Degree Theory

The Brouwer Degree :

Def. (i) $x_0 \in U$ is a regular point of $f(x)$ if $df(x_0)$ is nonsigular; Otherwise x_0 is called a critical point of f.

(ii) $y_0 \in R^n$ is called a regular value of f if $f^{-1}(y_0)$ contains no critical points of f; otherwise y_0 is called a critical value.

Motivation: If $f(x) \neq y_0$ for all $x \in \partial U$. Suppose that $f \in C'(\bar{U}; R^n)$. Let y_0 be a regular value of f, then the set

$$f^{-1}(y_0) = \{x \in \bar{U} : f(x) = y_0\}$$

is finite. In this case, we define the degree of f at y_0 as

$$d(y_0) = \sum_{x \in f^{-1}(y_0)} \operatorname{sgn}[\det df_x].$$

Note that $d(y_0)$ is an integer, positive, negative or zero.

Extension : (in two directions):

(i) to functions $f \in C^1(\bar{U}; R^n)$ at points $y_0 = f(x_0)$, where df_{x_0} is singular.

(ii) to functions $f \in C(\bar{U}; R^n)$, i.e., continous functions.

By approximation. (but it involves a good deal of work).

Alternate approach:

Def: Let $f \in C^1(U; R^n)$ and $y_0 \in R^n \setminus f(\partial U)$. Let $\mu = \phi(y)dy$ be a C^∞ n-form on R^n having compact support $K \subset R^n \setminus f(\partial U)$, such that $y_0 \in K$ and $\int \mu = 1$.

We define the degree of f at y_0 to be

$$d(f, U, y_0) = \int_U \mu \circ f,$$

where differential forms $\mu = \phi(y) dy_1 \wedge dy_2 \wedge \cdots \wedge dy_n$; which satisfy these conditions, are called admissible for y_0 and f.

Lemma (Well-defined):

Suppose that μ and η are admissible for y_0 and f, then

$$\int_U \mu \circ f = \int_U \eta \circ f .$$

Proof: Notice that $\mu - \eta = \phi(y)\,dy$ is a C^∞ n-form on R^n having compact support K satisfying $\int \mu = 0$.

Then there exists $(n-1)$-form w having compact support U with $\mu - \eta = dw$ (Ex: by induction).

$$\therefore \int_U \mu \circ f - \int_U \eta \circ f = \int_U (\mu - \eta) \circ f = \int_U dw \circ f = \int_{\partial U} w \circ f = 0 .$$

Basic Properties:

Property 1: If $|y_1 - y_0|$ is small, then $d(f, U, y_1) = d(f, U, y_0)$.

Proof: If μ is admissible for y_0, then μ is admissible for y_1, when $|y_0 - y_1|$ is sufficiently small. #

Property 2. If y_0 is a regular value for f, then $d(f, U, y_0) = d(y_0)$. In particular, $d(f, U, y_0) = 0$ if $y_0 \bar{\in} f(\bar{U})$.

Proof: Let $f^{-1}(y_0) = \{x_1, x_2, \cdots, x_m\}$. By the inverse function theorem, each x_i has a neighborhood N_i on which f is a homeomorphism. We choose these so small that $N_i \cap N_j = \phi$ if $i \neq j$.

Set $N = \bigcap_i f(N_i)$; then $y_0 \in N$.

Let μ have support in N and let it be admissible for y_0 and f.

Then
$$d(f, U, y_0) = \int_U \mu \circ f = \sum_{i=1}^{m} \int_{N_i} \mu \circ f$$
$$= \sum_{i=1}^{m} \int_{f(N_i)} \operatorname{sgn}[\det df_{x_i}] \mu$$
$$= \sum_{i=1}^{m} \operatorname{sgn}[\det df_{x_i}] \int_N \mu \qquad \left(\because \int_{f(N_i)\setminus N} \mu = 0\right).$$
$$= \sum_{i=1}^{m} \operatorname{sgn}[\det df_{x_i}] = d(y_0).$$

Finally, the last statement is immediate, since if $y_0 \bar{\in} f(\bar{U})$, then y_0 is a regular value and $d(y_0) = 0$. #

Corollary 12.5 : If y is a regular value of f lying in the same component of y_0 (in $R^n \setminus f(\partial U)$), then $d(y) = d(f, U, y_0)$.

