Existence, Uniqueness and Representation of Second Order Backward SDEs With Jumps

Nabil Kazi-Tani (Ecole Polytechnique and AXA Group Risk Management)
Joint work with Dylan Possamai and Chao Zhou
July 3rd 2012, Oxford
We want to define a notion of model uncertainty in a model with jumps.

One possible solution is to define 2nd order BSDEs with jumps.

A backward SDE with jumps, in the standard Lipschitz case, takes the following form:

\[ Y_t = \xi + \int_t^T F_s(Y_s, Z_s, U_s) \, ds - \int_t^T Z_s \, dB^c_s - \int_t^T \int_E U_s(x)(\mu_B^d - \nu)(ds, dx), \quad \mathbb{P}\text{-a.s.} \]

We shall consider the following second order backward SDE with jumps (2BSDEJ for short), for $0 \leq t \leq T$

\[ Y_t = \xi + \int_t^T F_s(Y_s, Z_s, U_s, \alpha, \nu)\,ds - \int_t^T Z_s\,dB_s^c \]

\[ - \int_t^T \int_E U_s(x)(\mu_{Bd} - \nu)(ds, dx) + K_T - K_t, \quad \mathbb{P}\text{-a.s}, \forall \mathbb{P} \in \mathcal{P}. \]
We shall consider the following second order backward SDE with jumps (2BSDEJ for short), for $0 \leq t \leq T$

$$Y_t = \xi + \int_t^T F_s(Y_s, Z_s, U_s, \alpha, \nu) ds - \int_t^T Z_s dB^c_s$$
$$- \int_t^T \int_E U_s(x)(\mu_B - \nu)(ds, dx) + K_T - K_t, \text{ } \mathbb{P}\text{-a.s }, \forall \mathbb{P} \in \mathcal{P}. $$

What are the measures $\mathbb{P} \in \mathcal{P}$ ?
What are the measures $\mathbb{P} \in \mathcal{P}$?

$B$ is the canonical process defined on $\Omega = \mathcal{D}([0, T], \mathbb{R}^d)$. For $\alpha$ and $\nu$ satisfying mild integrability conditions, let $\mathbb{P}^{\alpha, \nu}$ be a probability measure on $\mathcal{D}$ such that $B$ is a semimartingale under $\mathbb{P}^{\alpha, \nu}$ with characteristics

$$
\left(- \int_0^T \int_E x 1_{|x| > 1} \nu_s(dx)ds, \int_0^T \alpha_s ds, \nu_s(dx)ds\right).
$$

2BSDEs with jumps
What are the measures $\mathbb{P} \in \mathcal{P}$?

$B$ is the canonical process defined on $\Omega = \mathbb{D}([0, T], \mathbb{R}^d)$.
For $\alpha$ and $\nu$ satisfying mild integrability conditions, let $\mathbb{P}^{\alpha, \nu}$ be a probability measure on $\mathbb{D}$ such that $B$ is a semimartingale under $\mathbb{P}^{\alpha, \nu}$ with characteristics

$$\left(- \int_0^T \int_E x 1_{|x| > 1} \nu_s(dx) ds, \int_0^T \alpha_s ds, \nu_s(dx) ds \right).$$

$\mathbb{P}^{\alpha, \nu}$ is the solution to the martingale problem on $\mathbb{D}$ associated to $(\alpha, \nu)$. 
We shall consider the following second order backward SDE with jumps (2BSDEJ for short), for $0 \leq t \leq T$

$$Y_t = \xi + \int_t^T F_s(Y_s, Z_s, U_s, \alpha, \nu) ds - \int_t^T Z_s dB_s^c$$

$$- \int_t^T \int_E U_s(x)(\mu_B d - \nu)(ds, dx) + K_T - K_t, \ \mathbb{P}^{\alpha,\nu}-a.s, \ \forall \mathbb{P}^{\alpha,\nu} \in \mathcal{P}.$$
The form of the equation

