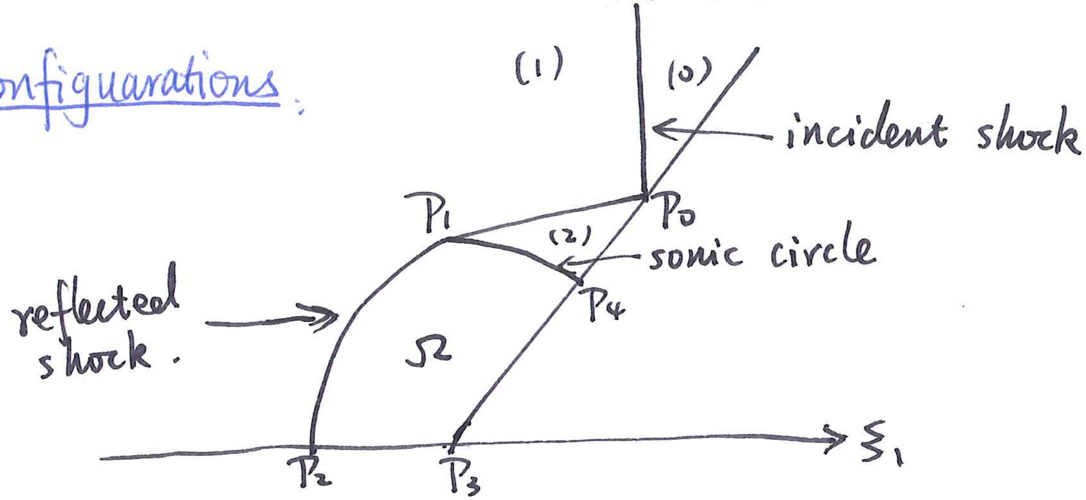
Mathematics of Shock Reflection-Diffraction Problem.

Configuarations:



Equations in Ω:

$$\operatorname{div}\left(\rho(|D\varphi|^2, \varphi) D\varphi\right) + 2\rho(|D\varphi|^2, \varphi) = 0.$$

with $\rho(|D\varphi|^2, \varphi) = \left(\rho_0^{\gamma-1} - (\gamma-1)\left(\varphi + \frac{1}{2}|D\varphi|^2\right)\right)^{\frac{1}{\gamma-1}}$.

Equation is elliptic if and only if

$$|D\varphi| < c_*(\varphi, \rho_0, \gamma) := \sqrt{\frac{2}{\gamma+1}\left(\rho_0^{\gamma-1} - (\gamma-1)\varphi\right)}.$$

Uniform states:

$$\varphi_0(\xi) = -\frac{|\xi|^2}{2} \qquad \text{for } \xi_1 > \xi_1^0$$

$$\varphi_1(\xi) = -\frac{|\xi|^2}{2} + u_1(\xi_1 - \xi_1^0) \qquad \text{for } \xi_1 < \xi_1^0,$$

Location of the incident shock:

$$\xi_1^0 = \rho_1 \sqrt{\frac{2(\rho_1^{\gamma-1} - \rho_0^{\gamma-1})}{(\gamma-1)(\rho_1^2 - \rho_0^2)}} = \frac{\rho_1 u_1}{\rho_1 - \rho_0} > 0.$$

$$\varphi_2(\xi) = -\frac{|\xi|^2}{2} + u_2(\xi_1 - \xi_1^0) + u_2 \operatorname{tg}\theta_w (\xi_2 - \xi_1^0 \operatorname{tg}\theta_w),$$

where $u_2 > 0$, $\rho_2 > \rho_1$.

Boundary conditions:

On Γ_{wedge}: $D\varphi \cdot \nu = 0$,

On Γ_{sonic}: $\varphi = \varphi_2$ (Should prove $D\varphi = D\varphi_2$).

On Γ_{shock}: $[\varphi]_s = 0$ (To determine the free boundary).

$[\rho(|D\varphi|^2, \varphi) D\varphi \cdot \nu]_s = 0$ (Boundary condition for φ).

Additional requirment:

To define ρ: $\rho_0^{\gamma-1} - (\gamma-1)(\varphi + \frac{1}{2}|D\varphi|^2) > 0$.

Definition of transonic admissible solutions:

Let $\gamma>1$, $\rho_1>\rho_0>0$ and $u_1>0$ be constants. Fix a wedge angle $\theta_w\in[\theta_w^s,\frac{\pi}{2})$. A function $\varphi\in C^{0,1}(\Lambda)$ is called an admissible solution of regular reflection problem if φ is a solution to the previous problem satisfying:

i). There exists the (shock) curve Γ_{shock} with endpoints P_1 and P_2, where $P_2=(\xi_{1P_2},0)$ with $\xi_{1P_2}<\min(0,u_1-c_1)$, $\xi_{1P_2}\leq\xi_{1P_1}$, and curve Γ_{shock} satisfies

- $\Gamma_{shock}\subset(\Lambda\setminus\overline{B_{c_1}(u_1,0)})\cap\{\xi_{1P_2}\leq\xi_1\leq\xi_{1P_1}\}$, where $C_1=\partial B_{c_1}(u_1,0)$ is the sonic circle of state (1).
- Γ_{shock} is C^2 in its relative interior. Moreover, denote by Γ_{shock}^{ext} the curve $\Gamma_{shock}^{ext}=\Gamma_{shock}\cup\Gamma_{shock}^{-}\cup\{P_2\}$, where Γ_{shock}^{-} is the reflection of Γ_{shock} with respect to $\{\xi_2=0\}$. Then Γ_{shock}^{ext} is C^1 at its relative interior (including P_2), in the sense that for any P in the relative interior of Γ_{shock}^{ext}, there exists $r>0$, $f\in C^1(\mathbb{R})$ and orthonormal coordinate system S,T in $\mathbb{R}^2$ such that $\Gamma_{shock}\cap B_r(P)=\{S=f(T)\}\cap B_r(P)$. Moreover, if $P\neq P_2$, then $f\in C^2$ can be chosen for sufficiently small r.

