Nonlinear Degenerate Elliptic Equations.

Difference between nonlinear and linear:

Linear: $2x u_{xx} + u_{yy} - u_x = 0$

if $u = Ax^\alpha \Rightarrow \alpha = \frac{3}{2}$ and $u = x^{\frac{3}{2}}$.

Nonlinear: $(2x - (\gamma+1) u_x) u_{xx} + u_{yy} - u_x = 0$

if $u = Ax^\alpha \Rightarrow \alpha = 2$ and $u = \frac{x^2}{2(\gamma+1)}$.

Sometimes "Nonlinearity" gives more regularity.

Prototype: $(2x - (\gamma+1) u_x) u_{xx} + \frac{1}{c_2^2} u_{yy} - u_x = 0$

in $x > 0$ and near $x = 0$.

(Main terms of the potential flow equation).

Linearized equations: $2x u_{xx} + \frac{1}{c_2^2} u_{yy} - u_x = 0$ (Keldysh equation).

Problem: $N(u)=\sum_{i,j=1}^{2} A_{ij}(Du,x)\,u_{x_ix_j}+\sum_{i=1}^{2} A_i(Du,x)\,u_{x_i}=f$ in Q

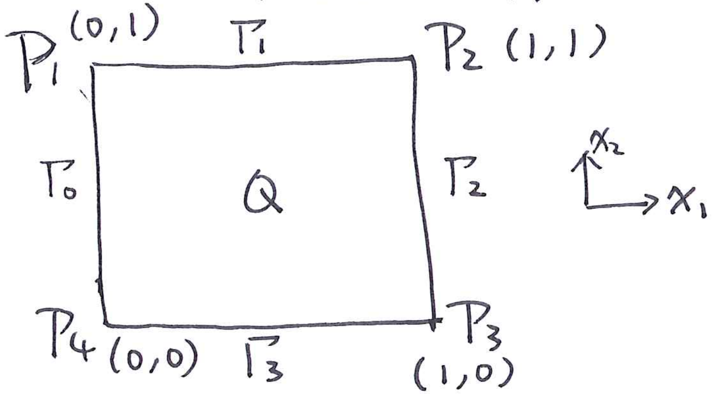
$B(Du,u,x)=0$ on Γ_1

$u=0$ on Γ_0

$b_1^{(k)}(x)u_{x_1}+b_2^{(k)}(x)u_{x_2}=g_k$ on Γ_k.

where $Q=(0,1)\times(0,1)$, $\partial Q=\Gamma_0\cup\Gamma_1\cup\Gamma_2\cup\Gamma_3$,

Γ_i ends at P_i and P_{i+1}, $\Gamma_0=P_4P_1$,



$P_1=(0,1)$, $P_2=(1,1)$

$P_3=(1,0)$, $P_4(0,0)$

Only near Γ_0 (degenerate line) be considered.

Assumptions (On equations) :

① Nonisotropic ellipticity: $\lambda|\mu|^2 \le \sum_{i,j=1}^{2} A_{ij}(P,x)\frac{\mu_i\mu_j}{x_1^{2-\frac{i+j}{2}}} \le \lambda^{-1}|\mu|^2$

for all $x\in Q$, $P,\mu\in\mathbb{R}^2$.

② Functions $(A_{ij}, A_i)(P,x)$ are independent of P on $Q\cap\{x_1\ge\varepsilon\}$, and $C^{1,\alpha}$ on $\{x_1\ge\varepsilon\}\cap Q$.

③ For any $P\in\mathbb{R}^2$,

$$\|(A_{ij},A_i)(P,\cdot)\|_{C^\alpha(\overline{Q\cap\{x_1\le\varepsilon\}})}+\|(D_PA_{ij}, D_PA_i)(P,\cdot)\|_{L^\infty(Q\cap\{x_1<2\varepsilon\})}\le M.$$

④ A_{11}, A_{22} and A_1 are independent of P_2 on $x_1<2\varepsilon$, and

$$|A_{ii}(P,(x_1,x_2))-A_{ii}(0,(0,x_2))|\le M|x_1|^\alpha.$$

⑤ A_{12}, A_{21} and A_2 are independent of P on $x_1<2\varepsilon$, and

$$\|A_{12},A_{21},A_2\|_{C^\alpha(\overline{Q\cap\{x_1<s\}})}\le Ms^{\frac{1}{2}-\alpha}, \text{ for all } s\in(0,2\varepsilon),\ \alpha\in(0,\tfrac{1}{2}).$$

⑥ $A_1(P,x)\le-\lambda$ for all $P\in\mathbb{R}^2$, $x\in Q\cap\{x_1<\varepsilon\}$.

Remark: equation $(2x-(\gamma+1)u_x)u_{xx}+u_{yy}-u_x=0$ in $x>0$ near $x=0$ satisfies the assumptions above.