We shall consider the following second order backward SDE with jumps (2BSDEJ for short), for $0 \leq t \leq T$

$$Y_t = \xi + \int_t^T F_s(Y_s, Z_s, U_s, \alpha, \nu) \, ds - \int_t^T Z_s \, dB^c_s$$

$$- \int_t^T \int_E U_s(x)(\mu_B - \nu)(ds, dx) + K_T - K_t, \quad \mathbb{P}^{\alpha, \nu} \text{-a.s}, \forall \mathbb{P}^{\alpha, \nu} \in \mathcal{P}.$$
We shall consider the following second order backward SDE with jumps (2BSDEJ for short), for $0 \leq t \leq T$

\[
Y_t = \xi + \int_t^T F_s(Y_s, Z_s, U_s, \hat{\alpha}, \hat{\nu}) ds - \int_t^T Z_s dB_s^c \\
- \int_t^T \int_E U_s(x)(\mu B^d - \hat{\nu})(ds, dx) + K_T - K_t, \quad \mathbb{P}^{\alpha, \nu}-a.s, \forall \mathbb{P}^{\alpha, \nu} \in \mathcal{P}.
\]
We shall consider the following second order backward SDE with jumps (2BSDEJ for short), for $0 \leq t \leq T$

\[
Y_t = \xi + \int_t^T F_s(Y_s, Z_s, U_s, \hat{\alpha}, \hat{\nu}) ds - \int_t^T Z_s dB_s^c \\
- \int_t^T \int_E U_s(x)(\mu_Bd - \hat{\nu})(ds, dx) + K_T - K_t, \ \mathbb{P}^{\alpha,\nu}-a.s, \forall \mathbb{P}^{\alpha,\nu} \in \mathcal{P}.
\]

$(\hat{\alpha}, \hat{\nu}) = (\alpha, \nu), \ \mathbb{P}^{\alpha,\nu}-a.s, \forall \mathbb{P}^{\alpha,\nu} \in \mathcal{P}.$
Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a given measurable space. Let \(\mathcal{P}\) be a set of non necessarily dominated probability measures and let \(\{X^\mathbb{P}, \mathbb{P} \in \mathcal{P}\}\) be a family of random variables indexed by \(\mathcal{P}\).

**Definition**

An *aggregator* of the family \(\{X^\mathbb{P}, \mathbb{P} \in \mathcal{P}\}\) is a random variable \(\hat{X}\) such that

\[
\hat{X} = X^\mathbb{P}, \quad \mathbb{P} - \text{a.s}, \text{ for every } \mathbb{P} \in \mathcal{P}.
\]
Introduction

Aggregation issues and Martingale problems

The form of the equations

Some applications

Aggregation, a very simple example

Example

Let $\mathbb{P}_1$ be the Wiener measure, and let $\mathbb{P}_2$ the law of $\sqrt{2}B$ under $\mathbb{P}_1$. Then

$$\int_0^t B_s dB_s = B_t^2 - t, \; \mathbb{P}_1\text{-a.s. and}$$

$$\int_0^t B_s dB_s = B_t^2 - 2t, \; \mathbb{P}_2\text{-a.s.}$$
Cohen, S.N. (2011) *Quasi-sure analysis, aggregation and dual representations of sublinear expectations in general spaces*, preprint arXiv:1110.2592v2. gave general conditions on a set $\mathbb{P}$ of probability measures such that any consistent family of processes indexed by $\mathbb{P}$ has an aggregator.
Proposition

There exists a set $\mathcal{P}$ of probability measures such that

- Every $\mathbb{P}$ in $\mathcal{P}$ satisfies the martingale representation property and the Blumenthal $0-1$ law.
- Every family of progressively measurable processes indexed by $\mathcal{P}$, and satisfying the consistency condition has a $\mathbb{P}$-q.s unique aggregator.
- $\mathcal{P}$ is stable by concatenation and bifurcation.
The 2BSDEJ

\[ Y_t = \xi + \int_t^T F_s(Y_s, Z_s, U_s, \hat{\alpha}, \hat{\nu}) \, ds - \int_t^T Z_s \, dB_s^c \\
- \int_t^T \int_E U_s(x)(\mu_B^d - \hat{\nu})(ds, dx) + K_T - K_t, \quad \mathbb{P}^{\alpha,\nu} - \text{a.s}, \forall \mathbb{P}^{\alpha,\nu} \in \mathcal{P}. \]
Assumptions

(i) The domains \( D_{F_t}^1(y,z,u) = D_{F_t}^1 \) and \( D_{F_t}^2(y,z,u) = D_{F_t}^2 \) are independent of \((\omega, y, z, u)\).