Definition of transonic admissible solutions (Continued):

Denote by Γ_{sonic} the arc P_1P_4 of sonic circle C_2 of state (2), and denote the line segments $\Gamma_{sym} = P_2P_3$, $\Gamma_{wedge} = P_3P_4$. From above, it follows that Γ_{shock}, Γ_{sonic}, Γ_{sym}, Γ_{wedge} do not have common points except their endpoints P_1, P_2, P_3, P_4. Thus $\overline{\Gamma_{shock}} \cup \overline{\Gamma_{sym}} \cup \overline{\Gamma_{wedge}} \cup \overline{\Gamma_{sonic}}$ is a closed curve without self-intersections. Denote by Ω the open bounded domain restricted by this curve. Note that $\Omega \subset \Lambda$,
$\partial\Omega = \overline{\Gamma_{shock}} \cup \overline{\Gamma_{sym}} \cup \overline{\Gamma_{wedge}} \cup \overline{\Gamma_{sonic}}$, and $\partial\Omega \cap \partial\Lambda = \overline{\Gamma_{sym}} \cup \overline{\Gamma_{wedge}}$.

ii): φ satisfies:

$$\varphi \in C^{0,1}(\Lambda) \cap C^1(\overline{\Lambda} \setminus \overline{P_0P_1P_2}),$$

$$\varphi \in C^3(\overline{\Omega} \setminus (\overline{\Gamma_{sonic}} \cup \{P_2, P_3\})) \cap C^1(\overline{\Omega}),$$

$$\varphi = \begin{cases} \varphi_0 & \text{for } \xi_1 > \xi_1^0 \text{ and } \xi_2 > \xi_1 \tan\theta_w, \\ \varphi_1 & \text{for } \xi_1 < \xi_1^0 \text{ and above the reflection shock } P_0P_1P_2 \\ \varphi_2 & \text{in } P_0P_1P_4. \end{cases}$$

iii): equation (2.2.8) is strictly elliptic in $\overline{\Omega} \setminus \overline{\Gamma_{sonic}}$:

$$|D\varphi| < c(|D\varphi|^2, \varphi) \quad \text{in } \overline{\Omega} \setminus \overline{\Gamma_{sonic}}.$$

Definition of transonic admissible solutions (Continued):

iv): $\varphi_1 \geq \varphi \geq \varphi_2$ in Ω;

v): Let e_{S_1} be a unit vector parallel to S_1 and oriented so that $e_{S_1} \cdot e_{\xi_1} < 0$. That is

$$e_{S_1} = \frac{P_1 - P_0}{|P_1 - P_0|} = -\frac{(v_2, u_1 - u_2)}{\sqrt{(u_1-u_2)^2 + v_2^2}}.$$

Then
$$\partial_{e_{S_1}} (\varphi_1 - \varphi) \leq 0 \quad \text{in } \Omega,$$
$$\partial_{\xi_2} (\varphi_1 - \varphi) \leq 0 \quad \text{in } \Omega.$$

Remark 1: For any θ_w sufficiently close to $\frac{\pi}{2}$, the solutions are transonic admissible solutions.
(Refer: G.-Q. Chen and M. Feldman, Global solutions of shock reflection by large-angle wedges for potential flow. Ann. of Math. (2) 171 (2010), 1067–1182.)

Remark 2: (velocity jump across Γ_{shock}).

Notice that φ_1 is pseudo-supersonic on Γ_{shock}, while φ is pseudo-~~~~subsonic on Γ_{shock}. Thus

$$D\varphi \neq D\varphi_1 \quad \text{on } \overline{\Gamma_{shock}}.$$

Remark 3: ($\Gamma_{sym} \cup \{P_2\}$ are interior points).

Let $\Omega^-(\Gamma^-_{sonic})$ be the reflection of $\Omega(\Gamma_{sonic})$ with respect to $\{\xi_2=0\}$, and $\Omega^{ext}=\Omega\cup\Omega^-\cup\Gamma_{sym}$, $\Gamma^{ext}_{sonic}=\Gamma_{sonic}\cup\Gamma^-_{sonic}$.