In particular, the degree is an integer-value function.

Property 3 (Homotopy Invariance) : Let $\{f_t(\cdot)\}$ be a continuous one-parameter family of mappings taking $\bar{U} \times [0,1]$ into R^n, which is C' on U, for each fixed t in $[0,1]$. Assume that $y_0 \bar{\in} f_t(\partial U)$, $0 \le t \le 1$. Then $d(f_t, U, y_0)$ is independent of it.

Proof : Let $Y = \{f_t(U) : x \in \partial U, 0 \le t \le 1\}$, then $y_0 \bar{\in} Y$, and Y is compact. Let μ be admissible for y_0 and f, where $(\operatorname{spt}\mu) \cap Y = \phi$. Then

$$d(f_t, U, y_0) = \int_U \mu \circ f_t ,$$

and this function is easily seen to be continuous in t. Since the degree is integer value, it must be constant in t. #

Property 4 (Dependence Only on Boundary Values):

If $f|_{\partial U} = g|_{\partial U}$, and $y_0 \notin f(\partial U)$ ~~and~~ $= g(\partial U)$, then

$$d(f, U, y_0) = d(g, U, y_0).$$

Proof: Apply Property 3 to the family $tf + (1-t)g$, $0 \le t \le 1$.

Since f and g agree on ∂U, the hypotheses of Property 3 are satisfied. #

Property 5: Let $\{U_i\}$ be a countable family of disjoint open sets contained in U. Let $y_0 \notin f(\bar{U} \setminus \bigcup_i U_i)$. Then $d(f, U_i, y_0)$ is zero for all but a finite number of i, and $d(f, U, y_0) = \sum_i d(f, U_i, y_0)$.

Proof: $\bar{U} \setminus \bigcup_i U_i$ is closed and thus compact; hence $f(\bar{U} \setminus \bigcup_i U_i)$ is compact.

Let N be a neighborhood of y_0 disjoint from this latter compact set, and let y be a regular value in N. Then $d(f, U, y_0) = d(f, U, y)$, and for each i, $d(f, U_i, y_0) = d(f, U_i, y)$. $\because$ $f^{-1}(y)$ in finite, it must be contained in a finite number of the U_i's, say $U_1, \cdots, U_k$. Then if $g^{-1}(y) = \{x_1, \cdots, x_k\}$, where $x_i \in U_i$, and

$$d(f, U, y_0) = d(f, U, y) = d(y) = \sum_{i=1}^{k} d(f, U_i, y) = \sum_{i=1}^{k} d(f, U_i, y_0).$$

Property 6: Let Q be a closed set in $\bar{U}$, and suppose $y_0 \bar{\in} f(Q)$; then

$$d(f, U, y_0) = d(f, U \setminus Q, y_0).$$

Proof: If we set $U_1 = U \setminus Q$, then the result follows immediately from Property 5. #

Property 7: Let U and $\tilde{U}$ be bounded open subsets of dimensions n and m, respectively, and suppose $f \in C^1(\bar{U}, R^n)$, and $\tilde{f} \in C^1(\bar{\tilde{U}}, R^m)$. Then if $y_0 \in R^n \setminus f(\partial U)$, and $\tilde{y}_0 \in R^m \setminus \tilde{f}(\partial \tilde{U})$,

$$d(f \times \tilde{f}, U \times \tilde{U}, (y_0, \tilde{y}_0)) = d(f, U, y_0)\, d(\tilde{f}, \tilde{U}, \tilde{y}_0). \qquad (*)$$