(ii) For fixed \((y, z, a, \nu)\), \( F \) is \( \mathbb{F} \)-progressively measurable in \( D_{F_t}^1 \times D_{F_t}^2 \).

(iii) The following uniform Lipschitz-type property holds. For all \((y, y', z, z', u, t, a, \nu, \omega)\)

\[
|F_t(\omega, y, z, u, a, \nu) - F_t(\omega, y', z', u, a, \nu)| \leq C \left( |y - y'| + |a^{1/2} (z - z')| \right).
\]
Assumptions

(i) The domains $D_{F_t}^1(y,z,u) = D_{F_t}^1$ and $D_{F_t}^2(y,z,u) = D_{F_t}^2$ are independent of $(\omega, y, z, u)$.

(ii) For fixed $(y, z, a, \nu)$, $F$ is $\mathbb{F}$-progressively measurable in $D_{F_t}^1 \times D_{F_t}^2$.

(iii) The following uniform Lipschitz-type property holds. For all $(y, y', z, z', u, t, a, \nu, \omega)$

\[ |F_t(\omega, y, z, u, a, \nu) - F_t(\omega, y', z', u, a, \nu)| \leq C \left( |y - y'| + |a^{1/2} (z - z')| \right). \]

(iv) For all $(t, \omega, y, z, u^1, u^2, a, \nu)$, there exist two processes $\gamma$ and $\gamma'$ such that

\[ F_t(\omega, y, z, u^1, a, \nu) - F_t(\omega, y, z, u^2, a, \nu) \leq \int_E \left( u^1(e) - u^2(e) \right) \gamma_t(e) \nu(de), \]

\[ \int_E \left( u^1(e) - u^2(e) \right) \gamma'_t(e) \nu(de) \leq F_t(\omega, y, z, u^1, a, \nu) - F_t(\omega, y, z, u^2, a, \nu) \text{ and} \]

\[ c_1(1 \wedge |x|) \leq \gamma_t(x) \leq c_2(1 \wedge |x|) \text{ where } -1 < c_1 \leq 0, \quad c_2 \geq 0, \]

\[ c'_1(1 \wedge |x|) \leq \gamma'_t(x) \leq c'_2(1 \wedge |x|) \text{ where } -1 < c'_1 \leq 0, \quad c'_2 \geq 0. \]
(i) The domains $D_{F_t}^1(y,z,u) = D_{F_t}^1$ and $D_{F_t}^2(y,z,u) = D_{F_t}^2$ are independent of $(\omega, y, z, u)$.

(ii) For fixed $(y, z, a, \nu)$, $F$ is $\mathbb{F}$-progressively measurable in $D_{F_t}^1 \times D_{F_t}^2$.

(iii) The following uniform Lipschitz-type property holds. For all $(y, y', z, z', u, t, a, \nu, \omega)$

$$|F_t(\omega, y, z, u, a, \nu) - F_t(\omega, y', z', u, a, \nu)| \leq C \left( |y - y'| + |a^{1/2} (z - z')| \right).$$

