Stochastic Processes and Applications (2nd Ed.): Errata–Corrigenda.

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Please email further errors to samuel.cohen@maths.ox.ac.uk. Thanks to the following people, who have kindly pointed out errors: Pietro Siorpaes, Gonçalo Simões, Victor Fedyashov, Martin Jönsson, Nicola Doninelli, Alexandros Saplaouras, Vladimir Vovk, Andi Wang, Willem van Zuijlen

Mathematical errors

- p32, Lemma 1.5.18: The proof of this standard result is incorrect (in particular, the claim that $F^n(x)$ forms a Cauchy sequence). A correct proof can be found in [160], p216.

- p43, lines 15–23: A better proof that g is in L^q is as follows. Let ϕ_n be a sequence of nonnegative simple functions increasing pointwise to $gI_{\{g\geq 0\}}$. As ϕ_n is simple, we know that

  $$\int_S \phi_n^q d\mu = \int_S \phi_n^{1+q/p} d\mu \leq \int_S \phi_n^{q/p} g\, d\mu = F(\phi_n^{q/p}).$$

  By boundedness of F, for some $c \in \mathbb{R}$,

  $$\int_S \phi_n^q d\mu \leq F(\phi_n^{q/p}) \leq c\Big(\int_S \phi_n^q d\mu\Big)^{1/p}.$$

  Rearranging, we obtain $\big(\int_S \phi_n^q d\mu\big)^{1/q} \leq c$, that is, ϕ_n is a sequence uniformly bounded in L^q. By the monotone convergence theorem, this implies that $gI_{\{g\geq 0\}} \in L^q$.

  We can now repeat this argument for a sequence increasing to $(-g)I_{\{g<0\}}$, and so conclude that $(-g)I_{\{g<0\}} \in L^q$ and, therefore, $g \in L^q$.

- p52, Definition 2.1.11: The definition given is for 'pairwise independence'. In general a finite collection of events (one per random variable) should be allowed to appear simultaneously.

- p64, Theorem 2.5.10: The proof stated contains an error. A correct and simpler version (similar to Theorem 4.4.6) is as follows: From Jensen's inequality, we know that, for any $\lambda \geq 0$,

  $$I(\mathcal{G},\lambda) := E\big[I_{\{|E[X|\mathcal{G}]|>\lambda\}}|E[X|\mathcal{G}]|\big] \leq E\big[I_{\{|E[X|\mathcal{G}]|>\lambda\}}|X|\big].$$

By Markov's inequality,

$$P\big(|E[X|\mathcal{G}]| > \lambda\big) \leq \frac{E\big[|E[X|\mathcal{G}]|\big]}{\lambda} \leq \frac{E\big[|X|\big]}{\lambda},$$

which tends to zero uniformly in $\mathcal{G}$ as $\lambda \to \infty$. As the measure $\nu(A) = E[I_A|X|]$ is absolutely continuous with respect to P, from Lemma 1.6.2 we see that $I(\mathcal{G}, \lambda)$ tends to zero uniformly in n as $\lambda \to \infty$. Hence, $\{E[X|\mathcal{G}]\}_{\mathcal{G}\in\mathfrak{G}}$ is uniformly integrable.

