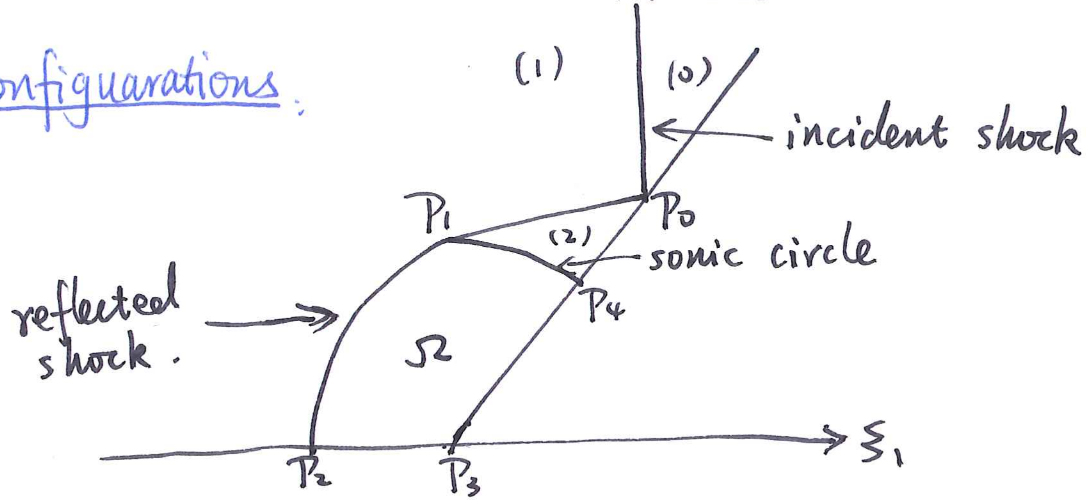
Mathematics of Shock Reflection-Diffraction Problem · II

Configuarations:



Equations in Ω: $\operatorname{div}\left(\rho(|D\varphi|^2,\varphi)D\varphi\right)+2\rho(|D\varphi|^2,\varphi)=0.$

with $\rho(|D\varphi|^2,\varphi)=\left(\rho_0^{\gamma-1}-(\gamma-1)(\varphi+\frac{1}{2}|D\varphi|^2)\right)^{\frac{1}{\gamma-1}}$.

Equation is elliptic if and only if

$$|D\varphi|<c_*(\varphi,\rho_0,\gamma):=\sqrt{\frac{2}{\gamma+1}(\rho_0^{\gamma-1}-(\gamma-1)\varphi)}.$$

Uniform states: $\varphi_0(\xi) = -\frac{|\xi|^2}{2}$ for $\xi_1 > \xi_1^0$

$\varphi_1(\xi) = -\frac{|\xi|^2}{2} + u_1(\xi_1 - \xi_1^0)$ for $\xi_1 < \xi_1^0$,

Location of the incident shock:

$$\xi_1^0 = \rho_1 \sqrt{\frac{2(\rho_1^{\gamma-1} - \rho_0^{\gamma-1})}{(\gamma-1)(\rho_1^2 - \rho_0^2)}} = \frac{\rho_1 u_1}{\rho_1 - \rho_0} > 0.$$

$$\varphi_2(\xi) = -\frac{|\xi|^2}{2} + u_2(\xi_1 - \xi_1^0) + u_2 \operatorname{tg}\theta_w (\xi_2 - \xi_1^0 \operatorname{tg}\theta_w),$$

where $u_2 > 0$, $\rho_2 > \rho_1$.

Boundary conditions:

On Γ_{wedge}: $D\varphi \cdot \nu = 0$,

On Γ_{sonic}: $\varphi = \varphi_2$ (Should prove $D\varphi = D\varphi_2$).

On Γ_{shock}: $[\varphi]_S = 0$ (To determine the free boundary).

$[\rho(|D\varphi|^2, \varphi) D\varphi \cdot \nu]_S = 0$ (Boundary condition for φ).

Additional requirment:

To define ρ: $\rho_0^{\gamma-1} - (\gamma-1)(\varphi + \frac{1}{2}|D\varphi|^2) > 0$.