(iv) For all $(t, \omega, y, z, u^1, u^2, a, \nu)$, there exist two processes $\gamma$ and $\gamma'$ such that

$$F_t(\omega, y, z, u^1, a, \nu) - F_t(\omega, y, z, u^2, a, \nu) \leq \int_E \left( u^1(e) - u^2(e) \right) \gamma_t(e) \nu(de),$$

$$\int_E \left( u^1(e) - u^2(e) \right) \gamma'_t(e) \nu(de) \leq F_t(\omega, y, z, u^1, a, \nu) - F_t(\omega, y, z, u^2, a, \nu) \text{ and}$$

$$c_1(1 \wedge |x|) \leq \gamma_t(x) \leq c_2(1 \wedge |x|) \text{ where } -1 < c_1 \leq 0, \ c_2 \geq 0,$$

$$c'_1(1 \wedge |x|) \leq \gamma'_t(x) \leq c'_2(1 \wedge |x|) \text{ where } -1 < c'_1 \leq 0, \ c'_2 \geq 0.$$

(v) $F$ is uniformly continuous in $\omega$ for the $\| \cdot \|_{\infty}$ norm.
The form of the equation

**Definition**

We say $(Y, Z, U) \in D^{2, \kappa} \times H^{2, \kappa} \times J^{2, \kappa}$ is a solution to a 2BSDEJ if

- $Y_T = \xi$, $\mathbb{P}$-a.s., $\forall \mathbb{P} \in \mathcal{P}$.
- For all $\mathbb{P} \in \mathcal{P}$ and $0 \leq t \leq T$, the process $K^\mathbb{P}$ defined below is predictable and has non-decreasing paths $\mathbb{P} - a.s.$

$$K_t^\mathbb{P} := Y_0 - Y_t - \int_0^t \hat{F}_s(Y_s, Z_s, U_s) \, ds + \int_0^t Z_s \, dB_s^c + \int_0^t \int_E U_s(x) \tilde{\mu}_B \, (ds, dx).$$  \hspace{1cm} (1)

- The family $\{K^\mathbb{P}, \mathbb{P} \in \mathcal{P}\}$ satisfies the minimum condition

$$K_t^\mathbb{P} = \text{ess inf} \mathbb{P} \in \mathcal{P}(t^+, \mathbb{P}) \mathbb{E}_t^{\mathbb{P}'} \left[ K_T^{\mathbb{P}'} \right], \quad 0 \leq t \leq T, \quad \mathbb{P} - a.s., \quad \forall \mathbb{P} \in \mathcal{P}_H^\kappa. \hspace{1cm} (2)$$

2BSDEs with jumps
A wellposedness result

**Theorem**

There exists a unique solution \((Y, Z, U)\) to the previously defined 2BSDE with jumps. Moreover, for any \(P \in \mathcal{P}\) and \(0 \leq t_1 < t_2 \leq T\),

\[
Y_{t_1} = \text{ess sup}_{P'} P' \left( t_1, Y_{t_1}(t_2, Y_{t_2}) \right), \quad P - \text{a.s.} \tag{3}
\]

where, for any \(P \in \mathcal{P}\), \(\mathbb{F}^+\)-stopping time \(\tau\), and \(\mathcal{F}^+_\tau\)-measurable random variable \(\xi \in L^2(P)\), \((y^P(\tau, \xi), z^P(\tau, \xi))\) denotes the solution to the following standard BSDE on \(0 \leq t \leq \tau\)

\[
y^P_t = \xi - \int_t^\tau \hat{F}_s(y^P_s, z^P_s, u^P_s) \, ds + \int_t^\tau z^P_s \, dB^c_s + \int_t^\tau \int_E u^P_s(x) \tilde{\mu}_B(d\tau, dx), \quad P - \text{a.s.} \tag{4}
\]
Robust utility maximization problem

The market:
\[
\frac{dS_t}{S_t} = b_t \, dt + \sigma_t \, dB_t + \int E_\beta_t(x) \, \mu(B) \, d(B_t, dx), \quad P\text{-a.s.}
\]
(5)

The value function \( V \) of the maximization problem can be written as
\[
V_\xi(x) := \sup_{\pi \in C} \inf_{P \in \mathcal{P}} \mathbb{E}_P[-\exp(-\eta(X_{\pi}^T - \xi))].
\]
where \( C := \{ (\pi_t) \text{ which are predictable and take values in } C \} \), is our set of admissible strategies.