- p85, Problem 3.4.3: "evanescent sets" should be "evanescent sets and their complements"
- p102, Remark 4.6.4: This remark is false as stated (Fatou's inequality does not show the stated relationship, which is generally false). However, it is true if we restrict our attention to *uniformly integrable* nonnegative supermartingales, in which case the result follows from the Vitali convergence theorem.
- p103, final line: "martingale" should be "uniformly integrable martingale".
- p112, Theorem 5.1.8: We should be more precise here, as X_{t+} and X_{t-} are being used to indicate the right and left limits of X taken using sequences in the rationals, as in Lemma 5.1.5 (though these could be replaced with some other dense countable set). We do not guarantee the existence of right and left limits for X when taken over the reals.
- p116, Remark 5.4.2: As on p102, this should only apply to uniformly integrable processes.
- p132, Definition 5.6.2: This definition should be for a general right-continuous process X, rather than for a right-continuous uniformly integrable supermartingale (otherwise Lemma 5.6.6 becomes unclear).
- p132, Lemma 5.6.5: Should state that "Every càdlàg martingale is in $\mathcal{M}_{\mathrm{loc}}$, that is, is locally a uniformly integrable martingale."
- p146, The comment is made in Remark 6.2.17 that $T - \epsilon$ is generally not a stopping time, so $X_{T-\epsilon}$ is only $\mathcal{F}_{T-}$-measurable, not $\mathcal{F}_{T-\epsilon}$-measurable. The point is that $\mathcal{F}_{T-\epsilon}$ is not well defined (we don't have a notion of the σ-algebra at a random time), which should be made more clearly.
- p147, line 17: The equation $\{T_A \leq T\} = \{T = 0\} \cup \big(\bigcap_n \{S_n < T\}\big)$ is incorrect, as $\{T = 0\}$ should be $\{T_A = 0\} = \{T = 0\} \cap A$. However, $\{T_A = 0\} \in \mathcal{F}_0 \subset \mathcal{F}_{T-}$, so the argument then proceeds as written.
- p377, Theorem 15.2.8: The assumption of the theorem that $\langle Y \rangle$ exists under P is unnecessary, as the changes to the proof on the following page will make clear.
- p378, In the second displayed equation on the page, we should have that, using the BDG inequality, as $n \to \infty$ we have

$$cE^Q\Big[\Big(\int_{[0,\infty[} (H^n - H^m)^2 d[\tilde{Y}]\Big)^{1/2}\Big] \leq \|X^n - X^m\|_{\mathcal{H}^1(Q)} \to 0.$$

Therefore, at least for a subsequence, H^n converges pointwise $d[\tilde{Y}] \times dP$-a.e. As H^n is predictable, this implies that H^n converges pointwise almost everywhere on the predictable support of $d[\tilde{Y}] \times dP$ (that is, the support of $d[\tilde{Y}] \times dP$ considered as a measure on the predictable σ-algebra, which agrees with the support of $d\langle \tilde{Y}\rangle \times dP$ if $\langle \tilde{Y}\rangle$ exists). Taking a limit, we have a predictable process H with the desired properties, and the proof follows.

- p407, Theorem 16.2.6: The proof of this theorem, as stated, requires us to know that the space of semimartingales under the stated operator norm is a Banach space, otherwise Lemma 1.5.9 does not apply. The difficulty is in verifying that the space is complete, and agrees with $\mathcal{H}^p_{\mathcal{S}}$. The following argument can be used to check this.

  Let $\|X\|_{op} = \sup_{H:|H|\leq 1}\{\|H \bullet X\|_{S^p}\}$ be the stated operator norm. Suppose X^n is a Cauchy sequence under this norm. It is clear that X^n is Cauchy in the semimartingale topology, so converges (in semimartingale topology) to some semimartingale X. Let $Y^{k,n} = X^{n+k} - X^k$. Fatou's lemma shows that for any simple H with $|H| \leq 1$,

  $$\|H \bullet (X - X^k)\|_{S^p} \leq \liminf_n \|H \bullet (X^n - X^k)\|_{S^p} \leq \liminf_n \|Y^{k,n}\|_{op}.$$

  Taking the supremum over such H we have

  $$\|X - X^k\|_{op} \leq \liminf_n \|Y^{k,n}\|_{op}.$$

  As X^k is Cauchy, $\liminf_n \|Y^{k,n}\|_{op} \to 0$ as $k \to 0$, so we have $\|X - X^k\|_{op} \to 0$. Therefore, X^k converges to X in the operator norm, and the completeness is proven.

  Whenever $\|X\|_{op} < \infty$, we know that X is a special semimartingale, so we can write $X = M + A$, where M is a local martingale and A is a predictable process. Taking $H = \mathrm{sign}(dA)$, we know that $H^2 \equiv 1$, so $\|H \bullet X\|_{op} = \|X\|_{op}$, and $H \bullet X = H \bullet M + H \bullet A$. However, $H \bullet A$ is increasing, so after localizing, by Garsia's inequality, $\|A\|_{op} = \|H \bullet A\|_{op} \leq \|X\|_{op} < \infty$. The proof as stated then shows that $A \in \mathcal{H}^p_{\mathcal{S}}$, and hence $X \in \mathcal{H}^p_{\mathcal{S}}$, so our spaces agree, and the equivalence of the norms follows as stated.