<u>Assumptions on boundary conditions</u>:

① <u>Uniform obliqueness</u>: $D_{p_2} B(p, z, x_1) \le -\lambda$ for all $(p, z, x_1) \in \mathbb{R}^2 \times \mathbb{R} \times [0,1]$ on Γ_1

② <u>Regularity of coefficients</u>: $\| B(0,0,\cdot), D^k_{p,z} B(p,z,\cdot)\|_{C^{1,\alpha}} \le M$, for all $(p,z) \in \mathbb{R}^2 \times \mathbb{R}$,

$k = 1, 2, 3$, $\quad D_z B(p, z, x_1) \le -\lambda$ for all $(p, z, x_1) \in \mathbb{R}^2 \times \mathbb{R} \times [0,1]$.

$D_{p_1} B(p_1, p_2, z, x_1) \le -\lambda$ for all $x_1 \in (0, \varepsilon)$, $(p, z) \in \mathbb{R}^2 \times \mathbb{R}$.

③ <u>Almost linear structure</u>: there exists

$$L(p, z, x_1) = b_1^{(1)}(x_1) p_1 + b_2^{(1)}(x_1) p_2 + b_0^{(1)}(x_1) z + g_1(x_1).$$

defined for $s \in (0,1)$, $(p, z) \in \mathbb{R}^2 \times \mathbb{R}$, with $b_i^{(1)}, g_1 \in C^{-\alpha, \{1\}}_{1,\alpha,(0,1)}$, and there exists a function v with $\|v\|^{-1-\alpha, \{P_1\} \cup \overline{\Gamma_2}}_{2,\alpha,Q} \le M$, such that

$$|B(p, z, x_1) - L(p, z, x_1)| \le \sigma(|p - Dv(x_1, 1)| + |z - v(x_1, 1)|),$$

$$|D_{p_k} B(p, z, x_1) - b_k(x_1)| \le \sigma \quad \text{for all } (p, z, x_1) \in \mathbb{R}^2 \times \mathbb{R} \times (0,1).$$

④ $b_i^{(k)}$: <u>Obliqueness</u>: $(b_1^{(k)}, b_2^{(k)}) \cdot \nu^{(k)} \ge \lambda$ on Γ_k, for $k = 1, 2, 3$.

$b_1^{(3)} \le 0$ on Γ_3, $\quad b_1^{(3)} = 0$ on $\Gamma_3 \cap \{0 < x_1 < \varepsilon\}$.

$$\left| \frac{(b_1^{(k)}, b_2^{(k)})}{|(b_1^{(k)}, b_2^{(k)})|}(P_k) - \frac{(b_1^{(k-1)}, b_2^{(k-1)})}{|(b_1^{(k-1)}, b_2^{(k-1)})|}(P_k) \right| \ge \lambda \quad \text{for } k = 2, 3.$$

$\|b_i^{(2)}\|_{C^\alpha(\overline{\Gamma_2})} \le M$, and $\|b_i^{(k)}\|^{-\alpha, \partial\Gamma_k}_{C_{1,\alpha,\Gamma_k}} \le M$ for $k = 1, 2$.

Assumptions on the right-hand sides:

$$\|f\|_{1,\alpha,Q}^{-\alpha,\Gamma_2} + \|g_1\|_{1,\alpha,(0,1)}^{-\alpha,\{1\}} + \|g_2\|_{C^\alpha([0,1])} + \|g_3\|_{1,\alpha,(0,1)}^{-\alpha,\{1\}} \leq M.$$

$f \equiv 0$ in $Q \cap \{x_1 < \varepsilon\}$, $B(0,0,\cdot) \equiv 0$ on $\Gamma_1 \cap \{x_1 < \varepsilon\}$, $g_3 \equiv 0$ on $\Gamma_3 \cap \{x_1 < \varepsilon\}$.

Existence: Proposition: there exists a unique solution

$u \in C(\bar{Q}) \cap C^{1,\alpha}(\bar{Q} \setminus \Gamma_0) \cap C^{2,\alpha}(Q)$, satisfying that for all $s \in (0, \frac{1}{10})$,

$\|u\|_{2,\alpha,Q_s}^{-1-\alpha_0,\bar{\Gamma}_2} \leq C_s$, where $Q_s = Q \cap \{x_1 > s\}$ and $|u(x_1,x_2)| \leq C x_1$ in Q.

Proof: by vanishing viscosity method.

Define $N_\delta(Du, x) = N(Du, x) + \lambda \delta u_{x_1 x_1}$.

then for N_δ, Nonisotropic ellipticity:

$$\lambda |\mu|^2 \leq \sum_{i,j=1}^{2} A_{ij}(p,x) \frac{\mu_i \mu_j}{(\max(x_1,\delta))^{2-\frac{i+j}{2}}} \leq \lambda^{-1} |\mu|^2$$

for all $x \in Q$, $p, \mu \in R^2$.