2BSDEs with jumps
Robust utility maximization problem

The market:

\[
\frac{dS_t}{S_t} = b_t dt + dB^c_t + \int_E \beta_t(x) \mu_B(dt, dx), \text{P-a.s. } \forall \text{P} \in \mathcal{P}.
\]
Robust utility maximization problem

- The market:

\[
\frac{dS_t}{S_t} = b_t dt + dB_t^c + \int_E \beta_t(x) \mu_{B^d}(dt, dx), \mathbb{P} \text{-a.s. } \forall \mathbb{P} \in \mathcal{P}.
\]  

- The value function \( V \) of the maximization problem can be written as

\[
V^\xi(x) := \sup_{\pi \in \mathcal{C}} \inf_{\mathbb{P} \in \mathcal{P}} \mathbb{E}^\mathbb{P} \left[ - \exp \left( -\eta(X_T^\pi - \xi) \right) \right].
\]

where

\[
\mathcal{C} := \{ (\pi_t) \text{ which are predictable and take values in } \mathcal{C} \},
\]

is our set of admissible strategies.
Robust utility maximization problem

Proposition

Assume that $\exp(\eta \xi) \in \mathcal{L}_{H}^{2, \kappa}$. Then the value function of the previous optimization problem is given by

$$V^{\xi}(x) = -e^{-\eta x} Y_0,$$

where $Y_0$ is defined as the initial value of the unique solution $(Y, Z, U) \in D_{2, \kappa} \times H_{2, \kappa} \times J_{2, \kappa}$ of the following BSDE:

$$Y_t = e^{\eta \xi} + \int_{T}^{t} \hat{F}_s(Y_s, Z_s, U_s) \, ds - \int_{T}^{t} Z_s \, dB_c + \int_{T}^{t} \tilde{E} U_s(x) \tilde{\mu}_B \, (ds, dx) + \mathcal{K}_P T - \mathcal{K}_P t,$$
Robust utility maximization problem

**Proposition**

Assume that $\exp(\eta \xi) \in L^{2,\kappa}_H$. Then the value function of the previous optimization problem is given by

$$V^\xi(x) = -e^{-\eta x} Y_0,$$

where $Y_0$ is defined as the initial value of the unique solution $(Y, Z, U) \in D^{2,\kappa} \times H^{2,\kappa} \times J^{2,\kappa}$ of the following 2BSDE

$$Y_t = e^{\eta \xi} + \int_t^T \tilde{F}_s(Y_s, Z_s, U_s) ds - \int_t^T Z_s dB_s^c$$

$$- \int_t^T \int_E U_s(x) \tilde{\mu}_B(d s, d x) + K_T^P - K_t^P,$$
Robust utility maximization problem

Proposition

Assume that $\exp(\eta \xi) \in \mathcal{L}_H^{2,\kappa}$. Then the value function of the previous optimization problem is given by

$$V^\xi(x) = -e^{-\eta x} Y_0,$$

where $Y_0$ is defined as the initial value of the unique solution $(Y, Z, U) \in \mathcal{D}^{2,\kappa} \times \mathcal{H}^{2,\kappa} \times \mathcal{J}^{2,\kappa}$ of the following 2BSDEJ

$$Y_t = e^{\eta \xi} + \int_t^T \tilde{F}_s(Y_s, Z_s, U_s) ds - \int_t^T Z_s dB_s^c$$

$$- \int_t^T \int_E U_s(x) \tilde{\mu}_B (ds, dx) + K_T^P - K_t^P,$$

where the generator is defined as follows

$$\tilde{F}_t(\omega, y, z, u) := F_t(\omega, y, z, u, \hat{a}_t, \hat{\nu}_t),$$

(6)

