Representation of Dynamic Time-Consistent Convex Risk Measures with Jumps

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Outlines

- Risk measures (VaR, \ldots); Dynamic time-consistent convex risk measures
- Backward Stochastic Differential Equations and $g$-expectation with jumps
- Relation between $g$-expectation and dynamic time-consistent convex risk measure
- Integral representation of the minimal penalty term
- An example
Risk Measures in the Literature

\((\Omega, \mathcal{F}, P)\): Probability space

\(X: \text{R.V.}\)

- **Standard Deviation:** \(R(X) := E[(X - E[X])^2]\)
e.g. Markowitz, Portfolio Selection, 1952;

- **Value at Risk (J.P. Morgan):** \(\forall \alpha \in (0, 1)\)

\[\text{VaR}(\alpha) := \inf \{x : P(X - X_0 \leq x) \geq \alpha\},\]

\[\text{CVaR}(\alpha) := E[X - X_0 \mid X - X_0 \leq \text{VaR}(\alpha)];\]

- **Stone Family of risk measures (1970s)**

\[R(k, \bar{X}, X^*) := \left(E[|X - X^*|^k \mathbb{I}_X \leq \bar{X}]\right)^{\frac{1}{k}};\]

\[\cdots\cdots\]
Coherent risk measures

Artzner, Delbaen, Eber and Heath, Math. Finance, 1999

$$\rho(\cdot) : \mathbb{L}^\infty(\Omega, \mathcal{F}, P) \to \mathbb{R},$$

satisfies four axioms:

- $$\rho(X) \leq 0, \forall X \geq 0,$$
- Subadditivity: $$\rho(X + Y) \leq \rho(X) + \rho(Y),$$
- Translation invariance: $$\rho(X + c) = \rho(X) - c,$$
- Positive homogeneity: $$\rho(\lambda X) = \lambda \rho(X), \forall \lambda \geq 0,$$

If $$\rho$$ satisfies Fatou Property: $$\rho(X) \leq \lim_n \rho(X_n), \forall X_n \to X,$$

$$\rho(X) = \sup_{Q \in \mathcal{P}_0} \{ E_Q[-X] \},$$

$$\mathcal{P}_0$$ is a closed and convex set of probabilities.
Convex risk measures

Föllmer and Schied, Finance and Stochastics, 2002

- Replace "positive homogenity" by the convexity:

\[ \rho(\lambda X + (1 - \lambda)Y) \leq \lambda \rho(X) + (1 - \lambda)\rho(Y), \quad \forall \lambda \in [0, 1]. \]

Then

\[ \rho(X) = \sup_{Q \in \mathcal{P}} \{ E_Q[-X] - C(Q) \}, \]

\( \mathcal{P} \) contains all the probabilities, and

\[ C(Q) := \sup_{X \in L^\infty} \{ E_Q[-X] - \rho(X) \} \]

is called the minimal penalty term.
Dynamic Time-Consistent Convex Risk Measures

Detlefsen and Scandolo, Finance and Stochastics, 2005

\((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,T]}, P)\): filtered probability space.

A family of mappings

\[
\rho_{t,s}(\cdot) : \mathbb{L}^2(\mathcal{F}_s) \rightarrow \mathbb{L}^2(\mathcal{F}_t), \quad 0 \leq t \leq s \leq T
\]

(A1) Monotonicity: \(\forall X, Y \in \mathbb{L}^2(\mathcal{F}_s), \ X \geq Y, \ \rho_{t,s}(X) \leq \rho_{t,s}(Y)\);

(A2) Translation invariance: \(\forall Z \in \mathbb{L}^2(\mathcal{F}_t),\)

\[
\rho_{t,s}(X + Z) = \rho_{t,s}(X) - Z;
\]

(A3) Convexity: for all \(\beta \in [0, 1], \ X, Y \in \mathbb{L}^2(\mathcal{F}_s),\)

\[
\rho_{t,s}(\beta X + (1 - \beta)Y) \leq \beta \rho_{t,s}(X) + (1 - \beta)\rho_{t,s}(Y);
\]
(A4) Normalization: $\rho_{t,s}(0) = 0$.

(A5) Time consistency: $\rho_{t,s}(X) = \rho_{t,r}(-\rho_{r,s}(X))$, $\forall r \in [t, s]$.

Definition

$(\rho_{t,s}(\cdot))_{0 \leq t \leq s \leq T}$ satisfying (A1)-(A5) is called a dynamic time-consistent convex risk measure (DTC risk measure).