For Ns: Lemma: Let $\lambda>0$, $M<\infty$, $\delta\in(0,\frac{1}{2})$, $\beta\in(0,1)$. Let $f\in C(\bar{Q}\setminus\bar{\Gamma}_2)$,

$g_k\in C(\bar{\Gamma}_k)$ with

$$\sup_{x\in Q}\left((1-x_1)^\beta|f(x)|\right)+\|g_1\|_{L^\infty(0,1)}+\|g_2\|_{L^\infty(0,1)}+\|g_3\|_{L^\infty(0,1)}\le M_1,$$

Let $u\in C^2(Q)\cap C^1(\bar{Q}\setminus\Gamma_0)\cap C(\bar{Q})$ be a solution, then there exists C depending only on $\lambda, M, M_1, \varepsilon, \beta$ (independent on δ). such that.

$$\|u\|_{C(\bar{Q})}\le C, \qquad |u(x_1,x_2)|\le Cx_1 \quad \text{in } Q.$$

Proof: Step 1 (To prove $\|u\|_{C(\bar{Q})}\le C$):

Consider linear equation and boundary condition:

$$\begin{cases} \mathcal{L}u=\sum_{i,j=1}^{2}a_{ij}(x)u_{x_ix_j}+\sum_{i=1}^{2}a_i(x)u_{x_i}=f & \text{in } Q.\\ \hat{B}^{(1)}u:=\hat{b}_1^{(1)}(x)u_{x_1}+\hat{b}_2^{(2)}(x)u_{x_2}+\hat{b}_0^{(1)}(x)u=\hat{g}_1 & \text{on } \Gamma_1 \end{cases}$$

where $a_{ij}(x)=A_{ij}(Du(x),x)$, $\quad a_i(x)=A_i(Du(x),x)$ for $i,j=1,2$.

$\hat{b}_i^{(1)}(x)=\int_0^1 D_{p_i}B(tDu(x),tu(x),x)\,dt$ for $i=0,1,2$, where $p_0=z$,

$\hat{g}_1(x)=B(0,0,x)$.

then from assumptions: a_{ij} satisfy ellipticity and bounds

$$\|a_{ij},a_i\|_{L^\infty(Q)}\le\lambda^{-1}.$$

Also from assumptions, $\hat{b}_i^{(1)}$ and $\hat{g}_1$ satisfy

$$\hat{b}_2^{(1)} \le -\lambda \quad \text{on } \Gamma_1 \ (\text{Obliqueness}).$$

and $\quad \|\hat{b}_i^{(1)}, \hat{g}_1\|_{L^\infty(\Gamma_1)} \le M \quad$ for $i = 0, 1, 2$.

$$\hat{b}_0^{(1)} \le -\lambda \qquad \text{on } \Gamma_1$$

Define comparison function

$$V(x) = W(x_1) - \frac{M_1}{\lambda\varepsilon(2-\beta)(1-\beta)}(1-x_1)^{2-\beta} - Nx_2 .$$

where $N>0$ and $w(x_1)$ satisfying

$$w' > 0 \quad , \quad w'' < 0 \qquad \text{on } [0,1].$$

will be determined later.

<u>First on Γ_3:</u> $\because$ $b_2^{(3)} \ge \lambda$ and $b_1^{(3)} \le 0$.

$$\therefore B^{(3)}v := b_1^{(3)} V_{x_1} + b_2^{(3)} V_{x_2} = b_1^{(3)}\left(w'(x_1) + \frac{M_1}{\lambda\varepsilon(1-\beta)}(1-x_1)^{1-\beta}\right) - b_2^{(3)} N \le -\lambda N.$$

$\therefore$ choose $N = -\frac{\|g_3\|_\infty}{\lambda}$, then $\quad B^{(3)} v \le g_3 \quad$ on Γ_3.

<u>Equation</u>:

$$\mathcal{L}v - f = a_{11}\left(w'' - M_1(\lambda\varepsilon)^{-1}(1-x_1)^{-\beta}\right) + a_1\left(w'(x_1) + \frac{M_1}{\lambda\varepsilon(1-\beta)}(1-x_1)^{1-\beta}\right) + a_2 N - f$$

$$\le x_1\lambda w'' + a_1 w' + \lambda^{-1}\left(N + \frac{M_1}{\lambda(1-\beta)}\right) + \left(-M_1\frac{x_1}{\varepsilon}(1-x_1)^{-\beta} - f\right)$$

$(\because a_{11} \ge \lambda x_1 \ , \ w' > 0, \ w'' < 0)$

Equation (Continued):

in $Q\cap\{x_1\geq\varepsilon\}$: $\quad -M_1\frac{x_1}{\varepsilon}(1-x_1)^{-\beta}-f\leq -M_1(1-x_1)^{-\beta}-f\leq 0.$

$\therefore\ \mathcal{L}v\leq f$ in $Q\cap\{x_1\geq\varepsilon\}$, if $\quad \varepsilon\lambda w''+\lambda^{-1}w'+\hat{N}=0$,

where $\hat{N}:=\lambda^{-1}\left(N+\frac{M_1}{\lambda(1-\beta)}\right)$.