Boundary value problem for the iteration:

$$\mathcal{N}(\hat\psi) := A_{11}\hat\psi_{\xi_1\xi_1} + 2A_{12}\hat\psi_{\xi_1\xi_2} + A_{22}\hat\psi_{\xi_2\xi_2} = 0 \qquad \text{in } \Omega.$$

$$\mathcal{M}(D\hat\psi, \hat\psi, \xi_1, \xi_2) = 0 \qquad \text{on } \Gamma_{shock}$$

$$\hat\psi = 0 \qquad \text{on } \Gamma_{sonic}$$

$$\hat\psi_\nu = 0 \qquad \text{on } \Gamma_{wedge}$$

$$\hat\psi_\nu = -v_2 \qquad \text{on } \Gamma_{sym}.$$

where $\hat\psi = \hat\varphi - \varphi_2$, $\psi = \varphi - \varphi_2$. For each $(\varphi, \theta_w) \in K$, we can define Ω, Γ_{sonic}, Γ_{shock}.

Equations:

Goal: ① It is strictly elliptic inside the domain Ω with elliptic degeneracy at the sonic arc Γ_{sonic};

② For a fixed point $\hat\psi = \psi$, equation coincides with the original equation.

In $\Omega \setminus D_{\varepsilon/10}$: $(c^2 - \varphi_{\xi_1}^2)\hat\psi_{\xi_1\xi_1} - 2\varphi_{\xi_1}\varphi_{\xi_2}\hat\psi_{\xi_1\xi_2} + (c^2 - \varphi_{\xi_2}^2)\hat\psi_{\xi_2\xi_2} = 0.$

with $c^2 = c^2(|D\varphi|^2, \varphi, \xi_1, \xi_2)$.

In $D_{\varepsilon/10}$: the original equation $\Leftrightarrow I_1 + I_2 + I_3 + I_4 = 0$.

where
$$I_1 := (c^2(D\hat\psi, \hat\psi, \xi_1, \xi_2) - (\xi_1^2 + \xi_2^2))\Delta\hat\psi,$$
$$I_2 := \xi_2^2\hat\psi_{\xi_1\xi_1} + \xi_1^2\hat\psi_{\xi_2\xi_2} - 2\xi_1\xi_2\hat\psi_{\xi_1\xi_2},$$
$$I_3 := 2(\xi_1\hat\psi_{\xi_1}\hat\psi_{\xi_1\xi_1} + (\xi_1\hat\psi_{\xi_2} + \xi_2\hat\psi_{\xi_1})\hat\psi_{\xi_1\xi_2} + \xi_2\hat\psi_{\xi_2}\hat\psi_{\xi_2\xi_2}),$$
$$I_4 := -\frac{1}{2}(\hat\psi_{\xi_1}(|D\hat\psi|^2)_{\xi_1} + \hat\psi_{\xi_2}(|D\hat\psi|^2)_{\xi_2})$$

Equations (Continued):

In polar coordinate:

$$I_1 = \left(c_2^2 - r^2 + (\gamma-1)\left(r\hat{\varphi}_r - \frac{1}{2}|D\hat{\varphi}|^2 - \hat{\varphi}\right)\right)\Delta\hat{\varphi},$$

$$I_2 = \hat{\varphi}_{\theta\theta} + r\hat{\varphi}_r,$$

$$I_3 = 2r\hat{\varphi}_r\hat{\varphi}_{rr} + \frac{2}{r}\hat{\varphi}_\theta\hat{\varphi}_{r\theta} - \frac{2}{r^2}\hat{\varphi}_\theta^2,$$

$$I_4 = -\hat{\varphi}_r^2\hat{\varphi}_{rr} - \frac{1}{2}\left(\hat{\varphi}_r\left(\frac{1}{r^2}\hat{\varphi}_\theta^2\right)_r + \frac{1}{r^2}\hat{\varphi}_\theta\left(|D\hat{\varphi}|^2\right)_\theta\right)$$

$\therefore$ the term $(\gamma+1)\hat{\varphi}_x$ in the coefficient of $\hat{\varphi}_{xx}$ in the sum of the coefficient $(\gamma-1)r\hat{\varphi}_r$ in I_1 and $2r\hat{\varphi}_r$ in I_3.

Cut-off:

$$\xi_1(s) = \begin{cases} s, & \text{if } |s| < \dfrac{2-\frac{\mu_0}{5}}{1+\gamma} \\[2ex] \dfrac{2-\frac{\mu_0}{10}}{1+\gamma}\operatorname{sign}(s), & \text{if } |s| > \dfrac{2}{\gamma+1}. \end{cases}$$

so that $\quad \xi_1'(s) \geq 0, \quad \xi_1(-s) = -\xi_1(s) \quad$ on $\mathbb{R}$;

$\quad \xi_1''(s) \leq 0 \quad$ on $\{s \geq 0\}$.