(A6) Continuity from below: $X_n \uparrow X$, $P$-a.s.

$$\lim_{n \to \infty} \rho_{t,s}(X_n) = \rho_{t,s}(X), \ P$-a.s.;$$

(A7) $C_{t,s}(P) = 0$, where

$$C_{t,s}(Q) := \text{ess sup}_{X \in \mathbb{L}^\infty(\mathcal{F}_s)} \{E_Q[-X|\mathcal{F}_t] - \rho_{t,s}(X)\}, \forall Q \ll P$$

is the minimal penalty term of $\rho_{t,s}$. 
Representation of DTC Risk Measures

**Proposition (Klöpple and Schweizer (2007), Bion-Nadal (2009))**

\[
\rho_{t,s}(X) = \text{ess sup}_{Q \in \mathcal{P}_t} \{E_Q[-X | \mathcal{F}_t] - C_{t,s}(Q)\},
\]

where \( \mathcal{P}_t = \{Q \sim P \mid Q = P \text{ on } \mathcal{F}_t\} \).

*Time-consistency is equivalent to*

\[
C_{t,s}(Q) = C_{t,r}(Q) + E_Q[C_{r,s}(Q) | \mathcal{F}_t], \quad 0 \leq t \leq r \leq s \leq T.
\]

**Under Brownian Filtration**

- Rosazza Gianin (2006), Risk measure via g-expectation, Insurance Mathematics and Economics
- Delbaen, Peng and Rosazza Gianin (2010), Representation of the penalty term of dynamic concave utilities, Finance and Stochastics
Backward Stochastic Differential Equation with Jumps

\[
\begin{cases}
    dY_t = -g(t, Y_t, Z_t, H_t) \, dt + Z_t \, dW_t + \int_{\mathbb{E}} H_t(e) \tilde{\mu}(dedt); \\
    Y_T = \xi.
\end{cases}
\]  

Denote \((Y, Z, H)\) as the solution, \(\mathcal{E}_g[\xi|\mathcal{F}_t] := Y(t)\) as the \(g\)-expectation.

\((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,T]}, P)\), usual conditions

- \(d\)-dimensional Brownian motion \(\{W_t\}_{t \in [0,T]}\)
- Poisson random measure \(\mu\) on \([0, T] \times \mathbb{E}, \mathbb{E} := \mathbb{R}\backslash\{0\}, \)

\[\tilde{\mu}(dtde) := \mu(dtde) - dt\lambda(de),\]

\[\int_{\mathbb{E}} (1 \wedge |e|^2)\lambda(de) < +\infty.\]
Assumptions on $g$

(H1) $|g(t, y, z, h) - g(t, \hat{y}, \hat{z}, \hat{h})| \leq L(|y - \hat{y}| + |z - \hat{z}| + \|h - \hat{h}\|)$;

(H2) $E \left[ \int_0^T |g_0(t)|^2 \, dt \right] < +\infty$, $g_0(t) := g(t, 0, 0, 0)$;

(H3) $\exists \kappa_1 \geq 0$, $\kappa_2 \in (-1, 0]$, such that

$$g(t, y, z, h) - g(t, y, z, \hat{h}) \leq \int_{\Omega} (h(e) - \hat{h}(e)) \gamma^{y, z, h, \hat{h}}_t(e) \lambda(de),$$

where

$$\kappa_2 (1 \wedge |e|) \leq \gamma^{y, z, h, \hat{h}}_t(e) \leq \kappa_1 (1 \wedge |e|),$$

(H4) $g(t, y, 0, 0) = 0$, a.e., a.s.;

(H5) $g$ is independent of $y$. 
Representation of the generator

**Proposition**

*Fixed* $x, p, y \in \mathbb{R}, \forall \varepsilon > 0, t + \varepsilon \leq T$. Consider the following FBSDE

\[
\begin{aligned}
X^t,x_s &= x + \int_t^s b(X^t,x_u)\,du + \int_t^s \sigma(X^t,x_u)\,dW_u \\
&\quad + \int_t^s \int_{\mathbb{E}} \eta(e, X^t,x_u)\tilde{\mu}(d\epsilon du), \quad s \in [t, t + \varepsilon],
\end{aligned}
\]