$\therefore\ w=-C_1e^{-\frac{x_1}{\varepsilon\lambda^2}}+C_2-\lambda\hat{N}x_1$, where C_1, C_2 are arbitrary constants.

if choose $\quad C_1\geq\varepsilon\lambda^3M\hat{N}e^{\frac{1}{\varepsilon\lambda^2}}$, then

$$w'(x_1)=\frac{C_1}{\varepsilon\lambda^2}e^{-\frac{x_1}{\varepsilon\lambda^2}}-\lambda\hat{N}\geq\frac{C_1}{\varepsilon\lambda^2}e^{-\frac{1}{\varepsilon\lambda^2}}-\lambda\hat{N}>0.$$

and $\quad w''(x_1)=-\frac{C_1}{\varepsilon^2\lambda^4}e^{-\frac{x_1}{\varepsilon\lambda^2}}<0\quad$ on R. ✓

in $Q\cap\{x_1<\varepsilon\}$: $\quad\because\ a_1\leq-\lambda,\quad \|f\|_{L^\infty(Q\cap\{x_1<\varepsilon\})}\leq 2M_1$,

$$\therefore\ \mathcal{L}v-f\leq a_1w'+\hat{N}-f\leq-\lambda w'+\hat{N}+\|f\|_{L^\infty(Q\cap\{x_1<\varepsilon\})}$$

$$\leq-\lambda\left(\frac{C_1}{\varepsilon\lambda^2}e^{-\frac{1}{\lambda^2}}-\lambda\hat{N}\right)+\lambda^{-1}N+2M_1<0.$$

if $C_1>0$ is sufficiently large.

$\Rightarrow\quad \mathcal{L}v-f\leq 0\quad$ in Q.

<u>on Γ_2</u>: $\because\ b_1^{(2)} \le -\lambda$ on $[0,1]$.

$\therefore\ B^{(2)}v = b_1^{(2)} v_{x_1} + b_2^{(2)} v_{x_2} = b_1^{(2)}\left(w'(x_1) + \frac{M_1}{\lambda\varepsilon(1-\beta)}(1-x_1)^{1-\beta}\right) - b_2^{(3)} N$

$\le -\lambda w'(1) + MN$

If $\frac{C_1}{\varepsilon\lambda^2} e^{-\frac{1}{\varepsilon\lambda^2}} - \lambda \hat{N} \ge \lambda^{-1}(MN + \|g_2\|_\infty)$.

then $w'(1) \ge \lambda^{-1}(MN + \|g_2\|_{L^\infty})$.

$\Rightarrow\ B^{(2)} v \le g_2$ on Γ_2

<u>on Γ_0</u>: If $C_2 \ge C_1 + \lambda\hat{N} + \frac{M_1}{\lambda\varepsilon(2-\beta)(1-\beta)} + N$,

then $v \ge 0$ in $Q = [0,1]\times[0,1]$.

<u>on Γ_1</u>: $\because\ \hat{b}_0^{(1)} \le -\lambda$ and $|\hat{b}_i^{(1)}| \le M$ on Γ_1,

$\therefore\ \hat{B}^{(1)} v = \hat{b}_1^{(1)}(x)\left(w'(x_1) + \frac{M_1}{\lambda\varepsilon(1-\beta)}(1-x_1)^{1-\beta}\right) - \hat{b}_2^{(1)} N$
$+ \hat{b}_0^{(1)}(x)\left(w(x_1) - \frac{M_1}{\lambda\varepsilon(1-\beta)(2-\beta)}(1-x_1)^{1-\beta} + N\right)$
$\le C(\lambda, \varepsilon, M, M_1, \beta) - \lambda C_2$.

$\therefore$ Choose C_2 large enough, then $\hat{B}^{(1)} v \le g_1$.

<u>Maximum Principle</u>: $\Rightarrow$

$$|u| \le v \le C_2 \quad \text{in } Q.$$

Step 2. (To prove $|u(x_1, x_2)| \le C x_1$ in Q):

Only need to prove in $Q \cap \{x_1 < \varepsilon\}$.

$\because$ $a_1 \le -\lambda$ in $Q \cap \{x_1 < \varepsilon\}$, $\hat{b}_1^{(1)}(x) \le -\lambda$ on $\Gamma_1 \cap \{x_1 < \varepsilon\}$.

Define $v = C_3 x_1$,

then $\mathcal{L}v = a_1 C_3 \le -2M_1 \le -\|f\|_{L^\infty(Q \cap \{0 < x_1 < \varepsilon\})}$, for sufficiently large $C_3 > 0$.

$\hat{B}^{(1)} v = \hat{b}_1^{(1)} C_3 + \hat{b}_0^{(1)} C_3 x_1 \le C_3(-\lambda + \hat{\varepsilon}\lambda^{-1}) \le -\frac{1}{2}\lambda C_3 \le -M_1 \le -\|\hat{g}_1\|_{L^\infty}$,

if $\hat{\varepsilon}$ small enough.

$\hat{b}_1^{(3)}(x) V_{x_1} + \hat{b}_2^{(3)}(x) V_{x_2} = b_1^{(3)} C_3 \le -M_1 \le -\|g_3\|_{L^\infty}$.

Maximum Principle $\Rightarrow$

$$|u| \le v = C_3 x_1 \quad \text{in } Q \cap \{x_1 < \varepsilon\}.$$

#.

For N_δ (Continued):

Lemma ($C^{2,\alpha}$-regularity):

There exists $\sigma, \delta_0 > 0$ and $\alpha_1 \in (0, \frac{1}{2})$, depending only on λ, M such that, let $\delta \in (0, \delta_0)$, $\alpha \in (0, \alpha_1)$, and let $u \in C^{-1-\alpha, \{P_1\} \cup \overline{\Gamma_2}}_{2,\alpha,Q}$ be a solution, then

$$\|u\|^{-1-\alpha, \{P_1\} \cup \overline{\Gamma_2}}_{2,\alpha,Q} \leq \hat{C}_\delta ,$$

where $\hat{C}_\delta$ depends only on λ, M, α, ε, σ, δ.