2BSDEs with jumps
Robust utility maximization problem

Proposition

where

\[ F_t(y, z, u, a, \nu) := \inf_{\pi \in C} \left\{ \left(-\eta b_t + \frac{\eta^2}{2} \pi a\right)\pi y - \eta \pi az \right\} \]

\[ + \int_E \left( e^{-\eta \pi \beta_t(x)} - 1 \right) (y + u(x)) \nu(dx) \]
Robust utility maximization problem

Proposition

where

\[ F_t(y, z, u, a, \nu) := \inf_{\pi \in \mathcal{C}} \left\{ (-\eta b_t + \frac{\eta^2}{2} \pi a)\pi y - \eta \pi az \right. \]

\[ + \int_E \left( e^{-\eta \pi \beta_t(x)} - 1 \right) (y + u(x))\nu(dx) \right\}. \]

Moreover, there exists an optimal trading strategy \(\pi^*\) realizing the supremum above.
The solution of a 2nd order BSDE with jumps, in the Markovian case, is the natural candidate for the probabilistic interpretation of fully non-linear PIDEs of the form

$$\frac{\partial}{\partial t} v(t, x) + h(t, x, v(t, x), Dv(t, x), D^2 v(t, x), v(t, \cdot)) = 0, \quad 0 \leq t \leq T,$$

$$v(T, x) = g(x).$$

where $h$ is the Fenchel-Legendre transform of the generator $f$ in $(a, \nu)$:

$$h(t, x, y, z, u, \gamma, v) = \sup_{(a, \nu) \in S_{d} \times D_2} \left\{ \frac{1}{2} a : \gamma + \int_{0}^{T} \int_{E} \tilde{v}(e) \nu(ds) - f(t, x, y, z, u, a, \nu) \right\}$$

with $\tilde{v}(e) := v(e + x) - v(x) - 1_{\{|e| \leq 1\}} e \cdot \nabla v(x)$. 

Paper in preparation!
Probabilistic counterpart of fully non-linear PIDEs

The solution of a 2nd order BSDE with jumps, in the Markovian case, is the natural candidate for the probabilistic interpretation of fully non-linear PIDEs of the form

\[ \partial_t v(t, x) + h \left( t, x, v(t, x), Dv(t, x), D^2 v(t, x), v(t, \cdot) \right) = 0, \quad 0 \leq t \leq T, \]
\[ v(T, x) = g(x). \]
The solution of a 2nd order BSDE with jumps, in the Markovian case, is the natural candidate for the probabilistic interpretation of fully non-linear PIDEs of the form

$$\partial_t v(t, x) + h \left( t, x, v(t, x), Dv(t, x), D^2 v(t, x), v(t, \cdot) \right) = 0, \quad 0 \leq t \leq T,$$

$$v(T, x) = g(x).$$

where $h$ is the Fenchel-Legendre transform of the generator $f$ in $(a, \nu)$:

$$h(t, x, y, z, u, \gamma, \nu) = \sup_{(a, \nu) \in S_d \times D_2} \left\{ \frac{1}{2} a : \gamma + \int_0^T \int_E \tilde{v}(e) \nu_s(de)ds - f(t, x, y, z, u, a, \nu) \right\}$$
The solution of a 2nd order BSDE with jumps, in the Markovian case, is the natural candidate for the probabilistic interpretation of fully non-linear PIDEs of the form

$$\partial_t v(t, x) + h \left( t, x, v(t, x), Dv(t, x), D^2 v(t, x), v(t, \cdot) \right) = 0, \ 0 \leq t \leq T,$$

$$v(T, x) = g(x).$$

where $h$ is the Fenchel-Legendre transform of the generator $f$ in $(a, \nu)$:

$$h(t, x, y, z, u, \gamma, \nu) = \sup_{(a, \nu) \in S_d \times D_2} \left\{ \frac{1}{2} a : \gamma + \int_0^T \int_E \tilde{v}(e) \nu_s(de)ds - f(t, x, y, z, u, a, \nu) \right\}$$

with

$$\tilde{v}(e) := v(e + x) - v(x) - \mathbf{1}_{\{|e| \leq 1\}} e.(\nabla v)(x).$$

Paper in preparation!
Thank you for your attention !