New equation: $\quad \hat{I}_1 + I_2 + \hat{I}_3 + \hat{I}_4 = 0,$

where

$$\hat{I}_1 := \left(c_2^2 - r^2 + (\gamma-1)r(c_2-r)\xi_1\left(\frac{\hat{\varphi}_r}{c_2-r}\right) - (\gamma-1)\left(\frac{1}{2}(c_2-r)^2\left(\xi_1\left(\frac{\hat{\varphi}_r}{c_2-r}\right)\right)^2 + \frac{1}{2r^2}\hat{\varphi}_\theta^2 + \hat{\varphi}\right)\right)\Delta\hat{\varphi},$$

$$\hat{I}_3 := 2r(c_2-r)\xi_1\left(\frac{\hat{\varphi}_r}{c_2-r}\right)\hat{\varphi}_{rr} + \frac{2}{r}\hat{\varphi}_\theta\hat{\varphi}_{r\theta} - \frac{2}{r^2}\hat{\varphi}_\theta^2.$$

$$\hat{I}_4 := -\left((c_2-r)\xi_1\left(\frac{\hat{\varphi}_r}{c_2-r}\right)\right)^2\hat{\varphi}_{rr} - \frac{1}{2}\left(\hat{\varphi}_r\left(\frac{1}{r^2}\hat{\varphi}_\theta^2\right)_r + \frac{1}{r^2}\hat{\varphi}_\theta\left(|D\hat{\varphi}|^2\right)_\theta\right).$$

Equations (Continued).

The modified equation coincides with the original equation if

$$|\hat{\psi}_r| < \frac{2-\frac{\mu_0}{\xi}}{1+\gamma}(c_2 - r).$$

In (x, y)-coordinate:

$$\left(2x - (\gamma+1)x\zeta_1\left(\frac{\hat{\psi}_x}{x}\right) + O_1^{\hat{\psi}}\right)\hat{\psi}_{xx} + O_2^{\hat{\psi}}\hat{\psi}_{xy} + \left(\frac{1}{c_2} + O_3^{\hat{\psi}}\right)\hat{\psi}_{yy}$$

$$-\left(1 + O_4^{\hat{\psi}}\right)\hat{\psi}_x + O_5^{\hat{\psi}}\hat{\psi}_y = 0,$$

with $|\hat{O}_1^{\hat{\psi}}(p, x, y)| \leq C|x|^{1+\delta^*}$, $|\hat{O}_k^{\hat{\psi}}(x, y)| \leq C|x|^{\frac{1}{2}+\delta^*}$ for $k = 2, \cdots, 5$.

for all $p \in \mathbb{R}^2$ and $(x, y) \in \Omega \cap D_\varepsilon$.

Remark: The modified equations satisfy all the properties listed in the previous lecture notes.

Boundary conditions on Γ_{shock}:

Goal: ① $\mathcal{M}(p, z, \xi)$ is a C^3-smooth as a function of (p, z, ξ), and does not depends on the smallness of Γ_{shock}.

② $\mathcal{M}(0, 0, \xi) = 0$ ($\mathcal{M}(p, z, \xi)$ is homogeneous).

③ For a fixed point $\psi = \hat{\psi}$, the boundary condition coincides with Rankine-Hugoniot condition between φ and φ_1 on Γ_{shock}.

Boundary conditions on Γ_{shock}:

R.-H. Condition: $\left(\rho(|D\varphi|^2,\varphi)D\varphi - \rho_1 D\varphi_1\right)\cdot \nu_{shock}=0$ on Γ_{shock}.

Notice $\varphi=\varphi_1$ on Γ_{shock} $\Rightarrow$ $\nu_{shock}=\dfrac{D(\varphi_1-\varphi)}{|D(\varphi_1-\varphi)|}$.

$$\Rightarrow\ M_0(P,z,\xi_1,\xi_2)=\left(\rho(P,z,\xi_1,\xi_2)(P+D\varphi_2(\xi_1,\xi_2))-\rho_1 D\varphi_1(\xi_1,\xi_2)\right)\cdot\frac{D(\varphi_1-\varphi_2)-P}{|D(\varphi_1-\varphi_2)-P|},$$

where $\rho(P,z,\xi_1,\xi_2)=\left(\rho_2^{\gamma-1}+(\gamma-1)(\xi_1P_1+\xi_2P_2-\frac{|P|^2}{2}-z)\right)^{\frac{1}{\gamma-1}}$,

then $\boxed{M_0(D\hat{\psi},\hat{\psi},\xi)=0}$ on Γ_{shock},

where $\hat{\psi}=\hat{\varphi}-\varphi_2$.