\[
\begin{aligned}
Y^t,x,p,y_s &= y + p(X^t,x_{t+\varepsilon} - x) + \int_s^{t+\varepsilon} g(u, Y^t,x,p,y_u, Z^{t,x,p,y}_u, H^{t,x,p,y}_u)\,du \\
&\quad - \int_s^{t+\varepsilon} Z^{t,x,p,y}_u\,dW_u - \int_s^{t+\varepsilon} \int_{\mathbb{E}} H^{t,x,p,y}(u, e)\tilde{\mu}(d\epsilon du), \\
&\quad s \in [t, t + \varepsilon],
\end{aligned}
\]
where \( g \) satisfies \((H1)-(H2)\)

\[
\begin{align*}
b : \mathbb{R} &\to \mathbb{R}, \quad \sigma : \mathbb{R} &\to \mathbb{R}^d, \quad \eta : \mathbb{E} \times \mathbb{R} &\to \mathbb{R}, \quad \eta(e, 0) \in L^2(\mathbb{E}), \text{ and} \\
\exists \ L > 0 \text{ such that} \\
|b(x_1) - b(x_2)| + |\sigma(x_1) - \sigma(x_2)| &\leq L|x_1 - x_2|, \quad \forall x_1, x_2 \in \mathbb{R}, \\
|\eta(e, x_1) - \eta(e, x_2)| &\leq L(1 \land |e|)|x_1 - x_2|, \quad \forall x_1, x_2 \in \mathbb{R},
\end{align*}
\]

then there exists \( A \subset [0, T] \) with full Lebesgue measure, such that \( \forall t \in A, \forall q \in [1, 2) \),

\[
L^q - \lim_{\varepsilon \downarrow 0} \frac{Y_{t,x,p,y} - y}{\varepsilon} = g(t, y, \sigma(x)p, \eta(\cdot, x)p) + b(x)p.
\]
Converse Comparison Theorem

**Theorem**

Suppose that $g_1$ and $g_2$ are two generators of BSDE (1), and they satisfy assumptions (H1), (H3) and (H4). If $\mathcal{E}_{g_1}[\xi|\mathcal{F}_t] \geq \mathcal{E}_{g_2}[\xi|\mathcal{F}_t]$ for all $\xi \in L^2(\mathcal{F}_T)$, then there exists a subset $S \subseteq [0, T]$ with $\nu([0, T] \setminus S) = 0$ ($\nu$ is the Lebesgue measure), such that for any $t \in S$,

$$g_1(t, y, z, h) \geq g_2(t, y, z, h), \quad \text{P-a.s.}$$

for all $y \in \mathbb{R}$, $z \in \mathbb{R}^d$, $h \in L^2(\mathbb{E})$. 

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Dynamic Convex Risk Measures
Corollary

Let $g$ satisfy the assumptions (H1), (H3) and (H4). Then for all $\beta \in [0, 1]$, the following are equivalent,

(1) $\forall \xi_1, \xi_2 \in L^2(\mathcal{F}_T)$,

$$ \mathcal{E}_g[\beta \xi_1 + (1 - \beta)\xi_2|\mathcal{F}_t] \leq \beta \mathcal{E}_g[\xi_1|\mathcal{F}_t] + (1 - \beta) \mathcal{E}_g[\xi_2|\mathcal{F}_t]; $$

(2) for all $y_1, y_2 \in \mathbb{R}, z_1, z_2 \in \mathbb{R}^d$ and $h_1, h_2 \in L^2(\mathbb{E})$,

$$ g(t, \beta y_1 + (1 - \beta)y_2, \beta z_1 + (1 - \beta)z_2, \beta h_1 + (1 - \beta)h_2) $$

$$ \leq \beta g(t, y_1, z_1, h_1) + (1 - \beta)g(t, y_2, z_2, h_2); $$
Proposition

Suppose that $g$ satisfies (H1)-(H3). Then the following are equivalent:

1. $\mathcal{E}_g[- \cdot | \mathcal{F}_t], t \in [0, T]$ is a DTC risk measure.
2. $g$ satisfies (H4) and (H5), and $g$ is jointly convex with respect to $z$ and $h$. 
Relation Between DTC Risk Measure and \( g \)-expectation

**Proposition**

If a DTC risk measure \( \rho_{t,T}(\cdot) \) is strictly monotone and \( \rho_{t,T}(-\cdot) \) is \( \mathcal{E}_{g_{\kappa_1,\kappa_2}} \)-dominated for some \( \kappa_1 \geq 0 \) and \( \kappa_2 \in (-1, 0] \), then