Then $\Gamma^{ext}_{shock}\subset\partial\Omega^{ext}$. Let φ^{ext} be the even extension of φ into Ω^{ext}, by ~~$\partial_\nu\varphi=0$~~ $\partial_\nu\varphi=0$ on Γ_{sym}, we get

$$\varphi^{ext}\in C^2(\overline{\Omega^{ext}}\setminus(\overline{\Gamma^{ext}_{sonic}}\cup\{P_2,P_3\}))\cap C^1(\overline{\Omega^{ext}}).$$

It is easy to ~~chec~~ check that φ^{ext} satisfies equations and the equations are strictly elliptic in $\overline{\Omega^{ext}}\setminus\overline{\Gamma^{ext}_{sonic}}$. Moreover φ^{ext} satisfies

$$\varphi^{ext}=\varphi_1,\quad \rho(|D\varphi^{ext}|^2,\varphi^{ext})D\varphi^{ext}\cdot\nu=\rho_1 D\varphi_1\cdot\nu \quad \text{on } \Gamma^{ext}_{shock}.$$

Remark 4: (Cone of monotonicity directions):

For all $e\in \mathrm{Cone}(e_{S_1}, e_{\xi_2})$, $e\neq 0$,

$$\partial_e(\varphi_1-\varphi)\le 0 \quad \text{in } \overline{\Omega}.$$

Where $\mathrm{Cone}(e,g)=\{ae+bg \mid a,b\ge 0\}$ for $e,g\in\mathbb{R}^2\setminus\{0\}$, with $e\neq cg$.

Remark 5 (Entropy Condition): If φ is a transonic admissible solution,

then $\quad \partial_\nu \varphi_1 > \partial_\nu \varphi > 0 \quad$ on Γ_{shock},

where ν is unit normal to Γ_{shock}, interior to Ω.

Proof: We have $D\varphi_1(P_2) = (u_1 - \xi_{1P_2}, 0) = (u_1 + |\xi_{1P_2}|, 0)$, and $\nu(P_2) = (1,0)$ along Γ_{shock}.

Thus $\quad \partial_\nu \varphi_1(P_2) > 0$.

Now we show that $\partial_\nu \varphi_1 > 0$ along Γ_{shock}.

If not, there exists a point $\hat{P} \in \Gamma_{shock}$, $\partial_\nu \varphi_1(\hat{P}) = 0$, then from

$\rho(|D\varphi|^2, \varphi)\partial_\nu \varphi = \rho_1 \partial_\nu \varphi_1$ on Γ_{shock}, either $\partial_\nu \varphi_0 = 0$ or $\rho = 0$ at $\hat{P}$.

From ellipticity, $\partial_\nu \varphi = 0$ at $\hat{P}$. So $D\varphi_1(\hat{P}) = D\varphi(\hat{P})$.

Contradict to Remark 2.

Now from Rankine-Hugoniot condition, $\partial_\nu \varphi > 0$ on Γ_{shock}.

Notice that equation is elliptic is equivalent to

$$|D\varphi| < \sqrt{\frac{2}{\gamma+1}\left(\rho_0^{\gamma-1} - (\gamma-1)\varphi\right)},$$

and $\varphi_\tau = \varphi_{1\tau}$, $\varphi = \varphi_1$ on Γ_{shock}.

so $\quad \partial_\nu \varphi_1 > \partial_\nu \varphi > 0 \quad$ on Γ_{shock}.

Main Results:

Th1: For case $u_1 \le c_1$. Let $\theta_w^s \in (0, \frac{\pi}{2})$ be the sonic angle. Then there exists $\alpha = \alpha(\rho_0, \rho_1, \gamma) \in (0, \frac{1}{2})$ such that, when $\theta_w \in (\theta_w^s, \frac{\pi}{2})$, there exists a global self-similar regular reflection-diffraction solution:

$$\Phi(t, x) = t\varphi(\tfrac{x}{t}) + (\tfrac{|x|^2}{2t}), \quad \text{for } \tfrac{x}{t} \in \Lambda, \ t > 0.$$

with $$\rho(t, x) = \left(\rho_0^{\gamma-1} - (\gamma-1)(\Phi_t + \tfrac{1}{2}|\nabla_x \Phi|^2)\right)^{\frac{1}{\gamma-1}},$$

where ~~$\varphi(\xi, \eta)$~~ $\varphi(\xi)$ is a transonic admissible solution.

Th2: For case $u_1 > c_1$, let $\theta_w^s \in (0, \frac{\pi}{2})$ be the sonic angle. There exists $\theta_w^{a*} \in [\theta_w^s, \frac{\pi}{2})$ and $\alpha \in (0, \frac{1}{2})$ depending only on (ρ_0, ρ_1, γ) such that the results of Th1 hold for each wedge angle $\theta_w \in (\theta_w^a, \frac{\pi}{2})$. If $\theta_w^a > \theta_w^s$, then for the wedge angle $\theta_w = \theta_w^a$, there exists an "attached" solution, i.e., a solution with the properties as in Th1 except that $P_2 = P_3$, and the reflected shock $P_0P_1P_2$ is Lipschitz up to its endpoints and is $C^{2,\beta}$ for any $\beta \in [0, \frac{1}{2})$ except the point P_2, and is C^∞ except the points P_1 and P_2.

Main Results:

Th3 (Optimal Regularity): The solution φ in Th1 and Th2 satisfies

(i) φ is $C^{2,\alpha}$ up to the sonic arc $\overline{P_1P_4}$ away from the point P_1 for any $\alpha\in(0,1)$. That is for any $\alpha\in(0,1)$ and any given $\xi_0\in\overline{P_1P_4}\setminus\{P_1\}$, there exist $K<\infty$ and $d>0$, depending only on $\rho_0, \rho_1, \gamma, \alpha$, and $\mathrm{dist}(\xi_0, \Gamma_{shock})$, so that

$$\|\varphi\|_{2,\alpha;\overline{B_d(\xi_0)\cap\Omega}}\le K;$$

(ii) For any $\xi_0\in\overline{P_1P_4}\setminus\{P_1\}$,

$$\lim_{\xi\to\xi_0,\,\xi\in\Omega}(D_{rr}\varphi-D_{rr}\varphi_2)=\frac{1}{\gamma+1},$$

where (r,θ) are the polar coordinates with center at (u_2,v_2).