Proof: $(y_0, \tilde{y}_0) \in R^{n+m} \setminus (f \times \tilde{f})(\partial(U \times \tilde{U}))$, so that the left-hand side of $(*)$ is defined. Let μ and $\tilde{\mu}$ be admissible for (y_0, f) and $(\tilde{y}_0, \tilde{f})$. Then $\mu \cdot \tilde{\mu}$ is an admissible $(n+m)$-form for $f \times \tilde{f}$ at $(y_0, \tilde{y}_0)$. Thus

$$\int_{U \times \tilde{U}} (\mu \wedge \tilde{\mu}) \circ (f \times \tilde{f}) = \int_U \mu \circ f \int_{\tilde{U}} \tilde{\mu} \circ \tilde{f}. \qquad \#.$$

Property 8: If the vectors $f(x)$ and $g(x)$ never point in opposite directions for $x \in \partial U$ (i.e., $f(x)+\lambda g(x) \neq 0$ for all $\lambda \geq 0$, $x \in \partial U$), then $d(f, U, 0) = d(g, U, 0)$, provided that the right hand side is defined, i.e., $0 \bar{\in} g(\partial U)$.

Proof. This is an immediate consequence of the homotopy invariance, using the homotopy $t[f(x)]+(1-t)[g(x)]$. #

Property 9: Let $f \in C(U, V)$, $g \in C(V, R^n)$, where U and V are bounded open subsets of R^n, and let $\{V_j\}$ be the set of open connected subsets of $V \setminus f(\partial U)$, whose closures are disjoint compact subsets contained in V. Then if $z_0 \in R^n \setminus (g \circ f)(\partial U)$,

$$d(g \circ f, U, z_0) = \sum_j d(f, U, V_j)\, d(g, V_j, z_0).$$

and the sum on the right is finite.

(Here $d(f, U, v)$ is constant for all $v \in V_j$, thus $d(f, U, V_j)$ is defined to be $d(f, U, v)$, $v \in V_j$).

Proof: For symplicity, we may suppose f and g are C^1 functions, and that z_0 is a regular value of both g and $g\circ f$. Then

$$d(g\circ f, U, z_0) = \sum_{\substack{u\in U \\ (g\circ f)(u)=z_0}} \operatorname{sgn}\det d(g\circ f)_u$$

$$= \sum_{u\in U,\ (g\circ f)(u)=z_0} \operatorname{sgn}\det dg_{f(u)} \cdot \operatorname{sgn}\det df_u$$

$$= \sum_{u\in V,\ g(v)=z_0} \operatorname{sgn}\det dg_v \cdot \sum_{\substack{u\in U \\ f(u)=v}} \operatorname{sgn}\det df_u$$

$$= \sum_{v\in V,\ g(v)=z_0} \operatorname{sgn}\det dg_v \ d(f, U, v).$$

If v is in a component of $V\setminus f(\partial U)$ having noncompact closure, then this component is disjoint from $f(\bar{U})$ so that $d(f, U, v)=0$ (by Property 2). Thus by Property 5, $d(f, U, v)=\sum_j d(f, U, V_j)$, and

$$d(g\circ f, U, z_0) = \sum_j d(f, U, V_j) \sum_{\substack{v\in V_j \\ g(v)=z_0}} \operatorname{sgn}\det dg_v$$

$$= \sum_j d(f, U, V_j)\, d(g, V_j, z_0).$$

#

Degree to continous functions:

Let $f \in C(U, R^n)$, and let $\{f_n\}$ be a sequence of functions in $C'(U, R^n)$ such that f_n converges uniformly to f on $\bar{U}$. If $y_0 \bar{\in} f(\partial U)$, then for n sufficiently large, $y_0 \bar{\in} f_n(\partial U)$ so that $d(f_n, U, y_0)$ is defined. We set

$$d(f, U, y_0) = \lim_{n \to \infty} d(f_n, U, y_0).$$

Lemma (Well-defined): The above limit exists and is independent of $\{f_n\}$.