Moreover, for $s \in (0, \frac{1}{10})$, denote $Q_s := Q \cap \{x_1 > s\}$. Then for each $s \in (0, \frac{1}{10})$, there exists C_s depending only on λ, M, α, ε, σ, s, but independent of δ, such that

$$\|u\|^{-1-\alpha, \overline{\Gamma_2}}_{2,\alpha,Q_s} \leq C_s .$$

For Ns (Continued):

Lemma (Existence): There exists a unique solution

$$u \in C^2(Q) \cap C^1(\bar{Q} \setminus \Gamma_0) \cap C(\bar{Q}),$$

which satisfies all the estimates listed in the previous lemma.

Proof: ~~...~~ By the method of continuity.

For $t \in [0,1]$, define operator P_t for $u \in C^1(\bar{Q}) \cap C^2(Q)$ by.

$$
\begin{aligned}
P_t(u) = \Big(& \sum_{i,j=1}^{2} A_{ij}(Du, x) u_{x_i x_j} + \sum_{i=1}^{2} A_i(Du, x) u_{x_i} - tf, \\
& \big(B(Du, u, x) - (1-t) B(0,0,x)\big)\Big|_{\Gamma_1}, \\
& \big(b_1^{(2)}(x) u_{x_1} + b_2^{(2)}(x) u_{x_2} - t g_2\big)\Big|_{\Gamma_2}, \\
& \big(b_1^{(3)}(x) u_{x_1} + b_2^{(3)}(x) u_{x_2} - t g_3\big)\Big|_{\Gamma_3} \Big).
\end{aligned}
$$

Define space:

$$C_D = \{ u \in C_{2,\alpha,Q}^{-1-\alpha, \{P_1\} \cup \bar{\Gamma}_2} \mid u_\nu = 0 \text{ on } \Gamma_3 \cap \{0 < x_1 < \varepsilon\},\ u = 0 \text{ on } \Gamma_0 \}.$$

$$C_P = \{ (f, g_1, g_2, g_3) \in C_{0,\alpha,\Omega}^{1-\alpha, \{P_1\} \cup \bar{\Gamma}_2} \times C_{1,\alpha,\Gamma_1}^{-\alpha, \partial\Gamma_1} \times C^\alpha(\bar{\Gamma}_2) \times C_{1,\alpha,\Gamma_3}^{-\alpha, \partial\Gamma_3} \mid g_3 = 0 \text{ on } \Gamma_3 \cap \{0 < x_1 < \varepsilon\} \}.$$

For N_S (Continued):

Proof of existence (Continued):

Sets C_D and C_T are Banach space.

Mapping $(t, u) \to P_t(u)$ is a Frechet-differentiable mapping from $[0,1] \times C_D$ to C_T.

Define $T = \{ t \in [0,1] \mid P_t(u_t) = 0 \text{ for some } u_t \in C_D \}$.

Step 1: $0 \in T$. Obviously.

Step 2: T is open.

Let $t_0 \in T$, then there exists a $u_{t_0} \in C_D$ with $P_{t_0}(u_{t_0}) = 0$.

From the unique existence of linearized equation at u_{t_0},

we have $D_u P_{t_0}(u_{t_0}) : C_D \to C_T$ is an isomorphism.

$\therefore$ By implicit function theorem, T is open.

Step 3: T is closed.

By uniform estimates.

$\therefore T = [0,1]$.

$\therefore 1 \in T$ #.

Pass limit $\delta \to 0$ $\Rightarrow$ Existence #.

Regularity:

① Hölder space with parabolic scaling:

Let $\sigma \in \mathbb{R}$ and $\sigma > 0$. Let m be a nonnegative integer. Let $\alpha \in (0,1)$ and denote $\delta_\alpha^{(par)}(z,\tilde{z}) = (|x-\tilde{x}|^2 + \min(x,\tilde{x})\,|y-\tilde{y}|^2)^{\frac{\alpha}{2}}$, where $z=(x,y)$, $\tilde{z}=(\tilde{x},\tilde{y})$.

Define: $\|u\|^{\sigma,(par)}_{m,0,D} := \sum_{0\le l+k\le m} \sup_{z\in D} \left(x^{k+\frac{l}{2}-\sigma} |\partial_x^k \partial_y^l u(z)| \right)$.

$$[u]^{\sigma,(par)}_{m,\alpha,D} := \sum_{l+k=m} \sup_{\substack{z,\tilde{z}\in D,\\ z\neq \tilde{z}}} \left(\min(x,\tilde{x})^{\alpha+k+\frac{l}{2}-\sigma} \frac{|\partial_x^k\partial_y^l u(z) - \partial_x^k \partial_y^l u(\tilde{z})|}{\delta_\alpha^{(par)}(z,\tilde{z})} \right).$$

$$\|u\|^{\sigma,(par)}_{m,\alpha,D} := \|u\|^{\sigma,(par)}_{m,0,D} + [u]^{\sigma,(par)}_{m,\alpha,D}$$

Denote: $C^{m,\alpha}_{\sigma,(par)}(D)$ the completion of the set $\{u \in C^\infty(D) \,|\, \|u\|^{\sigma,(par)}_{m,\alpha,D} < +\infty\}$.