(iii): $D^2\varphi$ has a jump across $\overline{P_1P_4}$: For any $\xi_0\in\overline{P_1P_4}\setminus\{P_1\}$,

$$\lim_{\xi\to\xi_0,\,\xi\in\Omega}D_{rr}\varphi-\lim_{\xi\to\xi_0,\,\xi\in\Lambda\setminus\Omega}D_{rr}\varphi=\frac{1}{\gamma+1}.$$

(iv): The limit

$$\lim_{\xi\to P_1,\,\xi\in\Omega}D^2\varphi$$

does not exist.

Reference: Bae, M.; Chen, G.-Q.; Feldman, M., Regularity of solutions to regular shock reflection for potential flow. Invent. Math. 175 (2009), no.3, 505-543.

Proof of Existence:

~~Defini~~ Notations:

① For $\sigma \in (0, \varepsilon_0]$ denote

$$D_\sigma := \{ \varphi_2 < \varphi_1 \} \cap \Lambda \cap N_{\varepsilon_1}(\Gamma_{sonic}) \cap \{ 0 < x < \sigma \}.$$

② For a set $S \in \{\Omega, \Gamma_{shock}, \Gamma_{wedge}\}$ and for integer $k \geq 0$ and real $\sigma > 0$, $\alpha \in (0,1)$, we define:

$$\|u\|^{*,\sigma}_{k,\alpha,S} = \|u\|^{-(k-1+\alpha),\Gamma_{sym}}_{k,\alpha,S\setminus D_{\frac{\varepsilon_0}{10}}} + \|u\|^{\sigma,(par)}_{k,\alpha,S\cap D_{\varepsilon_0}}.$$

$C^{k,\alpha}_{*,\sigma}(S)$ is the closure in the norm $\|\cdot\|^{*,\sigma}_{k,\alpha,S}$ of the set

$$\{ v \in C^\infty(S) \mid \|v\|^{*,\sigma}_{k,\alpha,S} < \infty \}.$$

Iteration set:

Iteration set $K \subset C^{2,\alpha}_{*,1+\delta^*}(\Omega) \times [\theta_w^*, \frac{\pi}{2}]$ is the set of all (φ, θ_w) satisfying the following properties:

① (φ, θ_w) satisfy the estimates $\|\varphi\|^{*,1+\delta^*}_{2,\alpha,\Omega} < \eta_1(\theta_w)$,

where $\eta_1 \in C(\mathbb{R})$ is defined by

$$\eta_1(\theta_w) = \begin{cases} \delta_1, & \text{if } \frac{\pi}{2} - \theta_w \leq \frac{\delta_1}{N_1}, \\ 10M, & \text{if } \frac{\pi}{2} - \theta_w \geq 2\frac{\delta_1}{N_1}, \\ \text{linear}, & \text{if } \frac{\pi}{2} - \theta_w \in (\frac{\delta_1}{N_1}, 2\frac{\delta_1}{N_1}). \end{cases}$$

where M is the constant defined by uniform estimates of the transonic admissible solutions.

Proof of Existence.

Iteration set (Continued):

② $\Gamma_{shock} \subset \Lambda(\theta_w) \setminus \overline{B_{c_1}(O_1)}$, and ~~$2C \geq$ dist(Γ_{shock},~~

$$2C \geq g_{shock} \geq \frac{1}{2C} .$$

where $\Gamma_{shock} = \{(s,t) \mid s \in (0, \hat{s}(\theta_w)),\ t = g_{shock}(s)\}$, and constant C is determined by the estimates of the transonic admissible solutions.

③ φ satisfies:

$$
\begin{aligned}
&(\varphi - \varphi_2) > \eta_2(\theta_w) && \text{in } \overline{\Omega} \setminus D_{\varepsilon/10} ;\\
&\partial_x(\varphi - \varphi_2) < \frac{2-\mu_0}{1+\gamma} x && \text{in } \Omega_\varepsilon \setminus \Omega_{\varepsilon/10} ;\\
&\partial_{e_{S_1}}(\varphi_1 - \varphi) < -\eta_2(\theta_w) && \text{in } \overline{\Omega} \setminus D_{\varepsilon/10} ;\\
&\partial_{\xi_2}(\varphi_1 - \varphi) < -\eta_2(\theta_w) && \text{in } \overline{\Omega} \setminus N_{\varepsilon/10}(\{\xi_2 = 0\}) ;\\
&\partial_\nu(\varphi_1 - \varphi) > \mu_1 && \text{on } \overline{\Gamma_{shock}} ,\\
&\partial_\nu \varphi > \mu_1 && \text{on } \overline{\Gamma_{shock}} ,
\end{aligned}
$$

where $\eta_2(\theta_w) = \delta_2 \min\left(\frac{\pi}{2} - \theta_w - \frac{\delta_1}{N_1^2}, \frac{\delta_1}{N_1^2}\right)$, and constants μ_k are choosen from uniform estimates of the transonic admissibe solutions.

