Mean-Variance Hedging on uncertain time horizon in a market with a jump

Thomas LIM

ENSIIE and Laboratoire Analyse et Probabilités d’Evry

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Joint work with Idris Kharroubi and Armand Ngoupeyou

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• Progressive enlargement of filtrations and BSDEs with jumps (with I. Kharroubi), forthcoming in Journal of Theoretical Probability.

• A decomposition approach for the discrete-time approximation of FBSDEs with a jump I: the Lipschitz case (with I. Kharroubi).

• A decomposition approach for the discrete-time approximation of FBSDEs with a jump II: the quadratic case (with I. Kharroubi).

• Mean-Variance Hedging on uncertain time horizon in a market with a jump (with I. Kharroubi and A. Ngoupeyou).
Mean-variance hedging in literature

\[ \inf_{\pi} \mathbb{E} \left[ \left( x + \int_{0}^{T} \pi_s dS_s - \xi \right)^2 \right] . \]

There exist two approaches to solve mean-variance hedging problem with a deterministic finite horizon:

- martingale theory and projection arguments: Delbaen-Schachermayer, Gouriéroux-Laurent-Pham, Schweizer, ... for the continuous case, and Arai for the semimartingale case,

- quadratic stochastic control and BSDE: Lim-Zhou, Lim, ... for the continuous case and the discontinuous case (driven by a Brownian motion and a Poisson process).

Jeanblanc-Mania-Santacroce-Schweizer combine tools from both approaches which allows them to work in a general semimartingale model.
Mean-variance hedging with random horizon

For some financial products (e.g. insurance, credit-risk) the horizon of the problem is not deterministic

\[
\inf_{\pi} \mathbb{E} \left[ \left( x + \int_0^{T \wedge \tau} \pi_s dS_s - \xi \right)^2 \right].
\]

We use a BSDE approach as in Lim and provide a solution to the mean-variance hedging problem with

- random horizon,
- dependent jump and continuous parts.

Theoretical issue: no result for our BSDEs in this framework.
Outline

1. Preliminaries and market model
   - The probability space
   - Financial model
   - Mean-variance hedging

2. Solution of the mean-variance problem by BSDEs
   - Martingale optimality principle
   - Related BSDEs
   - A verification Theorem

3. How to solve the BSDEs
Outline

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Settings

Let \((\Omega, \mathcal{G}, \mathbb{P})\) be a complete probability space equipped with

- \(W\) a standard Brownian motion with its natural filtration
  \(\mathbb{F} := (\mathcal{F}_t)_{t \geq 0}\),
- \(\tau\) a random time (we define the process \(H\) by \(H_t := 1_{\tau \leq t}\)).

\(\tau\) not always an \(\mathbb{F}\)-stopping time.

\(\Rightarrow\) \(\mathbb{G}\) smallest right continuous extension of \(\mathbb{F}\) that turns \(\tau\) into a \(\mathbb{G}\)-stopping time: \(\mathbb{G} := (\mathcal{G}_t)_{t \geq 0}\) where

\[
\mathcal{G}_t := \bigcap_{\varepsilon > 0} \tilde{\mathcal{G}}_{t+\varepsilon},
\]

for all \(t \geq 0\), with \(\tilde{\mathcal{G}}_s := \mathcal{F}_s \vee \sigma(1_{\tau \leq u}, u \in [0, s])\), for all \(s \geq 0\).
Assumption on $W$ and $\tau$

(\textbf{H}) The process $W$ remains a $\mathcal{G}$-Brownian motion.

(\textbf{H}_\tau) The process $H$ admits an $\mathcal{F}$-compensator of the form $\int_{0}^{\cdot\wedge\tau} \lambda_s ds$, i.e. $H - \int_{0}^{\cdot\wedge\tau} \lambda_s ds$ is a $\mathcal{G}$-martingale, where $\lambda$ is a bounded $\mathcal{P}(\mathcal{F})$-measurable process. We then denote by $M$ the $\mathcal{G}$-martingale defined by

$$M_t := H_t - \int_{0}^{t\wedge\tau} \lambda_s ds = H_t - \int_{0}^{t} \lambda_s^G ds,$$

for all $t \geq 0$, with $\lambda_t^G := (1 - H_t)\lambda_t$. 
Financial market

Financial market is composed by

- a riskless bond $B$ with zero interest rate: $B_t = 1$,
- a risky asset $S$ modeled by the stochastic differential equation

$$S_t = S_0 + \int_0^t S_u \left( \mu_u du + \sigma_u dW_u + \beta_u dM_u \right), \quad t \geq 0,$$

where $\mu$, $\sigma$ and $\beta$ are $\mathcal{P}(\mathbb{G})$-measurable processes satisfying (HS)

(i) $\mu$, $\sigma$ and $\beta$ are bounded,

(ii) there exists a constant $c > 0$ s.t.

$$\sigma_t \geq c, \quad \forall t \in [0, T], \quad \mathbb{P} - a.s.$$

(iii) $-1 \leq \beta_t, \quad \forall t \in [0, T], \quad \mathbb{P} - a.s.$
Admissible strategies