Homogeneous: Notice that $S_1=\{\varphi_1=\varphi_2\}$ and $P_1\in S_1$,

$\therefore\ (\varphi_1-\varphi_2)(\xi)=|(u_1-u_2,-v_2)|\,\nu_{S_1}\cdot(\xi-\xi_{P_1})$,

Moreover, $\varphi=\varphi_1$ on Γ_{shock}

$\Rightarrow\ \varphi_1-\varphi_2=\psi$ on Γ_{shock}.

$$\Rightarrow\ \xi\cdot\nu_{S_1}=\xi_{P_1}\cdot\nu_{S_1}+\frac{\psi(\xi)}{\sqrt{(u_1-u_2)^2+v_2^2}}\quad\text{on } \Gamma_{shock}.$$

$\Rightarrow\ \xi=F(\psi(\xi),\xi)$ on Γ_{shock},

$$\text{where } F(z,\xi)=\left(\xi_{P_1}\cdot\nu_{S_1}+\frac{\psi(\xi)}{\sqrt{(u_1-u_2)^2+v_2^2}}\right)\nu_{S_1}+(\xi\cdot\tau_{S_1})\tau_{S_1}.$$

Boundary conditions on Γ_{shock}:

Define: $$\mathcal{M}_1(P, \psi(\xi), \xi) = \mathcal{M}_0(P, \psi(\xi), F(\psi(\xi), \xi)).$$

Notice that $F(0,\xi) = (\xi_{P_1} \cdot \nu_{S_1})\nu_{S_1} + (\xi \cdot \tau_{S_1})\tau_{S_1} \in S_1$.

$$\Rightarrow \mathcal{M}_1(0, 0, \xi) = \mathcal{M}_0(0, 0, F(0,\xi))$$
$$= (\rho_2 D\varphi_2(F(0,\xi)) - \rho_1 D\varphi_1(F(0,\xi))) \cdot \nu_{S_1} = 0.$$

Cut-off: Let $\eta \in C^\infty(R)$, $\eta \equiv 1$ on $(-\infty, \frac{\sigma_1}{2})$, $\eta \equiv 0$ on (σ_1, ∞), and $\eta' \le 0$, then

$$\mathcal{M}(P, z, \xi) = (1 - \eta(|(P,z)|))\mathcal{M}_0(P, z, \xi) + \eta(|(P,z)|)\mathcal{M}_1(P, z, \xi)$$
for $(P, z, \xi) \in A_{\mathcal{M}}$.

Remark: Boundary conditions $\mathcal{M}(D\psi(\xi), \psi(\xi), \xi) = 0$ on Γ_{shock} satisfy all the requirement we expected.

Removing the cut-off:

In (x, y)-coordinate,

$$\Omega \cap D_\varepsilon = \{0 < x < \varepsilon,\ 0 < y < f_{shock}(x)\} \quad \text{for all } \varepsilon \in (0, \varepsilon_0).$$

$\psi_x \le \dfrac{2 - \frac{\mu_0}{5}}{1+\gamma} x$ in $\Omega \cap \{x \le \varepsilon\}$:

Let $A = \dfrac{2 - \frac{\mu_0}{5}}{1+\gamma}$, and let $v(x,y) = Ax - \psi_x(x,y)$.

$$\|\psi\|^{*,2}_{2,\alpha+\alpha',\Omega} \le C \Rightarrow |\psi(x,y)| \le Cx^2, \quad |\psi_x(x,y)| \le Cx, \quad |\psi_y(x,y)| \le Cx^{\frac{3}{2}}.$$

$\Rightarrow$ $\boxed{v = 0 \quad \text{on } \partial\Omega_\varepsilon \cap \{x=0\}}$.

and $|\psi_x| \le C(|\psi_y| + |\psi|) \le Cx^{\frac{3}{2}}$

$\Rightarrow$ $\boxed{v \ge 0 \quad \text{on } \Gamma_{shock} \cap \{0 < x < \varepsilon\}}$

Furthermore, $\psi_y = 0$ on Γ_{wedge}

$\Rightarrow$ $\boxed{v_y = 0 \quad \text{on } \Gamma_{wedge} \cap \{0 < x < \varepsilon\}}$