  (Thanks to Pietro Siorpaes for pointing out this difficulty, and for assistance in finding a simple proof of the missing steps.)

- p590, on the last line, the mean variation should be defined as

  $$\mathrm{MV}(X, \pi) = E\Big[|X_0| + \sum_{i=1}^{n} \big|E[X_{t_i} - X_{t_{i-1}}|\mathcal{F}_{t_{i-1}}]\big|\Big].$$

  (the conditioning on the right is wrong).

Typos,etc...

- p14, line 5: the line should begin "Next assume g is a"
- p27, Theorem 1.4.6: f should be a nonnegative measurable function

- p28, line 14: "topology" should be "topologies"
- p30, line -10: it should be clarified that the norm of (y, z) is $\|y\| + \|z\|$.
- p31, Lemma 1.5.16: in [160] (p.303), this result is proven only for the closed unit ball. The result as stated can be found in Whitely, R. *An elementary proof of the Eberlein–Šmulian theorem*, Mathematische Annalen, 172(2):116–118, 1967
- p33, line 12: $\mathcal{B}([0, \infty])$ should be $\mathcal{B}(\overline{\mathbb{R}})$.
- p42, line 1 and line 6: "measure space" should be "measurable space"
- p65, line -5: "$\mu_{\mathcal{G}}$" should be "$\mu|_{\mathcal{G}}$"
- p68, line 6 and line 8: "$k \in \mathbb{R}$" should be "$k > 0$"
- p100, line 12: the right hand side of the first line of this equation should be $-\int_{[0,\infty]} \lambda^p d\tilde{F}(\lambda)$, (not dF)
- p101, line 6: this line should read

$$\sup_n E[(-X_n)^-] = \sup_n E[X_n^+] \leq \sup_n E[|X_n|^p]^{1/p} = \sup_n \|X_n\|_p < \infty$$

- p106, line 21: the "to" before the closing parenthesis is redundant
- p107, line 7: to clarify, $n^{-\alpha}$ is the conditional variance of $Y_n|\mathcal{F}_{n-1}$, not its standard deviation.
- p111, line -5: to clarify, this condition is satisfied for any nonincreasing sequence of stopping times, provided X satisfies some integrability conditions (as given in Theorem 5.3.1)
- p125, line 14: "$\mathcal{F}_t = \sigma(X_s : s < t)$" should be "$\mathcal{F}_t = \sigma(X_s : s \leq t)$" for consistency (although it makes no difference, as our processes here are continuous)
- p125, line -10: "For a X" should be "For X a"
- p136, line 8: "$\{\tilde{\mathcal{F}}_t = \mathcal{F}_{c^2 t}\}_{t \geq 0}$" should be "$\{\tilde{\mathcal{F}}_t = \mathcal{F}_{c^{-2} t}\}_{t \geq 0}$"
- p140, line 13: $S(\omega)$ should be $T(\omega)$.
- p146, line 1: the final vertical bar should be on the next line.
- p148, line 9: The reference to "Theorem 3.1.13 and Theorem 6.1.4(iv)" should be simply a reference to "Theorem 3.1.15".
- p148, line -12 and -8: "Theorem 6.3.1" should be "Theorem 6.3.2". In case of confusion, it should also be clarified that, except in the first line of this proof, we are assuming that $\{S = T\} \in \mathcal{F}_{T-}$ for all predictable T.
- p149, line 21: when proving (i)⇒(iii), we consider T to be any predictable stopping time.

- p155, line -9 and -3: stochastic intervals in this example should be open on the right, that is, $[\![S,T[\![$ not $[\![S,T]\!]$.