We consider the set $\mathcal{A}$ of investment strategies which are $\mathcal{P}(\mathbb{G})$-measurable processes $\pi$ such that

$$
\mathbb{E} \left[ \int_0^{T \wedge \tau} |\pi_t|^2 dt \right] < \infty.
$$

We then define for an initial amount $x \in \mathbb{R}$ and a strategy $\pi$, the wealth $V^{x,\pi}$ associated with $(x, \pi)$ by the process

$$
V_t^{x,\pi} = x + \int_0^t \frac{\pi_r}{S_r} dS_r, \quad t \in [0, T \wedge \tau].
$$
Problem

For \( x \in \mathbb{R} \), the problem of mean-variance hedging consists in computing the quantity

\[
\inf_{\pi \in \mathcal{A}} \mathbb{E} \left[ \left| V_{T \wedge \tau}^{x, \pi} - \xi \right|^2 \right],
\]

where \( \xi \) is a bounded \( \mathcal{G}_{T \wedge \tau} \)-measurable random variable of the form

\[
\xi = \xi^b 1_{T < \tau} + \xi^a 1_{\tau \leq T},
\]

where \( \xi^b \) is a bounded \( \mathcal{F}_T \)-measurable random variable and \( \xi^a \) is a continuous \( \mathbb{F} \)-adapted process satisfying

\[
\text{ess sup}_{t \in [0, T]} |\xi^a_t| < +\infty.
\]
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3 How to solve the BSDEs
Sufficient conditions for optimality

We look for a family of processes \( \{(J_t^\pi)_{t \in [0,T]} : \pi \in \mathcal{A}\} \) satisfying

(i) \( J_{T \wedge \tau}^\pi = |V_{T \wedge \tau}^{x,\pi} - \xi|^2 \), for all \( \pi \in \mathcal{A} \).

(ii) \( J_0^{\pi_1} = J_0^{\pi_2} \), for all \( \pi_1, \pi_2 \in \mathcal{A} \).

(iii) \( J^\pi \) is a \( \mathbb{G} \)-submartingale for all \( \pi \in \mathcal{A} \).

(iv) There exists some \( \pi^* \in \mathcal{A} \) such that \( J_{T \wedge \tau}^{\pi^*} \) is a \( \mathbb{G} \)-martingale.

Under these conditions, we have for any \( \pi \in \mathcal{A} \)

\[
\mathbb{E}(J_{T \wedge \tau}^{\pi^*}) = J_0^{\pi^*} = J_0^{\pi} \leq \mathbb{E}(J_{T \wedge \tau}^{\pi}) .
\]
Solution of the mean-variance problem by BSDEs

Martingale optimality principle

Sufficient conditions for optimality

We look for a family of processes \( \{(J_t^\pi)_{t\in[0,T]} : \pi \in \mathcal{A}\} \) satisfying

(i) \( J_{T\wedge \tau}^\pi = |V_{T\wedge \tau}^{x,\pi} - \xi|^2 \), for all \( \pi \in \mathcal{A} \).

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\]

Therefore, we get

\[
J_0^{\pi^*} = \mathbb{E}[|V_{T\wedge \tau}^{x,\pi^*} - \xi|^2] = \inf_{\pi \in \mathcal{A}} \mathbb{E}[|V_{T\wedge \tau}^{x,\pi} - \xi|^2].
\]
Sufficient conditions for optimality

We look for a family of processes \( \{ (J_t^\pi)_{t \in [0, T]} : \pi \in \mathcal{A} \} \) satisfying

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Therefore, we get

\[
J_0^{\pi^*} = \mathbb{E}\left(\left| V_{T\wedge \tau}^{x,\pi^*} - \xi \right|^2\right) = \inf_{\pi \in \mathcal{A}} \mathbb{E}\left(\left| V_{T\wedge \tau}^{x,\pi} - \xi \right|^2\right).
\]
• $\mathcal{S}_\infty$ is the subset of $\mathbb{R}$-valued càdlàg $\mathcal{G}$-adapted processes $(Y_t)_{t \in [0,T]}$ essentially bounded

$$\|Y\|_{\mathcal{S}_\infty} := \sup_{t \in [0,T]} |Y_t| < \infty.$$ 

• $\mathcal{S}^\infty_\infty$ is the subset of $\mathcal{S}_\infty$ of processes $(Y_t)_{t \in [0,T]}$ valued in $(0, \infty)$, such that

$$\|\frac{1}{Y}\|_{\mathcal{S}_\infty} < \infty.$$ 

• $L^2_\mathcal{G}$ is the subset of $\mathbb{R}$-valued $\mathcal{P}(\mathcal{G})$-measurable processes $(Z_t)_{t \in [0,T]}$ such that

$$\|Z\|_{L^2} := \left(\mathbb{E}\left[\int_0^T |Z_t|^2 dt\right]\right)^{\frac{1}{2}} < \infty.$$ 

• $L^2(\lambda)$ is the subset of $\mathbb{R}$-valued $\mathcal{P}(\mathcal{G})$-measurable processes $(U_t)_{t \in [0,T]}$ such that

$$\|U\|_{L^2(\lambda)} := \left(\mathbb{E}\left[\int_0^{T \wedge \tau} \lambda_s |U_s|^2 ds\right]\right)^{\frac{1}{2}} < \infty.$$
Construction of $J^\pi$ using BSDEs