④ Uniform ellipticity in $\overline{\Omega} \setminus D_{\varepsilon/10}$:

$$|D\varphi| < c(|D\varphi|^2, \varphi) - \lambda \quad \text{in } \overline{\Omega} \setminus D_{\varepsilon/10} ,$$

where $\lambda = \lambda(\varepsilon) > 0$ is chosen in Remark ~~below~~ . below.

Proof of Existence:

Iteration Set (Continued):

⑤ Bounds on density: $\rho_{min} < \rho(|D\varphi|^2, \varphi) < \rho_{max}$ in $\overline{\Omega} \setminus D_{\varepsilon/10}$, with $\rho_{max} > \rho_{min} > 0$ defined by ~~$\rho_{min} = \frac{a}{2}\rho_1, \rho_{max}$~~ estimates of the admissible solutions.

⑥ Boundary value problem ⊗ defined by (φ, θ_w) below has a solution $\hat{\varphi} \in C^2(\Omega(u)) \cap C^1(\overline{\Omega}(u))$ with the following properties, ~~but~~ that

$$\| \varphi - \hat{\varphi} \|^{*, 1+\delta^*}_{2, \alpha/2, \Omega} < \delta_3 .$$

Remark 6. The choice of constants below will keep only the following dependences: δ^* is fixed above, and α depends only on the data and θ_w^*. Other constants, in addition to data and θ_w^*, have the following dependencies: Small ε depends on δ^*. Small δ_1 depends on δ^*, ε. Large N_1 depends on δ_1. Small δ_2 depends δ_1, N_1 and ε. Small λ depends on ε as defined in the next Remark. Then $\delta_3 > 0$ is made as small as needed, depending on all other constants.

Remark 7. Given $\varepsilon > 0$, by ellipticity estimate, there exists $\hat{\lambda} > 0$ such that ④ of iteration set holds strictly with $\hat{\lambda}$ for any admissible solution for $\theta_w \in [\theta_w^*, \frac{\pi}{2}]$. Then we choose $\lambda = \frac{\hat{\lambda}}{2}$.

Proof of Existence:

Remark 8: Since $\varphi=\varphi_1$ on Γ_{shock}, i.e., $|D_\nu(\varphi_1-\varphi)|=|D(\varphi_1-\varphi)|$, then $\partial_\nu(\varphi_1-\varphi)>\mu_1$ on $\overline{\Gamma_{shock}}$ is equivalent to $|D(\varphi_1-\varphi)|>\mu_1$ on $\overline{\Gamma_{shock}}$.

Remark 9: From definition of the constants ρ_{min}, ρ_{max}, and noting that $\rho(|D\varphi|^2,\varphi)=\rho_2$ on Γ_{sonic} for any admissible solution φ, it follows that

$$2\rho_{min}\leq\rho_2\leq\frac{\rho_{max}}{2}.$$

Furthermore, since for any admissible solution φ, at the point $P_1\in\overline{\Gamma_{shock}}$, there holds $(\varphi-\varphi_2)(P_1)=0$ and $D(\varphi-\varphi_2)(P_1)=0$, then

$$|(u_1-u_2,-v_2)|=|D(\varphi_1-\varphi_2)|>\mu_1.$$

Boundary Value Problem ⊗:

Fix $(\varphi,\theta_w)\in C^{2,\alpha}_{*,1+\delta^*}\times[\theta_w^*,\frac{\pi}{2}]$, satisfying properties in iteration set ①–⑤. For such (φ,θ_w) we obtain Ω, Γ_{shock}, Γ_{sonic}. Let $\psi=\varphi-\varphi_2$, $\hat\psi=\hat\varphi-\varphi_2$. The boundary value problem ⊗ for $\hat\psi$ is:

$$\mathcal{N}(\hat\psi)=A_{11}\hat\psi_{\xi_1\xi_1}+2A_{12}\hat\psi_{\xi_1\xi_2}+A_{22}\hat\psi_{\xi_2\xi_2}=0 \qquad \text{in } \Omega.$$

$\cancel{\mathcal{M}(\hat\psi)=0}$

$$\mathcal{M}(D\hat\psi,\hat\psi,\xi_1,\xi_2)=0 \qquad \text{on } \Gamma_{shock}.$$

$$\hat\psi=0 \qquad \text{on } \Gamma_{sonic}.$$

$$\hat\psi_\nu=0 \qquad \text{on } \Gamma_{wedge}.$$

$$\hat\psi_\nu=-v_2 \qquad \text{on } \Gamma_{sym}.$$

Proof of Existence:

Equation for the iteration: Cut-off, Nonlinear-linearization, then the coefficients $A_{ij}(P,\xi_1,\xi_2)$ for $P\in\mathbb{R}^2$, $(\xi_1,\xi_2)\in\Omega$, $i,j=1,2$ satisfy:

① For any $(\xi_1,\xi_2)\in\Omega$ and $P,\mu\in\mathbb{R}^2$,

$$\lambda_0(c_2-r)|\mu|^2\le\sum_{i,j=1}^{2}A_{ij}(P,\xi_1,\xi_2)\mu_i\mu_j\le\lambda_0^{-1}|\mu|^2\ ,\ \text{with}\quad r=\sqrt{\xi_1^2+\xi_2^2}\ ;$$

