_\varepsilon$, $\sigma = \{(x,y) \mid x^2+y^2=1, y\ge 0\}$. $[-1,1] := \{(x,0) \mid -1 \le x \le 1\}$.

~~then~~ $\because$ $Q_i(x,y;x_0,y_0)$ satisfy the adjoint equation for (x,y).

then $0 = \int_{\sigma+[-1,1]-C_\varepsilon} u\left[\left(v_y-\frac{2\beta}{y}v\right)dx - v_x dy\right] - v\left(u_y dx - u_x dy\right)$

where $v = G$, u is the solution.

Solutions to singular boundary problems (Continued):

$$\begin{cases} u_{xx}+u_{yy}+\frac{2\beta}{y}u_y=0 & \beta<\frac{1}{2} \quad \text{in } D. \\ u=\varphi & \text{on } \sigma \\ u=t(x) & \text{on } [-1,1]. \end{cases}$$

Let $\varepsilon \to 0$, then:

$$u(x_0,y_0)=\int_{-1}^{1} t(x)\left(G_y-\frac{2\beta}{y}G\right)\Big|_{y=0}dx+\int_\sigma \varphi\,(G_y dx-G_x dy)$$

Detailed formula:

$$\because\ q_{2y}=k_2 y_0^{1-2\beta}\beta r_1^{2\beta-2}F\left(1-\beta,1-\beta,2-2\beta,\frac{4yy_0}{r_1^2}\right)$$
$$+k_2 y y_0^{1-2\beta}\frac{\partial}{\partial y}\left[r_1^{2\beta-2}F\left(1-\beta,1-\beta,2-2\beta,\frac{4yy_0}{r_1^2}\right)\right]$$

$$\therefore\ \left(q_{2y}-\frac{2\beta}{y}q_2\right)_{y=0}=k_2(1-2\beta)\,y_0^{1-2\beta}\left[(x-x_0)^2+y_0^2\right]^{\beta-1}.$$

$$\therefore\ \int_{-1}^{1} t(x)\left(G_y-\frac{2\beta}{y}G\right)\Big|_{y=0}dx$$

$$=k_2(1-2\beta)\left\{\int_{-1}^{1}\frac{t(x)\,dx}{[(x-x_0)^2+y_0^2]^{1-\beta}}-\frac{1}{(x_0^2+y_0^2)^{1-\beta}}\int_{-1}^{1}\frac{t(x)\,dx}{[(x-\bar{x}_0)^2+\bar{y}_0^2]^{1-\beta}}\right\}.$$

Detailed formula (Continued):

$$\because \frac{\partial}{\partial y}\left[r_1^{2\beta-2} F\left(1-\beta,\ 1-\beta,\ 2-2\beta,\ \frac{4yy_0}{r_1^2}\right)\right]$$

$$= (2\beta-2) r_1^{2\beta-4} (y+y_0) F\left(1-\beta,\ 1-\beta,\ 2-2\beta,\ \frac{4yy_0}{r_1^2}\right)$$

$$+ \frac{1-\beta}{2} r_1^{2\beta-2} F\left(2-\beta,\ 2-\beta,\ 3-2\beta,\ \frac{4yy_0}{r_1^2}\right) \cdot \frac{4y_0}{r_1^2}\left[1 - \frac{2y(y+y_0)}{r_1^2}\right].$$

$$\therefore G_y\big|_\sigma = k_2 y y_0^{1-2\beta}(2\beta-2)(1-r_0^2) r_1^{2\beta-4} y \cdot F\left(1-\beta,\ 1-\beta,\ 2-2\beta,\ \frac{4yy_0}{r_1^2}\right)$$

$$+ k_2 y y_0^{1-2\beta}(2\beta-2)(1-r_0^2) r_1^{2\beta-4} y \frac{2yy_0}{r_1^2} F\left(2-\beta,\ 2-\beta,\ 3-2\beta,\ \frac{4yy_0}{r_1^2}\right).$$

By identity, $cF(a,b,c,z) - cF(a+1,b,c,z) + bzF(a+1,b+1,c+1,z) = 0$,

We have $G_y\big|_\sigma = -k_2 y^2 y_0^{1-2\beta}(1-r_0^2) r_1^{2\beta-4}(2-2\beta) F\left(2-\beta, 1-\beta, 2-2\beta, \frac{4yy_0}{r_1^2}\right)$.