To construct such a family $\{(J^\pi_t)_{t \in [0,T]}, \quad \pi \in \mathcal{A}\}$, we set

$$J^\pi_t := Y_t |V^\pi_{t \wedge \tau} - Y_t|^2 + \gamma_t, \quad t \geq 0,$$

where $(Y, Z, U)$, $(\mathcal{Y}, \mathcal{Z}, \mathcal{U})$ and $(\gamma, \Xi, \Theta)$ are solutions in $S^\infty_G \times L^2_G \times L^2(\lambda)$ to

$$Y_t = 1 + \int_{t \wedge \tau}^{T \wedge \tau} f(s, Y_s, Z_s, U_s) ds - \int_{t \wedge \tau}^{T \wedge \tau} Z_s dW_s - \int_{t \wedge \tau}^{T \wedge \tau} U_s dM_s, \quad (2)$$

$$\mathcal{Y}_t = \xi + \int_{t \wedge \tau}^{T \wedge \tau} g(s, \mathcal{Y}_s, \mathcal{Z}_s, \mathcal{U}_s) ds - \int_{t \wedge \tau}^{T \wedge \tau} \mathcal{Z}_s dW_s - \int_{t \wedge \tau}^{T \wedge \tau} \mathcal{U}_s dM_s, \quad (3)$$

$$\gamma_t = \int_{t \wedge \tau}^{T \wedge \tau} h(s, \gamma_s, \Xi_s, \Theta_s) ds - \int_{t \wedge \tau}^{T \wedge \tau} \Xi_s dW_s - \int_{t \wedge \tau}^{T \wedge \tau} \Theta_s dM_s, \quad (4)$$

for all $t \in [0, T]$. 
We are bounded to choose three functions $f$, $g$ and $h$ for which

- $J^\pi$ is a submartingale for all $\pi \in \mathcal{A}$,
- there exists $\pi^* \in \mathcal{A}$ such that $J^{\pi^*}$ is a martingale.

For that we would like to write $J^\pi$ as the sum of a martingale $M^\pi$ and a nondecreasing process $K^\pi$ that is constant for some $\pi^* \in \mathcal{A}$. 

\[ dJ^\pi_t = dM^\pi_t + dK^\pi_t, \]

where $M^\pi$ is a local martingale and $K^\pi$ is given by

\[ dK^\pi_t = K_t(\pi_t) dt = (At_t |\pi_t|^2 + B_t \pi_t + Ct_t) dt, \]

with

\[ A_t := |\sigma_t|^2 Y_t + \lambda G_t |\beta_t|^2 (U_t + Y_t), \]

\[ B_t := 2(V_{\pi_t} \wedge \tau - Y_t)(\mu_t Y_t + \sigma_t Z_t + \lambda G_t \beta_t U_t) - 2\sigma_t Y_t Z_t - 2\lambda G_t \beta_t U_t (Y_t + U_t), \]

\[ C_t := -f(t) |V_{\pi_t} \wedge \tau - Y_t|^2 + 2X_{\pi_t} (Y_t g(t) - Z_t Z_t - \lambda G_t U_t U_t) + Y_t |Z_t|^2 + \lambda G_t |U_t|^2 (U_t + Y_t) - h(t). \]
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$$A_t := |\sigma_t|^2 Y_t + \lambda_t^G |\beta_t|^2 (U_t + Y_t),$$

$$B_t := 2(V^\pi_{t\wedge \tau} - Y_t)(\mu_t Y_t + \sigma_t Z_t + \lambda_t^G \beta_t U_t) - 2\sigma_t Y_t Z_t - 2\lambda_t^G \beta_t U_t (Y_t + U_t),$$

$$C_t := -f(t)|V^\pi_{t\wedge \tau} - Y_t|^2 + 2X^\pi_t (Y_t g(t) - Z_t Z_t - \lambda_t^G U_t U_t) + Y_t |Z_t|^2$$

$$+ \lambda_t^G |U_t|^2 (U_t + Y_t) - h(t).$$
We are bounded to choose three functions $f$, $g$ and $h$ for which

- $J^\pi$ is a submartingale for all $\pi \in \mathcal{A}$,
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where $M^\pi$ is a local martingale and $K^\pi$ is given by

$$dK^\pi_t := K_t(\pi_t)dt = \left(A_t|\pi_t|^2 + B_t\pi_t + C_t\right)dt,$$

with

$$A_t := |\sigma_t|^2 Y_t + \lambda_t^G |\beta_t|^2 (U_t + Y_t),$$

$$B_t := 2(V_{t\wedge \tau}^\pi - \mathcal{Y}_t)(\mu_t Y_t + \sigma_t Z_t + \lambda_t^G \beta_t U_t) - 2\sigma_t Y_t Z_t - 2\lambda_t^G \beta_t U_t (Y_t + U_t),$$

$$C_t := -f(t)|V_{t\wedge \tau}^\pi - \mathcal{Y}_t|^2 + 2X_t^\pi (Y_t g(t) - Z_t Z_t - \lambda_t^G U_t U_t) + Y_t |Z_t|^2 + \lambda_t^G |U_t|^2 (U_t + Y_t) - h(t).$$
In order to obtain a nondecreasing process $K^\pi$ for any $\pi \in \mathcal{A}$ and that is constant for some $\pi^* \in \mathcal{A}$ it is obvious that $K_t$ has to satisfy $\min_{\pi \in \mathbb{R}} K_t(\pi) = 0$:

$$K_t := \min_{\pi \in \mathbb{R}} K_t(\pi) = C_t - \frac{|B_t|^2}{4A_t}.$$ 

We then obtain from the expressions of $A$, $B$ and $C$ that

$$K_t = A_t |V_{t \wedge \tau}^\pi - Y_t|^2 + B_t (V_{t \wedge \tau}^\pi - Y_t) + C_t,$$

with

$$A_t := -f(t) - \frac{\mu_t Y_t + \sigma_t Z_t + \lambda_t^G \beta_t U_t}{|\sigma_t|^2 Y_t + \lambda_t^G |\beta_t|^2 (U_t + Y_t)}$$,

$$B_t := 2 \left\{ \frac{(\mu_t Y_t + \sigma_t Z_t + \lambda_t^G \beta_t U_t)(\lambda_t^G \beta_t U_t(Y_t + U_t) + \sigma_t Y_t Z_t)}{|\sigma_t|^2 Y_t + \lambda_t^G |\beta_t|^2 (U_t + Y_t)} + g(t) Y_t - Z_t Z_t - \lambda_t^G U_t U_t \right\},$$

$$C_t := -h(t) + |Z_t|^2 Y_t + \lambda_t^G (U_t + Y_t)|U_t|^2 - \frac{\sigma_t Y_t Z_t + \lambda_t^G \beta_t U_t(U_t + Y_t)}{|\sigma_t|^2 Y_t + \lambda_t^G |\beta_t|^2 (U_t + Y_t)}.$$
Expressions of the generators

For that the family \((J^\pi)_\pi \in \mathcal{A}\) satisfies the conditions (iii) and (iv) we choose \(f\), \(g\) and \(h\) such that

\[
\mathcal{A}_t = 0, \ \mathcal{B}_t = 0 \ \text{and} \ \mathcal{C}_t = 0,
\]

for all \(t \in [0, T]\).

\[
\begin{align*}
\mathcal{A}_t &= -\frac{(\mu_t Y + \sigma_t Z + \lambda_t \beta_t U)^2}{|\sigma_t|^2 Y + \lambda_t |\beta_t|^2 (U+Y)}, \\
\mathcal{B}_t &= \frac{1}{Y_t} \left[ Z_t Z + \lambda_t U_t U - \frac{(\mu_t Y_t + \sigma_t Z_t + \lambda_t \beta_t U_t)(\sigma_t Y_t Z + \lambda_t \beta_t (U_t + Y_t) U)}{|\sigma_t|^2 Y_t + \lambda_t \beta_t^2 (U_t + Y_t)} \right], \\
\mathcal{C}_t &= |Z_t|^2 Y_t + \lambda_t (U_t + Y_t)|U_t|^2 - \frac{|\sigma_t Y_t Z_t + \lambda_t \beta_t U_t (U_t + Y_t)|^2}{|\sigma_t|^2 Y_t + \lambda_t |\beta_t|^2 (U_t + Y_t)}.
\end{align*}
\]

\[
\Rightarrow \quad \text{Nonstandard Decoupled BSDEs}
\]

**Theorem**

The BSDEs (2)-(3)-(4) admit solutions \((Y, Z, U), (Y, Z, U)\) and \((\Upsilon, \Xi, \Theta)\) in \(S_{G}^{\infty} \times L_{G}^{2} \times L^{2}(\lambda)\). Moreover \(Y \in S_{G}^{\infty, +}\).
Expressions of the generators

For that the family \((J^\pi)_{\pi \in \mathcal{A}}\) satisfies the conditions (iii) and (iv) we choose \(f\), \(g\) and \(h\) such that

\[
A_t = 0 \, , \, B_t = 0 \, \text{ and } \, C_t = 0 ,
\]

for all \(t \in [0, T]\).

\[
\begin{aligned}
f(t, Y, Z, U) &= -\frac{(\mu_t Y + \sigma_t Z + \lambda_t \beta_t U)^2}{|\sigma_t|^2 Y + \lambda_t |\beta_t|^2 (U+Y)} , \\
g(t, \Upsilon, \Xi, \Theta) &= \frac{1}{Y_t} \left[ Z_t \Upsilon + \lambda_t U_t U \right. - \frac{(\mu_t Y_t + \sigma_t Z_t + \lambda_t \beta_t U_t)(\sigma_t Y_t Z_t + \lambda_t \beta_t (U_t + Y_t) U_t)}{|\sigma_t|^2 Y_t + \lambda_t |\beta_t|^2 (U_t + Y_t)} \left. \right] , \\
h(t, \Upsilon, \Xi, \Theta) &= |Z_t|^2 Y_t + \lambda_t (U_t + Y_t)|U_t|^2 - \frac{|\sigma_t Y_t Z_t + \lambda_t \beta_t U_t(U_t + Y_t)|^2}{|\sigma_t|^2 Y_t + \lambda_t |\beta_t|^2 (U_t + Y_t)} .
\end{aligned}
\]

⇒ Nonstandard Decoupled BSDEs

Theorem

The BSDEs (2)-(3)-(4) admit solutions \((Y, Z, U), (\Upsilon, \Xi, U)\) and \((\Upsilon, \Xi, \Theta)\) in \(\mathcal{S}_G^\infty \times L^2_G \times L^2(\lambda)\). Moreover \(Y \in \mathcal{S}_G^\infty,^+\).
Optimal strategy-SDE of the optimal value portfolio