② $A_{ij}(P,\xi_1,\xi_2)=A^1_{ij}(\xi_1,\xi_2)$ for any $(\xi_1,\xi_2)\in\Omega\setminus D_\varepsilon$ and $P\in\mathbb{R}^2$, where

$$\|A^1_{ij}\|_{C^{-\alpha,\Gamma_{sym}}_{1,\alpha,\overline{\Omega}\setminus D_\varepsilon}}\le C\ ;$$

③ Functions $A_{ij}(P,\xi_1,\xi_2)$ satisfy for each $P\in\mathbb{R}^2$,

$$\|A_{ij}(P,\cdot,\cdot)\|_{C^\alpha(\overline{\Omega\cap D_{2\varepsilon}})}+\|D_pA_{ij}(P,\cdot,\cdot)\|_{L^\infty(\Omega\cap D_{2\varepsilon})}\le C\,,$$

$$A_{ij}(P,\cdot,\cdot)\ ,\ D_p^kA_{ij}(P,\cdot,\cdot)\in C^{1,\alpha}(\overline{\Omega\cap D_{2\varepsilon}}\setminus\overline{\Gamma_{sonic}})\ \text{for}\ k=1,2;$$

④ Coefficients $A_{ij}(P,\xi_1,\xi_2)$ can be extended by continuity to all $(\xi_1,\xi_2)\in\overline{\Omega}$ for each $P\in\mathbb{R}^2$. In particular, for $(\xi_1,\xi_2)\in\overline{\Gamma_{sonic}}$, the coefficients have the following explicit form:

$$A_{11}(P,\xi_1,\xi_2)=c_2^2-\xi_1^2\,,\qquad A_{22}(P,\xi_1,\xi_2)=c_2^2-\xi_2^2\,,\qquad A_{12}(P,\xi_1,\xi_2)=A_{21}(P,\xi_1,\xi_2)=-\xi_1\xi_2\,,$$

where we recall that we work in the shifted coordinates (ξ_1,ξ_2) with origin at O_2.

⑤ If $\hat\psi=\psi$ and $|\hat\psi_x|<\frac{2-\frac{\mu_0}{\varepsilon}}{1+\gamma}x$, then ψ satisfies the potential flow equation.

Proof of Existence:

Equation for the iteration (Continued):

In (x,y)-coordinates: $\hat{A}_{11}\hat{\psi}_{xx} + 2\hat{A}_{12}\hat{\psi}_{xy} + \hat{A}_{22}\hat{\psi}_{yy} + \hat{A}_1\hat{\psi}_x + \hat{A}_2\hat{\psi}_y = 0$,

where $\hat{A}_{ij} = \hat{A}_{ij}(D\hat{\psi}, x, y)$, $\hat{A}_i = \hat{A}_i(D\hat{\psi}, x, y)$, and $\hat{A}_{12} = \hat{A}_{21}$, satisfy:

① For any $(x,y) \in \Omega \cap D_{2\varepsilon}$, and $p, \kappa \in \mathbb{R}^2$,

$$\lambda_1 |\kappa|^2 \le \sum_{i,j=1}^{2} \hat{A}_{ij}(p,x,y)\frac{\kappa_i \kappa_j}{x^{2-\frac{i+j}{2}}} \le \frac{1}{\lambda_1}|\kappa|^2 \ ;$$

② For any $p \in \mathbb{R}^2$,

$$\|(\hat{A}_{ij}, \hat{A}_i)(p,\cdot,\cdot)\|_{C^\alpha(\overline{\Omega \cap D_{2\varepsilon}})} + \|(D_p\hat{A}_{ij}, D_p\hat{A}_i)(p,\cdot,\cdot)\|_{L^\infty(\Omega \cap D_{2\varepsilon})} \le C.$$

③ $\hat{A}_{11}$, $\hat{A}_{22}$, and $\hat{A}_1$ are independent of p_2, and

$$|\hat{A}_{ii}(p,x,y) - \hat{A}_{ii}(0,0,\tilde{y})| \le C|x|^\alpha \quad \text{for all } p \in \mathbb{R}^2, \ (x,y), (0,\bar{y}) \in \overline{\Omega \cap D_{2\varepsilon}}.$$

④ $\hat{A}_{12}$, $\hat{A}_{21}$, and $\hat{A}_2$ are independent of p, and

$$|(\hat{A}_{12}, \hat{A}_{21}, \hat{A}_2)(x,y)| \le C|x|^{\frac{1}{2}+\delta^*}, \quad \|\hat{A}_{12}, \hat{A}_{21}, \hat{A}_2\|_{C^\alpha(\overline{\Omega \cap D_{2\delta}})} \le C|\delta|^{\frac{1}{2}+\delta^*-\alpha} \ \text{for all } \delta \in (0,\varepsilon).$$