Rescale $u(x,y)$ in parabolic rectangles: $R(x,y) = \{(s,t) : |s-x| < \frac{x}{4},\ |t-y| < \frac{\sqrt{x}}{4}\}$ for $z=(x,y)$.

in $Q_1^{(z)} := \{(S,T)\in Q_1 : (x+\frac{x}{4}S,\ y+\frac{\sqrt{x}}{4}T) \in D_\varepsilon\}$ where $Q_1 = (-1,1)\times(-1,1)$,

define $u^{(z)}(S,T) := \frac{1}{x^\sigma} u(x+\frac{x}{4}S,\ y+\frac{\sqrt{x}}{4}T)$ for $(S,T)\in Q_1^{(z)}$.

Obviously, $C^{-1} \sup_{z\in D_{\frac{3}{4}\varepsilon}} \|u^{(z)}\|_{C^{m,\alpha}(\overline{Q_1^{(z)}})} \le \|u\|^{\sigma,(par)}_{m,\alpha,D_\varepsilon} \le C \sup_{z\in D_\varepsilon} \|u^{(z)}\|_{C^{m,\alpha}(\overline{Q_1^{(z)}})}$.

Regularity (Continued):

Property for $C^{m,\alpha}_{\sigma,(par)}(D)$: Let D be an open bounded subset of R^3,

Let m_1, m_2 be nonnegative integers. $\alpha_1, \alpha_2 \in (0,1)$ and $m_1+\alpha_1 > m_2+\alpha_2$.

Let $\sigma_1, \sigma_2 > 0$. Then $C^{m_1,\alpha_1}_{\sigma_1,(par)}(D)$ is compactly imbedded in $C^{m_2,\alpha_2}_{\sigma_2,(par)}(D)$.

② **Regularity theorem:**

Let domain $\Omega = \{(x,y) \mid x>0, y\in(0,f(x))\}$, where $f \in C^{1,\alpha}(\overline{R^1_+})$ and $f>0$ on R_+°.

Let $\Gamma_0 = \partial\Omega \cap \{x=0\}$, $\Gamma_n = \partial\Omega \cap \{y=0\}$, $\Gamma_f = \partial\Omega \cap \{x>0, y=f(x)\}$.

For $r>0$, denote $\Omega_r = \Omega \cap \{x<r\}$, $\Gamma_{n,r} = \Gamma_n \cap \{x<r\}$, $\Gamma_{f,r} = \Gamma_f \cap \{x>0, y=f(x)\}$.

Consider equations: $A_{11}u_{xx} + 2A_{12}u_{xy} + A_{22}u_{yy} + A_1 u_x + A_2 u_y = 0$ in Ω_r.

where $A_{ij} = A_{ij}(Du, u, x, y)$, $A_i = A_i(Du, u, x, y)$ satisfying

$$\frac{1}{\lambda}|K|^2 \geq \sum_{i,j=1}^{2} A_{ij}(P,z,x,y)\frac{K_iK_j}{x^{2-\frac{i+j}{2}}} \geq \lambda|K|^2,$$

for all $P\in R^2$, $z\in R$, $(x,y)\in\Omega_r$, $K=(K_1,K_2)\in R^2$.

Boundary conditions: $B(u_x, u_y, u, x, y) = 0$ on $\Gamma_{f,r}$.

$u_y = 0$ on $\Gamma_{n,r}$, $u=0$ on Γ_0.

Rescale: for every $(x_0, y_0)\in\Omega_r$.

$$A^{(x_0,y_0)}_{ij}(P_1,P_2,z,S,T) = x_0^{\frac{i+j}{2}-2} A_{ij}(4x_0P_1, 4x_0^{\frac{3}{2}}P_2, x_0^2 z, x, y)$$

$$A^{(x_0,y_0)}_{i}(P_1,P_2,z,S,T) = \frac{1}{4}x_0^{\frac{i-1}{2}} A_{i}(4x_0P_1, 4x_0^{\frac{3}{2}}P_2, x_0^2 z, x, y).$$

where $x = x_0(1+\frac{S}{4})$, $y = y_0 + \frac{\sqrt{x_0}}{4}T$.

Regularity (Continued):

Rescale (Continued):

$$B^{(x_0,y_0)}(P_1, P_2, z, S, T) = B(4x_0P_1, 4x_0^{\frac{3}{2}}P_2, x_0^2 z, x_0(1+\tfrac{S}{4}), y_0 + \tfrac{\sqrt{x_0}}{4}T).$$

$$\Gamma_1^{(x_0,y_0)} = \{(S,T)\in Q_1 : (x_0 + \tfrac{x_0}{4}S, y_0 + \tfrac{\sqrt{x_0}}{4}T) \in \Gamma_{f,r}\} \subset \partial Q_1^{(x_0,y_0)}.$$

<u>Th</u>: Let $f \in C^{1,\alpha}([0,r])$ satisfy $\|f\|_{C^{1,\alpha}([0,r])} \le \lambda^{-1}$ and $f > \lambda$ on R_+^1.