A candidate to be an optimal strategy is

$$\pi^*_t = \arg \min_{\pi \in \mathbb{R}} K_t(\pi),$$  \hspace{1cm} (5)

which gives the implicit equation in $\pi^*$

$$\pi^*_t = (\mathcal{Y}_t^* - V_{t-}^{x*,\pi^*})D_t + E_t,$$

with $D_t := \frac{\mu_t Y_t^* + \sigma_t Z_t + \lambda_t^G \beta_t U_t}{|\sigma|^2 Y_{t-}^* + \lambda_t^G \beta_t^2 (U_t + Y_{t-}^*)}$ and $E_t := \frac{\sigma_t Y_{t-}^* Z_t + \lambda_t^G \beta_t U_t (Y_{t-}^* + U_t)}{|\sigma|^2 Y_{t-}^* + \lambda_t^G \beta_t^2 (U_t + Y_{t-})}$.

Integrating each side of this equality w.r.t. $\frac{dS_t}{S_t}$ leads to the following SDE

$$V_t^* = x + \int_0^t (\mathcal{Y}_{r-}^* - V_{r-}^{x*})D_r \frac{dS_r}{S_r} + \int_0^t E_r \frac{dS_r}{S_r}, \quad t \in [0, T].$$  \hspace{1cm} (6)

Nonstandard SDE since $D$ and $E$ are not bounded.
Optimal strategy-SDE of the optimal value portfolio

Proposition

The SDE (6) admits a solution $V^*$ which satisfies

$$
\mathbb{E} \left[ \sup_{t \in [0, T \land \tau]} |V^*_t|^2 \right] < \infty.
$$
From Itô’s formula, we get

\[ dJ_t^\pi = dM_t^\pi + dK_t^\pi, \]

where \( M^\pi \) is a local martingale and \( K^\pi \) is given by

\[ dK_t^\pi := K_t(\pi_t)dt = (A_t|\pi_t|^2 + B_t\pi_t + C_t)dt, \]

with

\[
\begin{align*}
A_t &:= |\sigma_t|^2 Y_t + \lambda_t^G|\beta_t|^2(U_t + Y_t), \\
B_t &:= 2(V_{t\wedge \tau} - Y_t)(\mu_t Y_t + \sigma_t Z_t + \lambda_t^G \beta_t U_t) - 2\sigma_t Y_t Z_t - 2\lambda_t^G \beta_t U_t(Y_t + U_t), \\
C_t &:= -f(t)|V_{t\wedge \tau} - Y_t|^2 + 2X_t(\pi_t g(t) - Z_t Z_t - \lambda_t^G U_t U_t) + Y_t |Z_t|^2 + \lambda_t^G |U_t|^2(U_t + Y_t) - h(t).
\end{align*}
\]
Verification theorem

Theorem

The strategy $\pi^*$ given by (5) belongs to the set $\mathcal{A}$ and is optimal for the mean-variance problem (1)

$$
\mathbb{E}\left[\left| V_{T\wedge \tau}^{x, \pi^*} - \xi \right|^2 \right] = \min_{\pi \in \mathcal{A}} \mathbb{E}\left[\left| V_{T\wedge \tau}^{x, \pi} - \xi \right|^2 \right].
$$
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3. How to solve the BSDEs
A decomposition Approach: Data

We consider a BSDE of the form

\[ Y_t = \xi + \int_{t \land \tau}^{T \land \tau} F(s, Y_s, Z_s, U_s) \, ds - \int_{t \land \tau}^{T \land \tau} Z_s \, dW_s - \int_{t \land \tau}^{T \land \tau} U_s \, dH_s , \]  

(7)

- terminal condition

\[ \xi = \xi^b \mathbb{1}_{T < \tau} + \xi^a \mathbb{1}_{\tau \leq T} , \]

where \( \xi^b \) is an \( \mathcal{F}_T \)-measurable bounded r.v. and \( \xi^a \in \mathcal{S}_\mathbb{F}^\infty \),

- generator: \( F \) is a \( \mathcal{P}(\mathcal{G}) \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}) \)-measurable map and

\[ F(t, y, z, u) \mathbb{1}_{t \leq \tau} = F^b(t, y, z, u) \mathbb{1}_{t \leq \tau} , \quad t \geq 0 , \]

where \( F^b \) is a \( \mathcal{P}(\mathcal{F}) \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}) \)-measurable map.

We then introduce the following BSDE

\[ Y^b_t = \xi^b + \int_{t}^{T} F^b(s, Y^b_s, Z^b_s, \xi^a_s - Y^b_s) \, ds - \int_{t}^{T} Z^b_s \, dW_s . \]  

(8)
A decomposition Approach: Theorem

**Theorem**

Assume that BSDE (8) admits a solution \((Y^b, Z^b) \in S^{\infty}_F \times L^2_F\). Then BSDE

\[
Y_t = \xi + \int_{t \wedge \tau}^{T \wedge \tau} F(s, Y_s, Z_s, U_s) - \int_{t \wedge \tau}^{T \wedge \tau} Z_s dW_s - \int_{t \wedge \tau}^{T \wedge \tau} U_s dH_s ,
\]

