Abstract

We consider coherent sublinear expectations on a measurable space, without assuming the existence of a dominating probability measure. By considering a decomposition of the space in terms of the supports of the measures representing our sublinear expectation, we give a simple construction, in a quasi-sure sense, of the (linear) conditional expectations, and hence give a representation for the conditional sublinear expectation. We also show an aggregation property holds, and give an equivalence between consistency and a pasting property of measures.

1 Introduction

Decision making in the presence of uncertain outcomes is a fundamental human activity. In many cases, we need to make decisions, not only when we do not know what the outcome of our decision will be, but when we do not even know the probabilities of different outcomes. In this setting (commonly known as Knightian uncertainty, following [11]) the classical mathematical approach based on the mathematical expectation is insufficient. An alternative approach in this context is to take the ‘worst case’ under a range of different probability measures, which leads to a form of risk-averse decision making. This approach has strong axiomatic support (see Theorem 1) and is amenable to mathematical analysis.

When all the probability measures we consider agree on what events will occur with probability zero, this approach is, from a mathematical perspective, a relatively straightforward generalisation of the classical theory. On the other hand, when the measures do not agree in this manner (and more generally, when there is no dominating probability measure), then many difficulties arise, cutting to the heart of the mathematical theory of probability. In particular, results which are known to hold ‘with probability one’ in the classical setting

*Thanks to Terry Lyons and Freddy Delbaen for useful conversations during the preparation of this paper. In particular to F. Delbaen for the proof of Lemma 4.
(for example, the existence and uniqueness of the conditional expectation, martingale convergence results, the martingale representation theorem, etc...) may cease to be true in this more general setting.

In some ways, this issue may seem unreasonably abstract, however it arises even in the common case of the analysis of a Brownian motion, where the volatility is known only to lie within a given bound. This problem has been studied in various frameworks by various authors, for example, Lyons [12], Peng and coauthors [14, 6, 3], Soner, Touzi and Zhang [15, 16], Bion-Nadal and Kervarec [2] and Nutz [13], amongst many others.

In this type of analysis, the detailed structure of the mathematical spaces under consideration comes to the fore, and some technical details are needed. One option is to assume that the underlying measurable space can be viewed as a separable topological space (Ω, B(Ω)), and then to only consider those random variables which are quasi-continuous as functions Ω → R. This is the approach taken in Denis et al. [6]. This is in some ways unsatisfactory, as it implies that there are events (which can be easily assigned probabilities in the classical setting) which we refuse to consider when in the setting of uncertainty, purely due to insufficient continuity. Furthermore, by results of Bion-Nadal and Kervarec [2], for random variables in this class there exists a dominating probability measure, that is, there exists a measure θ* such that a (quasi-continuous) set is null for every test measure if and only if it is θ*-null. In this sense, the problem is avoided, as classical methods can be used.

A different assumption is made in Soner, Touzi and Zhang [16], where the set of test measures is assumed to be made up of measures in a particular separable class. In particular, they consider the measures induced on Wiener space by right-constant volatility processes satisfying some further restrictions (see Example 4). Under this assumption, they prove an aggregation property, with which much of the desired analysis can be performed. This approach is possibly unsatisfying as it is restricted to the problem of volatility uncertainty, and it is not apparent how this would generalise to other situations. For example, in discrete time (as one might obtain simply by taking the δ-skeleton of their setting), there is no process analogous to the volatility of the Wiener process, yet some regularity assumptions on the test measures are needed.

In this paper we seek to provide such regularity assumptions, in a manner consistent with [16]. We shall assume that θ, the set of test measures, permits a Hahn-like decomposition of the underlying space Ω, uniformly in all the measures in θ. A key step in the proof of the main aggregation result in [16] is to verify that a stronger version of our assumption holds; we show that this assumption is sufficient to guarantee their result holds (Theorem 4), and that with our assumption the proof is remarkably simple. On the other hand, our assumption has a natural interpretation in any space, rather than in the particular case of uncertain volatility. We shall also show that there are natural results regarding the pasting of measures and the representation of conditional sublinear expectations which follow directly from our assumption.