1. \( \exists g : \Omega \times [0, T] \times \mathbb{R}^d \times L^2(\mathbb{E}) \to \mathbb{R} \) such that

\[
\rho_{t,T}(\cdot) = \mathcal{E}_g[-\cdot|\mathcal{F}_t];
\]

2. \( g \) satisfies (H1)-(H5) and is jointly convex with respect to \( z \) and \( h \). Moreover, \( \kappa_1 \) is the Lipschitz coefficient on \( z \), \( \kappa_1 - \kappa_2 \) on \( h \), and \( \kappa_1 \) and \( \kappa_2 \) are the two coefficients in (H2).
\[ \mathcal{E}_{g_{\kappa_1,\kappa_2}} \text{-Domination} \]

**Definition**

\[ \phi[\cdot|\mathcal{F}_t] : L^2(\mathcal{F}_T) \rightarrow L^2(\mathcal{F}_t), \quad \forall t \leq T, \]

If for all \( \xi_1, \xi_2 \in L^2(\mathcal{F}_T), \)

\[ \phi[\xi_1 + \xi_2] - \phi[\xi_2] \leq \mathcal{E}_{g_{\kappa_1,\kappa_2}}[\xi_1], \]

where

\[ g_{\kappa_1,\kappa_2}(t, z, h) := \kappa_1 |z| + |\kappa_1| \int_E (1 \wedge |e|)h^+(e)\lambda(de) \]

\[ - \kappa_2 \int_E (1 \wedge |e|)h^-(e)\lambda(de). \]

Then \( \phi \) is \( \mathcal{E}_{g_{\kappa_1,\kappa_2}} \)-dominated.
Truncated DTC Risk Measure

- $\rho_{t,s}$: strictly monotone, $\mathcal{E}_{g_{\kappa_1,\kappa_2}}$-dominated, BSDE

What about a general $\rho_{t,s}$?

Define

$$\rho^n_{t,s}(X) := \text{ess sup}_{Q \in \mathcal{P}^n_t} \{ E_Q[-X|\mathcal{F}_t] - C_{t,s}(Q) \} ,$$

where

$$\mathcal{P}^n_t := \{ Q \in \mathcal{P}_t \mid |\theta(u,\omega)| \leq n,$$

$$-(1 - \frac{1}{n})(1 \wedge |e|) \leq \zeta(u,e,\omega) \leq n(1 \wedge |e|), \forall u \in [t,T] \}$$

with

$$\frac{dQ}{dP} = \mathcal{E} \exp \left\{ \int_0^T \theta_s dW_s + \int_0^T \int \zeta(e,s) \tilde{\mu}(deds) \right\} .$$
Let

\[ C_{t,s}^n(Q) := \begin{cases} 
C_{t,s}(Q), & Q \in \mathcal{P}_t^n; \\
+\infty, & \text{else},
\end{cases} \]

then

\[ \rho_{t,s}^n(X) := \text{ess sup}_{Q \in \mathcal{P}_t^n} \{ EQ[-X|\mathcal{F}_t] - C_{t,s}^n(Q) \}, \]

\[ \rho_{t,s}(\cdot) \text{ is } \mathcal{E}_{g_{n,-\frac{1}{n}}}-\text{dominated}. \]
Proposition

We have the following two assertions for $\rho_{t,s}^n$:

(1) $\rho_{t,s}^n$ is also a DTC risk measure satisfying (A1)-(A7) with $C_{t,s}^m$ being its minimal penalty term;

(2) $\exists g_n : [0, T] \times \Omega \times \mathbb{R}^d \times L^2(\mathbb{E}) \to \mathbb{R}$ satisfying (H1)-(H5) and jointly convex with respect to $z$ and $h$, such that

$$
\rho_{t,T}^n(X) = -X + \int_t^T g_n(s, Z_s, H_s) \, ds - \int_t^T Z_s \, dW_s \\
- \int_t^T \int_{\mathbb{E}} H_s(e) \tilde{\mu}(deds), \quad t \in [0, T].
$$
Proposition