Proof: Let $\delta = dist(y_0, f(\partial U))$. Let $\{g_n\}$ be another sequence of functions in $C'(U, R^n)$ with g_n converging uniformly to f in $\bar{U}$. We choose N large ~~such~~ such that $n \geq N$ implies that $\|f_n - f\|_\infty + \|g_n - f\|_\infty < \frac{\delta}{2}$.

If $y_0 = t f_n(x) + (1-t) g_n(x)$ for some $x \in \partial U$, $n \geq N$, $0 \leq t \leq 1$,

then $y_0 - f(x) = y_0 - t f(x) - (1-t) f(x) = t[f_n(x) - f(x)] + (1-t)[g_n(x) - f(x)]$,

so $|y_0 - f(x)| < \frac{\delta}{2}$. Impossible.

Hence for $n \geq N$, we apply the homotopy invariance property to the family $t f_n + (1-t) g_n$, $0 \leq t \leq 1$, and conclude that

$$d(f_n, U, y_0) = d(g_n, U, y_0)$$

Proof (Continued):

Thus if the limit exists, it is independent of $\{f_n\}$.

To see that the limit indeed exists, we merely apply the same argument to f_n and f_m with $m, n \geq N$.

Then we conclude that $d(f_n, U, y_0) = d(f_m, U, y_0)$.

Thus the sequence $\{d(f_n, U, y_0)\}$ is constant for $n \geq N$. #

Theorem: The properties 1-9 are valid for continuous functions. Moreover, if y_0 and y_1 are in the same component of $R^n \setminus f(\partial U)$, then $d(f, U, y_0) = d(f, U, y_1)$.

Property 10: If f and g are continuous functions, $y_0 \bar{\in} f(\partial U)$, and $f|_{\partial U} = g|_{\partial U}$, then $d(f, U, y_0) = d(g, U, y_0)$.

Property 11: If $\phi \in C(\partial U, R^n)$, and $y_0 \bar{\in} \phi(\partial U)$, then $d(\phi, U, y_0)$ depends only on the homotopy class of ϕ.

Applications of the Brouwer degree:

Th (Brouwer Fixed Point Theorem):

Let D be any set homeomorphic to an open ball in R^n, and let $\phi : \bar{D} \to \bar{D}$ be continuous. Then ϕ has a fixed point in $\bar{D}$; i.e., there is an $\bar{x} \in \bar{D}$ with $\phi(\bar{x}) = \bar{x}$.

Proof: Step 1: If there is a continuous mapping $f : \bar{D} \to \partial D$ such that $f|_{\partial D}$ is the identity, then since $0 \in R^n \setminus f(\partial D)$, $\therefore d(f, D, 0) = d(I, D, 0)$.

But $d(I, D, 0) = 1$, $\therefore d(f, D, 0) = 1$ $\therefore 0 \in f(D)$. contradiction.

Thus there is no such mapping.

Step 2: For simplicity, assume that D is the unit disk centered at the origin.

If ϕ had no fixed points, then for each $x \in D$, the points x and $\phi(x)$ define a line: $\lambda x + (1-\lambda)\phi(x)$, $\lambda \in R$.

Let $f(x)$ be the unique point on this line having norm 1, where $\lambda \geq 1$.

Then f maps $\bar{D}$ into ∂D continously, and $f|_{\partial D} = I$.

Contradiction #

Leray-Schauder Degree:

Counter example: Let l_2 denote the space of infinite sequence $x=(x_1, x_2, \cdots)$ with $\|x\|^2 = \sum |x_i|^2 < \infty$. Let D be the closed unit ball in l_2 and let T be the transformation on l_2 defined by $Tx = (\sqrt{1-\|x\|^2}, x_1, x_2, \cdots)$. T is clearly continuous, and if $\|x\| \le 1$, $\|Tx\|^2 = 1 - \|x\|^2 + \|x\|^2 = 1$.

Thus T maps D into ∂D. This implies that T cannot have any fixed points. For if x were a fixed point, then $\|x\| = \|Tx\| = 1$, and $(0, x_1, x_2, \cdots) = (x_1, x_2, \cdots)$. Contradiction.