- p159, line 13: "a complete filtration $\{F_t\}_{t\in[0,\infty[}$" should be "a filtration $\{F_t\}_{t\in[0,\infty[}$ satisfying the usual conditions"

- p165, line -7: "Corollary 7.2.5" should be "Corollary 7.1.9"

- p165, line -5: "$\phi(x) = \frac{\pi}{4}\arctan(x)$" should be "$\phi(x) = \frac{2}{\pi}\arctan(x)$"

- p167, line 12 and 13: "$I_{]\!]S,T[\![}$" should be "$I_{]\!]S,T]\!]}$" while "only on $[\![S]\!]$" should be "only on $[\![S]\!] \cup [\![T]\!]$"

- p167, line -1: it should be made clear the equality should hold for all $T \in \mathcal{T}_x$

- p170, line 3 and elsewhere on this page: "From Theorem 7.6.5" should be "From Theorem 7.6.5 and Remark 7.6.4"

- p171, line 6: "$A \subseteq [0,t[\times \Omega$" should be "$A \subseteq [0,\infty[\times \Omega$"

- p171, line 9: "Theorem 7.4.1 or otherwise" should be "Theorem 7.4.1, Theorem 7.3.3 or otherwise"

- p176, line 17: V should be $\mathcal{V}$.

- p186, line 14: the final equation has an excess parenthesis, it should read "$E[(X \bullet A)_\infty] = E[(Y \bullet A)_\infty]$"

- p194, line -3: $X^{T_n} \in \mathcal{A}_{\text{loc}}$ should be $X^{T_n} \in \mathcal{A}$

- p211, Definition 10.1.1: the index of X on the right of the displayed equations should be s in the first definition, and u in the second

- p214, lines 10 and 11: The sets considered are uniformly integrable with respect to n, not k. That is, it should read "$\{|X^n_{T_k} - X_{T_k}|\}_{n\in\mathbb{N}}$ is uniformly integrable. Therefore, $\{(Y^n)^*_{T_k}\}_{n\in\mathbb{N}}$ is uniformly integrable."

- p229, line -6 and p230 lines 4 and 9: On each occasion, $\big(X_s^2 I_{\{|X_s|\leq a\}} + |X_s| I_{\{|X_s|\leq a\}}\big)$ should be $\big(X_s^2 I_{\{|X_s|\leq a\}} + |X_s| I_{\{|X_s|> a\}}\big)$ (or similarly with stopping times)

- p247, line -1: ψ should be given by $\psi(\delta) = (4\delta/(\beta^2 - 1 - \delta^2))^2$

- p250, line -3: In the displayed equation of Remark 11.5.8, $\sup_t(\Delta M_t)^p$ should be $\sup_t |\Delta M_t|^p$

- p295, line 20: the integral on the right should be $\int_{]0,t]} \lambda(A,s)dF_s$ (rather than being taken over $]0,t[$

- p304, line 13: $\tilde{\Sigma}_x = \Sigma_x \otimes \mathcal{Z}$ should be $\tilde{\Sigma}_x = \Sigma_x \otimes \mathfrak{Z}$

- p318, line 14: the left hand side of the equation should be $M_\mu[\Delta X|\tilde{\Sigma}_p]$

- p326, in the Lévy–Khintchine Formula, the left hand side of the first equation should read $\int_{\mathbb{R}^d} e^{i\langle x,y\rangle}\mu(dy)$

- p346, line -1: the integral $\int_{]0,t]} X_{s-}dY$ should be $\int_{]0,t]} X_{s-}dY_s$
- p373, line 4: $(\Omega, \mathcal{F}, P)$ should be $(\Omega, \mathcal{F})$
- p468, line 3: ξ should be Y_T.
- p470, line 12: Y^* should be Y_t^*.
- p473, equation (19.3), the term $f(\omega, t, Y_t, Z_t, \Theta_t) - \tilde{f}(\omega, t, \tilde{Y}_t, \tilde{Z}_t, \tilde{\Theta}_t)$ should be $f^1(\omega, t, Y_t^1, Z_t^1, \Theta_t^1) - f^2(\omega, t, Y_t^2, Z_t^2, \Theta_t^2)$
- p477, line 21: $[Y, \Gamma]_t$ should be $d[Y, \Gamma]_t$
- p484, Remark 19.4.2: the domain of g (in the first bullet point) is $\mathcal{Z} \times [0, T] \times \mathbb{R}^d$, not $\mathbb{R}^n \times [0, T] \times \mathbb{R}^d$
- p486, line 26: $t \geq x$ should be $t' \geq t$.
- p487, line -2: $\frac{\partial v}{\partial t}(t, X_s^{(t,x)}) + \mathcal{L}_t(t, X_s^{(t,x)})$ should be $\frac{\partial v}{\partial s}(s, X_s^{(t,x)}) + \mathcal{L}_s(s, X_s^{(t,x)})$
- p488, line 2: the right hand side of this line should be