Similarly, $G_x\big|_\sigma = k_2 y_0^{1-2\beta}(1-r_0^2) r_1^{2\beta-4}(2\beta-2) xy F\left(2-\beta, 1-\beta, 2-2\beta, \frac{4yy_0}{r_1^2}\right)$.

$$\therefore \int_\sigma \varphi (G_y dx - G_x dy) = k_2 (2-2\beta) y_0^{1-2\beta}(1-r_0^2)\int_0^\pi \varphi(\theta) \frac{F\left(2-\beta, 1-\beta, 2-2\beta, \frac{4yy_0}{r_1^2}\right)}{[(x-x_0)^2+(y+y_0)^2]^{2-\beta}} \sin\theta\, d\theta.$$

Detailed formula:

$$u(x_0, y_0) = k_2(1-2\beta)\, y_0^{1-2\beta} \left\{ \int_{-1}^{1} \frac{t(x)\,dx}{[(x-x_0)^2+y_0^2]^{1-\beta}} - \frac{1}{(x_0^2+y_0^2)^{1-\beta}} \int_{-1}^{1} \frac{t(x)\,dx}{[(1-\bar{x}_0)^2+\bar{y}_0^{\,2}]^{1-\beta}} \right\}$$

$$+ k_2(2-2\beta)\, y_0^{1-2\beta} (1-r_0^2) \int_0^{\pi} \varphi(\theta)\, \frac{F\left(2-\beta, 1-\beta, 2-2\beta, \frac{4yy_0}{r_1^2}\right)}{[(x-x_0)^2+(y+y_0)^2]^{2-\beta}} \sin\theta \, d\theta .$$

Ex: to verify it is a solution.

Another Singular boundary problem:

$$\begin{cases} u_{xx} + u_{yy} + \frac{2\beta}{y} u_y = 0, & \beta > 0 \quad \text{in } D. \\ u|_\sigma = \varphi & \text{on } \sigma. \\ \lim\limits_{y \to 0} y^{2\beta} u_y = t(x), & \end{cases}$$

Solution: Similarly,

$$u(x_0, y_0) = -k_1 \int_{-1}^{1} \frac{t(x)\,dx}{[(x-x_0)^2+y_0^2]^{\beta}} + \frac{k_1}{(x_0^2+y_0^2)^{\beta}} \int_{-1}^{1} \frac{t(x)\,dx}{[(x-\bar{x}_0)^2+\bar{y}_0^{\,2}]^{\beta}}$$

$$+ 2k_1\beta(1-r_0^2) \int_0^{\pi} \varphi(\theta)\, \frac{F\left(1+\beta, \beta, 2\beta, \frac{4yy_0}{r_1^2}\right)}{[(x-x_0)^2+(y+y_0)^2]^{\beta+1}} \sin^{2\beta}\theta \, d\theta .$$

Another singular boundary problem (Continued):

$$\begin{cases} u_{xx} + u_{yy} + \frac{2\beta}{y} u_y = 0, & \beta < \frac{1}{2}, \quad y > 0. \\ u|_{y=0} = t(x), & -\infty < x < +\infty. \end{cases}$$

Solution: $u(x_0, y_0) = \int_{-\infty}^{+\infty} t(x) \left(Q_{2y} - \frac{2\beta}{y} Q_2 \right)\Big|_{y=0} dx$

$$= k_2 (1-2\beta) y_0^{1-2\beta} \int_{-\infty}^{+\infty} \frac{t(x)\,dx}{[(x-x_0)^2 + y_0^2]^{1-\beta}}.$$

Problem.
$$\begin{cases} u_{xx} + u_{yy} + \frac{2\beta}{y} u_y = 0, & \beta > 0, \quad y > 0. \\ \lim_{y \to 0} y^{2\beta} u_y = t(x), & -\infty < x < +\infty. \end{cases}$$

Solution: $u(x_0, y_0) = -k_1 \int_{-\infty}^{+\infty} \frac{t(x)\,dx}{[(x-x_0)^2 + y_0^2]^{\beta}}$.

Ex: to verify ~~them~~ them.