Define

\[ f_n(t, \omega, a, b) := \sup_{(z,h) \in \mathbb{R}^d \times L^2(E)} \left\{ \langle a, z \rangle + \langle b, h \rangle - g_n(t, \omega, z, h) \right\} \]

for all \((a, b) \in \mathbb{R}^n \times L^2(E)\). Here \(f_n\) can take the value \(+\infty\) and the integration here is defined to be extended. Then

\[ C_{t,s}^n(Q) = \mathbb{E}_Q \left[ \int_t^s f_n(r, \theta_r, \zeta_r) dr \mid \mathcal{F}_t \right], \quad \forall Q \sim P, \]

and

\[ \rho_{t,s}^n(X) = \text{ess sup}_{Q \in \mathcal{P}_t} \mathbb{E}_Q \left[ -X - \int_t^s f_n(r, \theta_r, \zeta_r) dr \mid \mathcal{F}_t \right]. \]
Limit function

Lemma

Define

\[ f(t, \omega, a, b) = \inf_n f_n(t, \omega, a, b), \]

then for any \((t, \omega, a, b)\), the following two are alternative:

(i) \( \exists n, \) such that \( f_n(t, \omega, a, b) < +\infty \), then, \( \forall m \geq n, \)
\( f_m(t, \omega, a, b) = f_n(t, \omega, a, b) = f(t, \omega, a, b); \)

(ii) \( \forall n, \) \( f_n(t, \omega, a, b) = +\infty \), then we define \( f(t, \omega, a, b) = +\infty. \)
\[ \hat{P} := \left\{ Q \sim P \mid \zeta(u, e) > -(1 \wedge |e|) \right\} \]

**Theorem**

Let \( \rho_{t,s}(\cdot) \) be a DTC risk measure satisfying assumption (A1)-(A7). Then, for any \( Q \in \hat{P} \), we have

\[
C_{t,s}(Q) \leq E \left[ \int_t^s f(r, \theta_r, \zeta_r)dr \middle| \mathcal{F}_t \right]
\]

with the equality “=” holding for \( Q \in \bigcup_{n=1}^{\infty} \mathcal{P}_t^n \).
Let $\mathbb{E} := \{1\}$, then $\mu(dtde) := N(dt)$ is a Poisson process.

**Theorem**

$$C_{t,s}(Q) = E_Q \left[ \int_t^s f(u, \theta_u, \zeta_u)du \right| \mathcal{F}_t], \ Q \sim P,$$

and

$$\rho_{t,s}(X) = \operatorname{ess sup}_{Q \in \mathcal{P}_t} E_Q \left[ -X - \int_t^s f(r, \theta_r, \zeta_r)dr \right| \mathcal{F}_t].$$

- If $\mathbb{E}$ is a finite set, similar equalities hold too.
Example: Loss Function

Loss function \( l : \mathbb{R} \to \mathbb{R} \), nondecreasing, convex

\[
\mathcal{A}_{t,T} = \{ X \in L^\infty(\mathcal{F}_T) \mid E_P[l(-X)|\mathcal{F}_t] \leq x_0 \}
\]

\[
\rho_{t,T}^\mathcal{A}(X) := \text{ess inf}\{ \xi \in L^\infty \mid \xi + X \in \mathcal{A} \}
\]

Time-consistent \( \Leftrightarrow l \) is a linear or exponential function

\[
l(x) := \exp\{x\}, \quad x_0 := 1.
\]

\[
\rho_{t,T}^\mathcal{A}(X) = \log (E_P[\exp\{-X\}|\mathcal{F}_t]) = \text{ess sup}_{Q \in \mathcal{P}_t} \left\{ E_Q[-X|\mathcal{F}_t] - C_{t,T}(Q) \right\},
\]

\[
C_{t,T}(Q) := E_Q \left[ \log \frac{dQ}{dP} \middle| \mathcal{F}_t \right] = E_Q \left[ \int_t^T \frac{\left| \theta_s \right|^2}{2} + \int_E \left( \zeta(s,e) \log(1 + \zeta(s,e)) \right) \right.
\]
Example (continue)

\[ f(t, a, b) = \frac{|a|^2}{2} + \int\mathcal{E} \left( b(e) \log(1 + b(e)) + \log(1 + b(e)) - b(e) \right) \nu(de). \]

Define

\[ g(t, z, h) = \frac{|z|^2}{2} + \int\mathcal{E} \left[ -h(e) + \exp\{h(e)\} - 1 \right] \lambda(de), \]

then, \( f \) is the conjugate function of \( g \), and vice versa.

\[ \rho_{t,T}(X) = -X + \int_t^T g(t, Z, H) \, ds - \int_t^T Z(s) \, dW_s \]

\[ + \int_t^T \int\mathcal{E} H(s, e) \tilde{\mu}(de \, ds). \]
Thank you!