\(t \in [0, T]\), admits a solution \((Y, Z, U) \in S^{\infty}_G \times L^2_G \times L^2(\lambda)\) given by

\[
Y_t = Y^b_t 1_{t < \tau} + \xi^a_t 1_{t \geq \tau} ,
\]

\[
Z_t = Z^b_t 1_{t \leq \tau} ,
\]

\[
U_t = (\xi^a_t - Y^b_t) 1_{t \leq \tau} ,
\]

for all \(t \in [0, T]\).
Expressions of the generators

\[
\begin{align*}
    f(t, Y, Z, U) &= -\frac{(\mu_t Y + \sigma_t Z + \lambda_t \beta_t U)^2}{|\sigma_t|^2 Y + \lambda_t |\beta_t|^2 (U + Y)}, \\
g(t, Y, Z, U) &= \frac{1}{Y_t} \left[ Z_t Z + \lambda_t U_t U - \frac{(\mu_t Y_t + \sigma_t Z_t + \lambda_t \beta_t U_t)(\sigma_t Y_t Z + \lambda_t \beta_t (U_t + Y_t) U)}{|\sigma_t|^2 Y_t + \lambda_t |\beta_t|^2 (U_t + Y_t)} \right], \\
h(t, \gamma, \Xi, \Theta) &= |Z_t|^2 Y_t + \lambda_t (U_t + Y_t)|U_t|^2 - \frac{|\sigma_t Y_t Z_t + \lambda_t \beta_t U_t (U_t + Y_t)|^2}{|\sigma_t|^2 Y_t + \lambda_t |\beta_t|^2 (U_t + Y_t)}. 
\end{align*}
\]
Solution to BSDE \((f, 1)\)

According to the general existence Theorem, we consider for coefficients \((f, 1)\) the BSDE in \(\mathbb{F}\): find \((Y^b, Z^b) \in S^\infty \times L^2\) such that

\[
\begin{aligned}
    dY^b_t &= \left\{ \frac{|(\mu_t - \lambda_t \beta_t)Y^b_t + \sigma_t Z^b_t + \lambda_t \beta_t|^2}{|\sigma_t|^2 Y^b_t + \lambda_t \beta_t^2} - \lambda_t + \lambda_t Y^b_t \right\} dt \\
    &+ Z^b_t \, dW_t , \\
    Y^b_T &= 1 .
\end{aligned}
\]

The generator of this BSDE can be written under the form

\[
\left\{ \frac{|\mu_t - \lambda_t \beta_t|^2}{|\sigma_t|^2} Y^b_t - \frac{\lambda_t |\beta_t|^2}{|\sigma_t|^4} |\mu_t - \lambda_t \beta_t|^2 - \lambda_t + \lambda_t Y^b_t + \frac{2(\mu_t - \lambda_t \beta_t)}{|\sigma_t|^2} (\sigma_t Z^b_t + \lambda_t \beta_t) \\
    + \frac{|\sigma_t Z^b_t + \lambda_t \beta_t + (\lambda_t \beta_t - \mu_t) \lambda_t |\beta_t|^2}{|\sigma_t|^2 Y^b_t + \lambda_t |\beta_t|^2} \right\} .
\]
How to solve the BSDEs

Introduction of a modified BSDE

Let \((Y^\varepsilon, Z^\varepsilon)\) be the solution in $S^\infty_{\mathcal{F}} \times L^2_{\mathcal{F}}$ to the BSDE

\[
\begin{align*}
    dY^\varepsilon_t &= \left\{ \frac{|\mu_t - \lambda_t \beta_t|^2}{|\sigma_t|^2} Y^\varepsilon_t - \frac{\lambda_t |\beta_t|^2}{|\sigma_t|^4} |\mu_t - \lambda_t \beta_t|^2 + \frac{2(\mu_t - \lambda_t \beta_t)}{|\sigma_t|^2} (\sigma_t Z^\varepsilon_t + \lambda_t \beta_t) \\
    & \quad - \lambda_t + \lambda_t Y^\varepsilon_t + \frac{|\sigma_t Z^\varepsilon_t + \lambda_t \beta_t + (\lambda_t \beta_t - \mu_t) \lambda_t |\beta_t|^2}{|\sigma_t|^2 (Y^\varepsilon_t \vee \varepsilon) + \lambda_t |\beta_t|^2} \right\} dt + Z^\varepsilon_t dW_t, \\
    Y^\varepsilon_T &= 1,
\end{align*}
\]

where \(\varepsilon\) is a positive constant such that

\[
\exp\left(-\int_0^T \left( \lambda_t + \frac{|\mu_t - \lambda_t \beta_t|^2}{|\sigma_t|^2} \right) dt \right) \geq \varepsilon, \quad \mathbb{P} - a.s.
\]