$\partial_x(\varphi - \varphi_2) < \dfrac{2-\mu_0}{1+\gamma} x$ in $\Omega_\varepsilon \setminus \Omega_{\varepsilon/10}$

$\Rightarrow$ $\boxed{v > 0 \quad \text{on } \partial\Omega \cap \{x = \tfrac{\varepsilon}{2}\}}$

Removing the cut-off:

$\psi_x \le \frac{2-\frac{\mu_0}{\xi}}{1+\gamma} x$ (Continued):

Equation:

$$a_{11} V_{xx} + a_{12} V_{xy} + a_{22} V_{yy} + b V_x + c V = -A((\gamma+1)A-1) + E(x,y).$$

where $a_{11} = 2x - (\gamma+1) x \xi_1 (\frac{\psi_x}{x}) + \hat{O}_1$, $\quad a_{12} = \hat{O}_2$, $\quad a_{22} = \frac{1}{C_2} + \hat{O}_3$.

$$c(x,y) = (\gamma+1)\frac{A}{x}\left(\xi_1'(A - \frac{v}{x}) - \int_0^1 \xi_1'(A - s\frac{v}{x})\,ds\right).$$

Notice that $\quad a_{11} \ge \frac{1}{6} x$, $\quad a_{22} \ge \frac{1}{2C_2}$, $\quad |a_{12}| \le \frac{1}{3\sqrt{C_2}} x^{\frac{1}{2}}$.

$$\therefore \quad 4 a_{11} a_{22} - (a_{12})^2 \ge \frac{2}{9C_2} x.$$

Furthermore: $\quad c(x,y) \le 0$.

$|E(x,y)| \le C x^{\frac{1}{2}}$, $\quad (\gamma+1)A > 1$.

$\Rightarrow \quad a_{11} V_{xx} + a_{12} V_{xy} + a_{22} V_{yy} + b V_x + c V < 0 \quad$ in $\Omega_{\varepsilon/2}$.

$\Rightarrow \quad V \ge 0 \quad$ on $\Omega_{\varepsilon/2}$

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Removing the cut-off:

$\psi_x \geq -\dfrac{2-\frac{\mu_0}{5}}{1+\nu} x \quad$ in $\Omega \cap D_\varepsilon$:

Since $S_1 = \{\varphi_1 = \varphi_2\}$, then $\partial_{e_{S_1}} \varphi_1 = \partial_{e_{S_1}} \varphi_2$, thus $\partial_{e_{S_1}}(\varphi - \varphi_2) = -\partial_{e_{S_1}}(\varphi_1 - \varphi)$.

$\therefore \ \partial_{e_{S_1}}(\varphi_1 - \varphi) \leq 0 \Rightarrow \boxed{\partial_{e_{S_1}} \psi \geq 0 \quad \text{in } \Omega}$

Notice that $\quad \partial_{e_{S_1}} \psi = (e_{S_1} \cdot e_x)\psi_x + (e_{S_1} \cdot e_y)\psi_y \quad$ in $\Omega \cap D_\varepsilon$,

where $e_x = -(\cos\theta, \sin\theta)$, $\quad e_y = (-\sin\theta, \cos\theta)$.

$\frac{M}{2} \geq -\partial_y(\varphi_1 - \varphi_2) \geq \frac{2}{M} \Rightarrow \frac{M}{2} \geq -a\,\nu_{S_1} \cdot e_y \geq \frac{2}{M} > 0$. $\quad \left(\because \nu_{S_1} = \dfrac{D(\varphi_1 - \varphi_2)}{|D(\varphi_1 - \varphi_2)|} := \dfrac{D(\varphi_1 - \varphi_2)}{a}\right)$.

$$\Rightarrow \boxed{\frac{M}{2} \geq -a\, e_{S_1} \cdot e_x \geq \frac{2}{M} \geq 0 \quad \text{in } \Omega \cap D_\varepsilon}$$

$$\therefore \quad \psi_x = \frac{\partial_{e_{S_1}} \psi}{e_{S_1} \cdot e_x} - \frac{e_{S_1} \cdot e_y}{e_{S_1} \cdot e_x} \psi_y \geq -\frac{e_{S_1} \cdot e_y}{e_{S_1} \cdot e_x} \psi_y \geq -\frac{aM}{2} c x^{\frac{3}{2}} \geq -\frac{2-\frac{\mu_0}{5}}{1+\nu} x .$$

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