Remark: This example shows that the Brouwer fixed point theorem is not valid in Banach spaces with merely a continuity hypothesis.

Definition: A continuous mapping T of a subset U of a Banach space B into B is called compact if $\overline{T(K)}$ is compact for every closed and bounded subset $K \subset U$.

Lemma: Let K be a closed and bounded subset of a Banach space B, and suppose that $T: K \to B$ is compact. Then T is a uniform limit (i.e., limit in the norm-topology on operators) of finite-dimensional mappings (i.e., mappings whose ranges are finite dimensional).

Proof: Let $\varepsilon > 0$ be given. Since $\overline{T(K)}$ is compact, it can be covered by open balls $N_1, \ldots, N_{j(\varepsilon)}$, each of radius ε, with centers $x_1, \cdots, x_{j(\varepsilon)}$, respectively. Let $\{\psi_i(x) : 1 \le i \le j(\varepsilon)\}$ be a partition of unity on $\overline{T(K)}$ subordinate to the cover $N_1, \cdots, N_{j(\varepsilon)}$; i.e., for each i, $\psi_i \ge 0$, $\operatorname{spt} \psi_i \subset N_i$, and if $x \in \overline{T(K)}$, $\sum \psi_i(x) = 1$. Set

$$T_\varepsilon(x) = \sum_{i=1}^{j(\varepsilon)} \psi_i(T(x)) x_i \quad , \quad x \in B .$$

Then clearly $T_\varepsilon(x)$ has finite range, and

$$\| T(x) - T_\varepsilon(x) \| = \| \sum_{i=1}^{j(\varepsilon)} \psi_i(T(x)) [x_i - T(x)] \| ,$$

If $\psi_i(T(x)) \neq 0$, then $\psi_i(T(x)) > 0$ $\therefore$ $x \in N_i$ $(\because \operatorname{spt} \psi_i \subset N_i)$

so $\| x_i - T(x) \| < \varepsilon$

Therefore $\| T(x) - T_\varepsilon(x) \| < \varepsilon$ #.

Remark: We call T_ε defined here an ε-approximation of T.

Leray-Schauder degree:

We are now going to extend the concept of degree to mappings of the form $T = I - K$, where I is the identity and K is compact.

Let U be a bounded open subset of a Banach space B and let T map $\bar{U}$ into B. Obviously $T(\partial U)$ is a closed set.

Let $y_0 \in B \setminus T(\partial U)$, then $\operatorname{dist}(y_0, T(\partial U)) = \delta > 0$.

Let $\varepsilon < \frac{\delta}{2}$, and let K_ε be an ε-approximation of K which has range in the finite-dimensional space V_ε containing y_0.

Set $T_\varepsilon = I - K_\varepsilon$. Then $T_\varepsilon(x) \neq y_0$ if $x \in \partial U$.

Thus for the mapping

$$T_\varepsilon |_{V_\varepsilon \cap \bar{U}} : V_\varepsilon \cap \bar{U} \longrightarrow V_\varepsilon,$$

the degree $d(T_\varepsilon, V_\varepsilon \cap \bar{U}, y_0)$ is defined.

We now set

$$\boxed{d(T, U, y_0) = d(T_\varepsilon, V_\varepsilon \cap U, y_0).}$$

It is called the Leray-Schauder degree.

To show it is well defined, we need a lemma.

<u>Lemma</u>: Let U be a bounded open subset of $R^n = R^{n_1} \oplus R^{n_2}$, $n_1 + n_2 = n$. Let f be a mapping from $\bar{U}$ into R^n of the form $I + \phi$, where $\phi : \bar{U} \to R^{n_1} \times \{0\}$. Let $y_0 \in R^{n_1} \times \{0\} \setminus f(\partial U)$; then

$$d(f, U, y_0) = d(f|_{U_1}, U_1, y_0), \quad \text{where } U_1 = U \cap R^{n_1}.$$

<u>Proof</u>: It suffices to prove the lemma for $f \in C^1(U, R^n)$, and then extend it to continuous f by the usual approximation arguments. Also assume $y_0 = 0$. We write $x \in R^n$ uniquely as $x = x_1 + x_2$, where $x_i \in R^{n_i}$.