Question: \(Y^\varepsilon \geq \varepsilon\)?
Introduction of a modified BSDE

Let \((Y^\varepsilon, Z^\varepsilon)\) be the solution in \(S^\infty \times L^2\) to the BSDE

\[
\begin{align*}
\begin{cases}
    dY^\varepsilon_t &= \left\{ \frac{|\mu_t - \lambda_t \beta_t|^2}{|\sigma_t|^2} Y^\varepsilon_t - \frac{\lambda_t |\beta_t|^2}{|\sigma_t|^4} |\mu_t - \lambda_t \beta_t|^2 + \frac{2(\mu_t - \lambda_t \beta_t)}{|\sigma_t|^2} (\sigma_t Z^\varepsilon_t + \lambda_t \beta_t) \right. \\
    &\quad - \lambda_t + \lambda_t Y^\varepsilon_t + \frac{\sigma_t Z^\varepsilon_t + \lambda_t \beta_t + (\lambda_t \beta_t - \mu_t) \frac{\lambda_t |\beta_t|^2}{|\sigma_t|^2}^2}{|\sigma_t|^2 (Y^\varepsilon_t \vee \varepsilon) + \lambda_t |\beta_t|^2} \left. \right\} dt + Z^\varepsilon_t dW_t,
    \\
    Y^\varepsilon_T &= 1.,
\end{cases}
\end{align*}
\]

where \(\varepsilon\) is a positive constant such that

\[
\exp \left( - \int_0^T \left( \lambda_t + \frac{|\mu_t - \lambda_t \beta_t|^2}{|\sigma_t|^2} \right) dt \right) \geq \varepsilon, \quad \mathbb{P} - a.s.
\]

Question: \(Y^\varepsilon \geq \varepsilon\)?
Change of probability

Define the process $L^\varepsilon$ by

$$L_t^\varepsilon := 2 \left( \frac{\mu_t - \lambda_t \beta_t}{\sigma_t} \right) + 2 \frac{\sigma_t (\lambda_t \beta_t + \frac{\lambda_t |\beta_t|^2}{|\sigma_t|^2} (\lambda_t \beta_t - \mu_t))}{|\sigma_t|^2 (Y_t^\varepsilon \vee \varepsilon) + \lambda_t |\beta_t|^2} + \frac{|\sigma_t|^2 Z_t^\varepsilon}{|\sigma_t|^2 (Y_t^\varepsilon \vee \varepsilon) + \lambda_t |\beta_t|^2}.$$

Since $L^\varepsilon \in BMO(\mathbb{P})$, we can apply Girsanov theorem:

$$\tilde{W}_t := W_t + \int_0^t L_s^\varepsilon ds,$$

is a Brownian motion under the probability $\mathbb{Q}$ defined by

$$\frac{d\mathbb{Q}}{d\mathbb{P}} \bigg|_{\mathcal{F}_T} := \mathcal{E} \left( \int_0^T L_t^\varepsilon dW_t \right).$$
Comparison under $Q$

\[
\begin{cases}
-dY^\varepsilon_t &= \left\{ \frac{\lambda_t|\beta_t|^2}{|\sigma_t|^4} \left| \mu_t - \lambda_t \beta_t \right|^2 - \frac{|\mu_t - \lambda_t \beta_t|^2}{|\sigma_t|^2} Y^\varepsilon_t - 2\lambda_t \beta_t \frac{(|\mu_t - \lambda_t \beta_t)|^2}{|\sigma_t|^2} \\
&\quad+ \lambda_t - \lambda_t Y^\varepsilon_t - \frac{|\lambda_t \beta_t + (\lambda_t \beta_t - \mu_t) \frac{\lambda_t |\beta_t|^2}{|\sigma_t|^2}|^2}{|\sigma_t|^2 (Y^\varepsilon_t \vee \varepsilon) + \lambda_t |\beta_t|^2} \right\} dt - Z^\varepsilon_t d\tilde{W}_t,
\end{cases}
\]

\[Y^\varepsilon_T = 1.\]

We remark that

\[
\text{generator} \geq -\lambda_t y - \frac{|\mu_t - \lambda_t \beta_t|^2}{|\sigma_t|^2} y.
\]

Therefore, we get from a comparison theorem that

\[
Y^\varepsilon_t \geq \mathbb{E}_Q \left[ \exp \left( - \int_t^T (\lambda_s + \frac{|\mu_s - \lambda_s \beta_s|^2}{|\sigma_s|^2}) ds \right) \big| \mathcal{F}_t \right] \geq \epsilon.
\]

Moreover $Z^\varepsilon \in BMO(\mathbb{P})$. 
Expressions of the generators

\[
\begin{align*}
    f(t, Y, Z, U) &= - \frac{(\mu_t Y + \sigma_t Z + \lambda_t \beta_t U)^2}{|\sigma_t|^2 Y + \lambda_t |\beta_t|^2 (U + Y)}, \\
    g(t, Y, Z, U) &= \frac{1}{Y_t} \left[ Z_t Z + \lambda_t U_t U - \frac{(\mu_t Y_t + \sigma_t Z_t + \lambda_t \beta_t U_t)(\sigma_t Y_t Z_t + \lambda_t \beta_t (U_t + Y_t) U_t)}{|\sigma_t|^2 Y_t + \lambda_t |\beta_t|^2 (U_t + Y_t)} \right], \\
    h(t, \gamma, \Xi, \Theta) &= |Z_t|^2 Y_t + \lambda_t (U_t + Y_t)|U_t|^2 - \frac{|\sigma_t Y_t Z_t + \lambda_t \beta_t U_t (U_t + Y_t)|^2}{|\sigma_t|^2 Y_t + \lambda_t |\beta_t|^2 (U_t + Y_t)}. 
\end{align*}
\]
Solution to BSDE \((g, \xi)\)