Let $\|(A_{ij}^{(x_0,y_0)}, A_i^{(x_0,y_0)})(P,z,\cdot,\cdot)\|_{C^\alpha(\overline{Q_1^{(x_0,y_0)}})} \le \lambda^{-1}$, for all $(P,z)\in R^2\times R$.

$\|D_{P,z}(A_{ij}^{(x_0,y_0)}, A_i^{(x_0,y_0)})(P,z,\cdot,\cdot)\|_{C(\overline{Q_1^{(x_0,y_0)}})} \le \lambda^{-1}$, for all $(P,z)\in R^2\times R$.

Let function B satisfy

$\partial_{P_1} B(P,z,x,y) \le -\lambda$ for all $(P,z)\in R^2\times R$, $(x,y)\in\Gamma_{f,r}$.

$B(0,0,x,y) = 0$ for all $(x,y)\in\Gamma_{f,r}$.

$\|D_{P,z} B^{(x_0,y_0)}(P,z,\cdot,\cdot)\|_{C^{1,\alpha}(\overline{\Gamma_1^{(x_0,y_0)}})} \le \lambda^{-1}$ for all $(P,z)\in R^2\times R$.

$\|D^2_{P,z} B\|_{C^1(R^2\times R\times\overline{\Gamma_1^{(x_0,y_0)}})} \le \lambda^{-1}$.

Let $u \in C(\overline{\Omega_r}) \cap C^{2,\alpha}(\overline{\Omega_r}\setminus\Gamma_0)$ be a solution with $|u| \le Mx^2$ in Ω.

Then there exists $r' \in (0, \frac{r}{2})$ small depending only on λ, M, α such that

$$\|u\|^{2,(par)}_{2,\alpha,\Omega_{r'}} \le C(\lambda, M, \alpha).$$

<u>Prototype</u>: $(2x - (\gamma+1)u_x)u_{xx} + \frac{1}{c_2^2}u_{yy} - u_x = 0.$

with estimate $|u| \le \dfrac{(1-\varepsilon)}{(\gamma+1)} x^2.$

Proof. **Step 1:** Let $r' \in (0, \frac{r}{2})$. For $z = (x,y) \in \Omega_{r'}$ and $\rho \in (0,1)$, define

$$\tilde{R}_{z,\rho} := \{(s,t) : |s-x| < \tfrac{\rho}{4}x, \ |t-y| < \tfrac{\rho}{4}\sqrt{x}\}, \quad R_{z,\rho} := \tilde{R}_{z,\rho} \cap \Omega_r.$$

Then for any $z \in \Omega_{r'}$, and $\rho \in (0,1)$,

$$R_{z,\rho} \subset \Omega_r \cap \{(s,t) : \tfrac{3}{4}x < s < \tfrac{5}{4}x\} \subset \Omega_r.$$

For any $z \in \Omega_{r'}$, we have at least one of the following three cases:

① $R_{z,\frac{1}{10}} = \tilde{R}_{z,\frac{1}{10}}$, ② $z \in R_{z_w,\frac{1}{2}}$ for $z_w = (x,0) \in \Gamma_{n,r'}$,

③ $z \in R_{z_s,\frac{1}{2}}$ for $z_s = (x, f(x)) \in \Gamma_{f,r'}$.

Thus it suffices to make the local estimates of Du and D^2u in the following rectangles:

Case ① $R_{(x_0,y_0),\frac{1}{20}}$ for $(x_0,y_0) \in \Omega_{r'}$ such that $R_{(x_0,y_0),\frac{1}{10}} = \tilde{R}_{(x_0,y_0),\frac{1}{10}}$;

Case ② $R_{(x_0,y_0),\frac{1}{4}}$ for $(x_0,y_0) \in \Gamma_{n,r'}$;

Case ③ $R_{(x_0,y_0),\frac{1}{4}}$ for $(x_0,y_0) \in \Gamma_{f,r'}$.

Step 2 (Case ①): $\because R_{(x_0,y_0),\frac{1}{10}} = \{(x_0 + \frac{x_0}{4}S, \ y_0 + \frac{\sqrt{x_0}}{4}T) : (S,T) \in Q_{\frac{1}{10}}\}$, where $Q_\rho := (-\rho,\rho)^2$ for $\rho > 0$.

Define $u^{(x_0,y_0)}(S,T) = \frac{1}{x_0^2} u(x_0 + \frac{x_0}{4}S, \ y_0 + \frac{\sqrt{x_0}}{4}T)$ for $(S,T) \in Q_{\frac{1}{10}}$.

Then $\|u^{(x_0,y_0)}\|_{C(\overline{Q_{\frac{1}{10}}})} \le 4M$.

and $u^{(x_0,y_0)}$ satisfies $\sum_{i,j=1}^{2} A_{ij}^{(x_0,y_0)} D_i D_j u^{(x_0,y_0)} + \sum_{i=1}^{2} A_i^{(x_0,y_0)} D_i u^{(x_0,y_0)} = 0.$ in $Q_{\frac{1}{10}}$.