We choose functions $f_i(x_i)$ in $C_0^\infty(R^n)$ having small support about $0 \in R^{n_i}$ and such that $\int_{R^{n_i}} f_i(x_i)\, dx_i = 1$, by definition,

$$d(f, U, 0) = \int_U (f_1 f_2) \circ f.$$

Thus ~~from~~ by $x + \phi(x) = [x_1 + \phi(x_1 + x_2)] + x_2$,

$$d(f, U, 0) = \int_{R^{n_1}} \int_{R^{n_2}} f_1\,(x_1 + \phi(x_1 + x_2)) f_2(x_2) \det(I + d\phi_{x_1})\, dx_1\, dx_2$$

$\therefore$ If $f_2 \to \delta$ (delta function), then

$$d(f, U, 0) = \int_{R^{n_1}} f_1(x_1 + \phi(x_1)) \det(I + d\phi_{x_1})$$
$$= \int_{R^{n_1}} f_1 \circ f = d(f|_{U_1}, U_1, 0). \quad \#$$

Well-defined:

We now show that the degree does not depend on V_ε.

Note first that if $V = V_\varepsilon \oplus N$, where N is finite dimensional, then it follows from the previous lemma that

$$d(T_\varepsilon, V \cap U, y_0) = d(T_\varepsilon, V_\varepsilon \cap U, y_0).$$

Thus if K_η is another approximation of K such that $K_\eta : U \to V_\eta$, $\eta < \frac{\delta}{2}$, and $V = V_\varepsilon \oplus V_\eta$, then by the lemma again,

$$d(T_\varepsilon, V_\varepsilon \cap U, y_0) = d(T_\varepsilon, V \cap U, y_0)$$

and

$$d(T_\eta, V_\eta \cap U, y_0) = d(T_\eta, V \cap U, y_0).$$

If $T_t = t T_\eta + (1-t) T_\varepsilon$, then $y_0 \bar{\in} T_t(\partial U)$ by δ, and by the homotopy invariance,

$$d(T_\varepsilon, V \cap U, y_0) = d(T_\eta, V \cap U, y_0).$$

so

$$d(T_\varepsilon, V_\varepsilon \cap U, y_0) = d(T_\eta, V_\eta \cap U, y_0). \qquad \#$$

<u>Th</u>: All of the properties of degree that hold in R^n are valid to Banach spaces. B.

Application:

Consider the problem: $\begin{cases} \Delta u + f(x, u, Du) = 0, & x \in \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$

where f is a C^∞ function.

Let $K = -\Delta^{-1}$, $F(u) = f(x, u, Du)$. ~~then K is a smoothing~~
then K is a compact operator in an appropriate function space.
If f does not grow too fast at infinity, we can show

$$\|u\|_1 \le C.$$

Let $B = \{u \in C^1(\Omega) : u = 0 \text{ on } \partial\bar{\Omega}\}$.
Let U be the ball $\|u\|_1 \le 1 + C$, contained in B,
and let $T : U \to B$ be defined by

$$T(u) = u - KF(u).$$

We seek a solution to the equation $T(u) = 0$.
If $u \in \partial U$, then $T(u) \ne 0$. Thus $d(T, U, 0)$ is defined.
In order to compute this degree, we consider the mappings

$$T_t(u) = u - tKF(u), \qquad 0 \le t \le 1.$$

By the homotopy property, $d(T, U, 0) = d(I, U, 0) = 1$.
Thus the problem has a solution $\bar{u}$ in Ω.
It is not too hard to show that under reasonable conditions on f,
$\bar{u}$ is smooth.

Keble Complexity Cluster Workshop on Mathematical modeling as a tool to bridge disciplines.

Date : 13th , November 2013

Venue : Pusey Room

Time : 2:30pm — 6:00 pm.

Organizers : Prof. G.-Q. Chen and Dr. A. Majumdar.