We consider the associated decomposed BSDE in \(\mathbb{F}\): find 
\((Y^b, Z^b) \in S^\infty_{\mathbb{F}} \times L^2_{\mathbb{F}}\) such that

\[
\begin{align*}
\frac{dY^b_t}{Y^b_t} &= \left\{ \left( \left( \mu_t - \lambda_t \beta_t \right) Y^b_t + \sigma_t Z^b_t + \lambda_t \beta_t \right) \left( \sigma_t Y^b_t Z^b_t + \lambda_t \beta_t \xi^a_t - \lambda_t \beta_t Y^b_t \right) \right. \\
&\quad \left. Y^b_t \left( |\sigma_t|^2 Y^b_t + \lambda_t |\beta_t|^2 \right) \\
&\quad - \frac{Z^b_t}{Y^b_t} Z^b_t - \frac{\lambda_t}{Y^b_t} \xi^a_t + \frac{\lambda_t}{Y^b_t} Y^b_t \right\} dt + Z^b_t dW_t, \\
Y^b_T &= \xi^b.
\end{align*}
\]
Change of probability

Define the process $\rho$ by $\rho_t := \frac{Z_t^b}{Y_t^b} - \frac{\sigma_t ((\mu_t - \lambda_t \beta_t) Y_t^b + \sigma_t Z_t^b + \lambda_t \beta_t)}{|\sigma_t|^2 Y_t^b + \lambda_t |\beta_t|^2}$. Since $\rho \in BMO(\mathbb{P})$, we can apply Girsanov theorem

$$W_t := W_t - \int_0^t \rho_s ds$$

is a $\tilde{Q}$-Brownian motion, where $\frac{d\tilde{Q}}{d\mathbb{P}} |_{\mathcal{F}_T} := \mathcal{E} \left( \int_0^T \rho_t dW_t \right)$.

Hence, BSDE can be written

$$\begin{cases} 
  dY_t^b &= a_t (Y_t^b - \xi^a_t) dt + Z_t^b d\tilde{W}_t, \\
  Y_{t \wedge \tau}^b &= \xi^b,
\end{cases}$$

with $a_t := \frac{\lambda_t |\sigma_t|^2 Y_t^b - \lambda_t \beta_t ((\mu_t - \lambda_t \beta_t) Y_t^b + \sigma_t Z_t^b)}{Y_t^b (|\sigma_t|^2 Y_t^b + \lambda_t |\beta_t|^2)}$.

We can prove that $Y^b$ defined by

$$Y_t^b := \mathbb{E}_{\tilde{Q}} \left[ \exp \left( - \int_t^T a_u du \right) \xi^b + \int_t^T \exp \left( - \int_t^s a_u du \right) a_s \xi^a_s ds \mid \mathcal{F}_t \right]$$

is solution of this BSDE.
Expressions of the generators

\[ \begin{align*}
  f(t, Y, Z, U) &= -\frac{(\mu_t Y + \sigma_t Z + \lambda_t \beta_t U)^2}{|\sigma_t|^2 Y + \lambda_t |\beta_t|^2 (U+Y)}, \\
  g(t, Y, Z, U) &= \frac{1}{Y_t} \left[ Z_t Z + \lambda_t U_t U - \frac{(\mu_t Y_t + \sigma_t Z_t + \lambda_t \beta_t U_t)(\sigma_t Y_t Z_t + \lambda_t \beta_t (U_t + Y_t) U_t)}{|\sigma_t|^2 Y_t + \lambda_t |\beta_t|^2 (U_t + Y_t)} \right], \\
  h(t, \gamma, \Xi, \Theta) &= |Z_t|^2 Y_t + \lambda_t (U_t + Y_t)|U_t|^2 - \frac{|\sigma_t Y_t Z_t + \lambda_t \beta_t U_t (U_t + Y_t)|^2}{|\sigma_t|^2 Y_t + \lambda_t |\beta_t|^2 (U_t + Y_t)}. 
\end{align*} \]
Solution to BSDE \((\mathfrak{h}, 0)\)

We consider the associated decomposed BSDE in \(\mathbb{F}\): find \((\Upsilon^b, \Theta^b) \in \mathcal{S}_\infty^\infty \times L_\mathbb{F}^2\) such that

\[
\Upsilon^b_t = \int_t^T \left( |Z^b_t|^2 Y^b_t + \lambda_t |\xi^a_t - \Upsilon^b_t|^2 - \frac{\sigma_t Y^b_t Z^b_t + \lambda_t \beta_t (\xi^a_t - \Upsilon^b_t)^2}{|\sigma_t|^2 Y^b_t + \lambda_t |\beta_t|^2} - \lambda_s \Upsilon_s \right) ds
- \int_{t \wedge \tau}^{T \wedge \tau} \Xi^b_s dW_s.
\]

We can prove that \(\Upsilon^b\) defined by

\[
\Upsilon^b_t := \mathbb{E} \left[ \int_t^T \exp \left( - \int_t^s \lambda_u du \right) R_s ds \left| \mathcal{F}_t \right. \right],
\]

where \(R_t := |Z^b_t|^2 Y^b_t + \lambda_t |\xi^a_t - \Upsilon^b_t|^2 - \frac{\sigma_t Y^b_t Z^b_t + \lambda_t \beta_t (\xi^a_t - \Upsilon^b_t)^2}{|\sigma_t|^2 Y^b_t + \lambda_t |\beta_t|^2}\), is solution of this BSDE.
Thanks!