Proof (Continued):

$\because \frac{1}{\lambda}|K|^2 \geq \sum_{i,j=1}^{2} A_{ij}^{(x_0,y_0)}(P,z,S,T)K_iK_j \geq \lambda|K|^2$,

for all $p \in R^2$, $z \in R^1$, $(S,T) \in Q_{\frac{1}{10}}$, $K=(K_1,K_2) \in R^2$.

$\therefore \quad \|u^{(x_0,y_0)}\|_{C^{2,\alpha}(\overline{Q_{\frac{1}{20}}})} \leq C.$

Step 3 (Case ②): Let $(x_0,y_0) \in \Gamma_{n,r'}$. $\because f \geq \lambda$

$\therefore$ we can choose $r' < \frac{1}{10}\lambda^2$ such that $\overline{R_{(x_0,y_0),1}} \cap \partial\Omega \subset \Gamma_{n,r}$.

$\therefore$ for any $p \in (0,1]$, $R_{(x_0,y_0),p} = \{(x_0+\frac{x_0}{4}S,\, y_0+\frac{\sqrt{x_0}}{4}T) : (S,T) \in Q_p \cap \{T>0\}\}$.

Define $u^{(x_0,y_0)}(S,T) = \frac{1}{x_0^2}u(x_0+\frac{x_0}{4}S,\, y_0+\frac{\sqrt{x_0}}{4}T)$,

then $\|u^{(x_0,y_0)}\|_{C(\overline{Q}_1 \cap \{T \geq 0\})} \leq 4M$

and $u^{(x_0,y_0)}$ satisfy a uniform elliptic equation as in step 2,

with boundary condition $\partial_T u^{(x_0,y_0)} = 0$ on $\{T=0\} \cap Q_1$.

$\therefore \quad \|u^{(x_0,y_0)}\|_{C^{2,\alpha}(\overline{Q}_{\frac{1}{2}} \cap \{T \geq 0\})} \leq C.$

Step 4 (Case ③): Let $(x_0,y_0) \in \Gamma_{f,r'}$ and choose $r' < \frac{1}{10}\lambda^2$, $\therefore \overline{R_{(x_0,y_0),1}} \cap \partial\Omega \subset \Gamma_{f,r}$.

Then for any $p \in (0,1]$, $R_{(x_0,y_0),p} = \{(x_0+\frac{x_0}{4}S,\, y_0+\frac{\sqrt{x_0}}{4}T) : (S,T) \in Q_p \cap \{T < F_{(x_0,y_0)}(S)\}\}$.

where $F_{(x_0,y_0)}(S) = 4\dfrac{f(x_0+\frac{x_0}{4}S) - f(x_0)}{\sqrt{x_0}}$.

Then we use $x_0 \in (0, r')$ to obtain

$$F_{(x_0,y_0)}(0) = 0$$

$$\|F_{(x_0,y_0)}\|_{C^1([-1,1])} \leq \frac{\|f'\|_{L^\infty([0,r])} x_0}{\sqrt{x_0}} \leq \lambda^{-1}\sqrt{r'} .$$

$$[F'_{(x_0,y_0)}]_{0,\alpha,(-1,1)} \leq \frac{[f']_{0,\alpha,(0,r)}\, x_0^{1+\alpha}}{4^\alpha \sqrt{x_0}} \leq C(\lambda)(r')^{\frac{1}{2}+\alpha} .$$

$$\therefore \quad \|F_{(x_0,y_0)}\|_{C^{1,\alpha}([-1,1])} \leq C(\lambda), \quad \text{where } r' < \lambda^2 .$$

Define $u^{(x_0,y_0)}(S,T) := \frac{1}{x_0^2} u(x_0 + \frac{x_0}{4}S,\ y_0 + \frac{\sqrt{x_0}}{4}T)$, for $(S,T) \in Q_1 \cap \{T < F_{(x_0,y_0)}(S)\}$.

$$\therefore \quad \|u^{(x_0,y_0)}\|_{C(\overline{Q}_1 \cap \{T \leq F_{(x_0,y_0)}(S)\})} \leq 4M .$$

On $T_{f,r}$, $u^{(x_0,y_0)}$ satisfies

$$B^{(x_0,y_0)}(Du^{(x_0,y_0)}, u^{(x_0,y_0)}, S, T) = 0 \quad \text{on } \{T = F_{(x_0,y_0)}(S)\} \cap Q_1 .$$

It can be written in the form

$$\partial_S u^{(x_0,y_0)} = \hat{B}^{(x_0,y_0)}(\partial_T u^{(x_0,y_0)}, u^{(x_0,y_0)}, S, T) ,$$

where $\hat{B}^{(x_0,y_0)}(p_2, z, S, T)$ satisfies $\partial_{p_2}\hat{B}^{(x_0,y_0)} = x_0^{\frac{1}{2}} \frac{B_{p_2}}{B_{p_1}}$.

From assumptions, we get

$$\|u^{(x_0,y_0)}\|_{C^{2,\alpha}(\overline{Q}_{\frac{1}{2}} \cap \{T \leq F_{(x_0,y_0)}(S)\})} \leq C$$

#.