  $$-f\big(s, X_s^{(t,x)}, v(s, X_s^{(t,x)}), \partial_x v(s, X_s^{(t,x)})\sigma(s, X_s^{(t,x)}), \tilde{v}(s, X_s^{(t,x)})\big)ds$$

  and the final line of the equation should also have t replaced by s (except in the superscript of X)
- p488, equation (19.9) should read

  $$0 = \frac{\partial v}{\partial t}(t, x) + \mathcal{L}_t v(t, x) + f\big(t, x, v(t, x), \partial_x v(t, x)\sigma(t, x)\big).$$

  and in the following equation, the final (s, x) should be (t, x).
- p522, line 8, the equation should be $M_t^{u^*} = E_{u^*}[M_\tau^{u^*} | \mathcal{F}_t]$.
- p523, Lemma 21.3.1: in the definition of J, the integral on the right should be $\int_{]t,T]} c(\omega, s, u_s)ds$, not $\int_{]t,T]} c(\omega, t, u_t)dt$
- p528, equation (21.2): X_t should be X_{t-} on the right hand side (this is particularly important in the final term)
- p528, equation (21.4), p529 line 8: there should be no dt at the end of the line, and the integral term should be

  $$\int_{\mathcal{Z}} g(\zeta, t, X_t)\big(\beta(\zeta; t, X_t, u_t) - 1\big)\nu(d\zeta).$$

- p530, Theorem 21.4.7: as f doesn't depend on ω directly, the conditions should hold dt almost everywhere (not $dP \times dt$ almost everywhere).
- p531, Definition 21.4.9: Whenever it does not have an argument, v should be evaluated at either (t, x) or (s, x), as appropriate.
- p553, equation (22.26), the final integral should be with respect to $d\tilde{Y}_u$, not $d\tilde{Y}_t$

- p555 lines 6–8: N_t^{ij} should be $\mathcal{J}_t^i$, J_t^i should be $\mathcal{O}_t^i$ and G_t^i should be $\mathcal{T}_t^i$, for consistency with later sections.
- p562, line 9: This equation should read

$$\sigma_t(\langle X_s, e_i\rangle X) = E_Q[\Lambda_t\langle X_s, e_i\rangle X|\mathcal{Y}_t] \propto E[\langle X_s, e_i\rangle X|\mathcal{Y}_t].$$

- p591, lines -3 to -4 (the final displayed equation on the page) should read

$$\begin{aligned}\sum_{i=1}^{n}\big|E[X_{t_i} - X_{t_{i-1}}|\mathcal{F}_{t_{i-1}}]\big| &\leq \sum_{i=1}^{n}\Big(E\big[|B_{t_i} - B_{t_{i-1}}|\,\big|\mathcal{F}_{t_{i-1}}\big] + E\big[|C_{t_i} - C_{t_{i-1}}|\,\big|\mathcal{F}_{t_{i-1}}\big]\Big)\\ &= E\big[B_{t_n}|\mathcal{F}_{t_{n-1}}\big] + E\big[C_{t_n}|\mathcal{F}_{t_{n-1}}\big]\end{aligned}$$

  (the conditioning on the right is wrong).
- p593, lines 16, 19 and 20: these equations all have expectations conditional on $\mathcal{F}_{t_i}$, which should be conditional on $\mathcal{F}_{s_i}$.
- p594, line -7: the limit should be as $n \to \infty$, not $n \to 0$.
- p608, line 6: isomporphic should be isomorphic
- p612, line 9: Strook should read Stroock
- p620, line 14: on the last line of the second displayed equation, $f(Z_t, \Theta_t)$ should be $|f(Z_t, \Theta_t)|$, so that the use of the monotone convergence theorem on line 15 is valid.
- p658, the entry $[\![T[\![$ should be $[\![T]\!]$