⑤ For each $(x_0,y_0) \in \Omega \cap D_{2\varepsilon}$ define

$$\hat{A}_{ij}^{(x_0,y_0)}(P_1,P_2,S,T) = x_0^{\frac{i+j}{2}-2}\hat{A}_{ij}(4x_0P_1, 4x_0^{\frac{3}{2}}P_2, x, y),$$

$$\hat{A}_i^{(x_0,y_0)}(P_1,P_2,S,T) = \frac{1}{4}x_0^{\frac{i-1}{2}}\hat{A}_i(4x_0P_1, 4x_0^{\frac{3}{2}}P_2, x, y), \quad \text{where } x = x_0(1+\tfrac{S}{4}), \ y = y_0 + \tfrac{\sqrt{x_0}}{4}T,$$

in the domain $Q_{1,1}^{(x_0,y_0)}$, satisfy:

$$\|\hat{A}_{ij}^{(x_0,y_0)}(P,\cdot,\cdot)\|_{C^{1,\alpha}(\overline{Q_{1,1}^{(x_0,y_0)}})} \le \lambda^{-1} \quad \text{for all } P \in \mathbb{R}^2,$$

$$\|D_{P,z}(\hat{A}_{ij}^{(x_0,y_0)}, \hat{A}_i^{(x_0,y_0)})(P,\cdot,\cdot)\|_{C^{1,\alpha}(\overline{Q_{1,1}^{(x_0,y_0)}})} \le \lambda^{-1} \quad \text{for all } P \in \mathbb{R}^2.$$

Proof of Existence:

The iteration set is open:

① The set K^{ext} is relatively open in $C^{2,\alpha}_{*,1+\delta^*}(\Omega)\times[\theta_w^*, \frac{\pi}{2}]$, where K^{ext} is the set of all $(\varphi, \theta_w)\in C^{2,\alpha}_{*,1+\delta^*}(\Omega)\times[\theta_w^*, \frac{\pi}{2}]$ which satisfy properties ①-⑤ of the definition of the iteration set K.

② For any $(\varphi^\#, \theta_w^\#)\in K$, there exists $\delta^\#>0$ small so that for any $(\varphi, \theta_w)\in K^{ext}$ which satisfy $\|\varphi^\#-\varphi\|_{C^1(\bar\Omega)}+\|\theta_w^\#-\theta_w\|\leq\delta^\#$, the solution $\hat\varphi$ of boundary value problem by (φ, θ_w), satisfies

$\|\hat\varphi\|^{*,2}_{2,\alpha,\Omega}<C$, where C depends only on the data and θ_w^*.

③ For $\delta^\#$ again small enough, the solution $\hat\varphi$ above actually satisfies

$$\|\hat\varphi-\varphi\|^{*,1+\delta^*}_{2,\frac{\alpha}{2},\Omega}<\delta_3,$$

when $|\theta_w-\theta_w^\#|$ small enough.

Thus the iteration set K is open.

Proof of Existence:

Extension of functions in weighted spaces:

Let $a>0$, $g\in C^{0,1}([0,a])$ with $g(s)>0$ on $[0,a]$. Let

$$\Omega^a_\infty=(0,a)\times(0,+\infty),\quad \Omega^a_g=\{(s,t)\in\Omega^a_\infty \mid 0<t<g(s)\}.$$

Furthermore, let $\Gamma^a_g=\{(s,g(s))\mid s\in(0,a)\}$.

For real $\sigma>0$, $\alpha\in(0,1)$, $\varepsilon>0$ denote

$$\|u\|^{*,\sigma}_{2,\alpha,\Omega^a_g}=\|u\|^{-(1+\alpha),\{a\}\times(0,+\infty)}_{2,\alpha,\Omega^a_g\cap\{s>\frac{\varepsilon}{10}\}}+\|u\|^{\sigma,(par)}_{2,\alpha,\Omega^a_\sigma\cap\{s<\varepsilon\}},$$

where parabolic norms are with respect to $\{s=0\}$.

Denote by $C^{2,\alpha}_{*,\sigma}(\Omega^a_g)$ the space

$$C^{2,\alpha}_{*,\sigma}(\Omega^a_g)=\{u\in C^1(\overline{\Omega^a_g})\cap C^2(\Omega^a_g)\mid \|u\|^{*,\sigma}_{2,\alpha,\Omega^a_g}<+\infty\}.$$

Th4: Let $a>0$. Let $g\in C^{0,1}([0,a])$ with $g(s)\geq\lambda>0$ on $[0,a]$. Then there exists an operator $\mathcal{E}^{(a)}_*: C^2(\Omega^a_g\cup\Gamma^a_g)\to C^2(\Omega^a_\infty)$ such that $\mathcal{E}^{(a)}_*(u)\equiv u$ on Ω^a_g for any $u\in C^2(\Omega^a_g\cup\Gamma^a_g)$, and furthermore, for any $\sigma>0$, $\alpha\in(0,1)$ and $\varepsilon>0$,

① $\mathcal{E}^{(a)}_*$ is a linear continuous operator from $C^{2,\alpha}_{*,\sigma}(\Omega^a_g)$ to $C^{2,\alpha}_{*,\sigma}(\Omega^a_g)$;

② There exists C depending only on $Lip[g]$, λ, α, ε, σ, a such that

$$\|\mathcal{E}^{(a)}_*(u)\|^{*,\sigma}_{2,\alpha,\Omega^a_\infty}\leq C\|u\|^{*,\sigma}_{2,\alpha,\Omega^a_g},\ \text{for all } u\in C^{2,\alpha}_{*,\sigma}(\Omega^a_g);$$

Proof of Existence:

Extension of functions in weighted spaces:

③ $\mathcal{E}_*^{(a)}(u) \equiv 0$ for all $t > m$, where $m > 0$ depends only on $\|g\|_{C^{0,1}([0,a])}, \lambda$;

④ If $g_i \in C^{0,1}([0,a_i])$, $g \in C^{0,1}([0,a])$, where $a_i \to a$, $g_i \to g$ uniformly on compact subsets of $[0,a]$, and if $u_i \in C_{*,\sigma}^{2,\alpha}(\Omega_{g_i}^a)$, $u \in C_{*,\sigma}^{2,\alpha}(\Omega_g^a)$ and $u_i \to u$ in C^2 on compact subsets of the open set Ω_g^a, and if there exists $L, M > 0$ such that $\mathrm{Lip}[g_i] \leq L$ and $\|u_i\|_{2,\alpha,\Omega_{g_i}^a}^{*,\sigma} \leq M$ for all i, then denoting $w_i(s,t) := \mathcal{E}_{*,g_i}^{(a_i)}(u_i)(\frac{a}{a_i}s, t)$, we have

$$w_i \to \mathcal{E}_{*,g}^{(a)}(u)$$

in $C_{*,\sigma'}^{2,\beta}(\Omega_\infty^a)$ for all $\beta \in (0,\alpha)$ and $\sigma' \in (0,\sigma)$.

The iteration map:

Iteration map $I: K \to C_{*,1+\delta^*}^{2,\alpha}(Q)$ is defined by $I^{(\Theta_w)}(\varphi) = \hat{\varphi}$, where $\hat{\varphi}$ is the solution of boundary value problem solved by $(\varphi, \Theta_w) \in K$, and then is extended to the fixed domain Q.

Compactness and continuity of the iteration map:

By elliptic estimates and uniqueness.

Proof of Existence:

Degree:

① $I^{(\frac{\pi}{2})}(\varphi) \equiv \varphi_2 \quad \Rightarrow \deg(I^{(\frac{\pi}{2})}; K) \neq 0$

② For any fixed point, i.e. $I^{(\theta_w)}(\varphi) = \varphi$,

φ is a admissible solution. (by uniform estimates).

$\Rightarrow$ Fixed points do not lies in ∂K.

③ $\{I^{(\theta_w)}\}$ is homotopy with respect to θ_w.

④ $I^{(\theta_w)}$ are compact for any θ_w.

$\Rightarrow$ Admissibe solutions exist for all $\theta_w \in [\theta_w^*, \frac{\pi}{2}]$.

Remark 10: For ②, we need to prove all the estimates listed in the definition of the transonic admissible solution and also need to remove all the cut-off functions we added to make sure that the solutions we obtained are the solutions to the potential flow equation.

Uniform estimates:

① The strict monotonicity for $\varphi_1-\varphi$:

$$\partial_e(\varphi_1-\varphi)<0 \quad \text{in } \overline{\Omega} \quad \text{for all } \quad e\in \mathrm{Cone}^\circ(e_{S_1}, e_\eta).$$

$\Rightarrow$ The shock is a graph for a cone of direction.

② The monotonicity cone for $\varphi-\varphi_2$:

$$\partial_e(\varphi-\varphi_2)\geq 0 \quad \text{in } \overline{\Omega} \quad \text{for all} \quad e\in \mathrm{Cone}^\circ(e_{S_1}, -\nu_{wedge}).$$

$\Rightarrow$ $\partial_x(\varphi-\varphi_2)>0$ near Γ_{sonic}, where $x=c_2-r$.

③ Uniform estimate of the size of Ω, the Lipschitz norm of the potential, and the density from above and below:

$$\Omega\subset B_C(0), \qquad \|\varphi\|_{C^{0,1}(\overline{\Omega})}\leq C,$$

$$a\rho_1\leq \rho\leq C \quad \text{in } \Omega \qquad \text{with} \quad a=\left(\frac{2}{\gamma+1}\right)^{\frac{1}{\gamma-1}}>0,$$

$$\rho_1<\rho\leq C \qquad \text{on} \quad \overline{\Gamma_{shock}\cup\{P_3\}}.$$

④ Uniform positive bound for the distance between the shock and the sonic circle of state (1): $\mathrm{dist}(\Gamma_{shock}, B_{c_1}(O_1))>\frac{1}{C}>0.$

⑤ Uniform positive lower bound for the distance between the shock and the wedge, and separation of the shock and the symmetry line:

1): $f'_{shock}(\xi)>\mu$ on $\{\xi\in(\xi_{P_2}, \min(\xi_{P_1}, 0)) : f_{shock}(\xi)\in(0, \frac{c_1}{2})\}$.

2): $\mathrm{dist}(\Gamma_{shock}, \Gamma_{wedge})>\frac{1}{C}$.

Uniform estimates:

⑥ Uniform estimate of the ellipticity of equation in Ω up to the shock:

Let $M^2 = \frac{|D\varphi|^2}{c^2}$, then

$$M^2(\xi,\eta) \le 1 - \mu\, \mathrm{dist}((\xi,\eta), \Gamma_{sonic}), \text{ for all } (\xi,\eta) \in \Omega(\varphi).$$

⑦ Regularity estimates:

1) Regularity estimates away from the sonic arc:

uniform ellipticity estimates with weight.

free boundary (partial hodograph).

2) Regularity estimates near sonic arc:

$$(\varphi - \varphi_2)_x \le \frac{2-\delta}{1+\gamma} x ,$$

$$|\varphi - \varphi_2| \le C x^2 \Rightarrow \quad \|\varphi - \varphi_2\|^{(par)}_{2,\alpha, N_\varepsilon(\Gamma_{sonic}) \cap \Omega} \le C.$$

① – ⑦: More than ~~[illegible]~~ 200 pages proof.

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