2 Sublinear expectations

The theory of sublinear expectations lies at the heart of our study. These operators can either be defined on probability spaces, when they are related
to the theory of BSDEs, or can be defined using the approach of quasi-sure analysis, for example the $G$-expectation of Peng [14] or the 2BSDEs of Soner, Touzi and Zhang [15, 16], amongst many others. In discrete time, the theory of sublinear expectations using a quasi-sure analysis is discussed in [3]. In this work, we shall use the approach of quasi-sure analysis, and shall be quite general about the types of probability spaces under consideration.

Let $(\Omega, \mathcal{F})$ be a measurable space, let $m\mathcal{F}$ denote the $\mathcal{F}/\mathcal{B}(\mathbb{R})$-measurable real valued functions. We wish to define a sublinear expectation on this space, that is, a map taking random variables to $\mathbb{R}$ satisfying some useful properties.

We begin by defining the space of random variables for which the expectation will be well defined.

**Definition 1.** Let $\mathcal{H}$ be a linear space of $\mathcal{F}$-measurable $\mathbb{R}$-valued functions on $\Omega$ containing the constants. We assume that $X \in \mathcal{H}$ implies $|X| \in \mathcal{H}$ and $1_A X \in \mathcal{H}$ for any $A \in \mathcal{F}$.

**Definition 2.** A map $\mathcal{E}: \mathcal{H} \to \mathbb{R}$ will be called a coherent sublinear expectation if, for all $X, Y \in \mathcal{H}$, it is

(i) (Monotone:) if $X \geq Y$ (for all $\omega$) we have $\mathcal{E}(X) \geq \mathcal{E}(Y)$,

(ii) (Constant invariant:) for constants $c$, $\mathcal{E}(c) = c$,

(iii) (Cash additive:) for constants $c$, $\mathcal{E}(X + c) = \mathcal{E}(X) + c$,

(iv) (Coherent:) for all constants $c > 0$, $\mathcal{E}(cX) = c\mathcal{E}(X)$, and

(v) (Sublinear:) $\mathcal{E}(X + Y) \leq \mathcal{E}(X) + \mathcal{E}(Y)$,

(vi) (Monotone continuous:) for $X_n$ a nonnegative sequence in $\mathcal{H}$ increasing pointwise to $X$, $\mathcal{E}(X_n) \uparrow \mathcal{E}(X)$.

Due to its convexity, a coherent sublinear expectation has a simple representation.

**Theorem 1** (See [4, Theorem 3.2], [14, Theorem I.2.1]). A coherent sublinear expectation has a representation

$$\mathcal{E}(X) = \sup_{\theta \in \Theta} E_{\theta}[X]$$

where $\Theta$ is a collection of $(\sigma$-additive) probability measures on $\Omega$. For simplicity, shall say that $\Theta$ represents $\mathcal{E}$.

Once we have this representation, it is natural to wonder how far we can extend $\mathcal{E}$ to functions not in $\mathcal{H}$. Clearly we can define $\mathcal{E}$ for every bounded $\mathcal{F}$-measurable function. As we will not, in general, know that our measures in $\Theta$ will be absolutely continuous (in fact, the focus of this paper is on the case where they are not), we cannot simply complete $\mathcal{F}$ under some reference measure, however this leads us to the following definition.

**Definition 3.** Let $\Theta$ be a collection of probability measures on $(\Omega, \mathcal{F})$. Let $\mathcal{F}^{\Theta}$ denote the completion of $\mathcal{F}$ under the measure $\theta$. We write

$$\mathcal{F}^{\Theta} = \bigcap_{\theta \in \Theta} \mathcal{F}^{\theta}.$$
The collection $F^\Theta$ is a $\sigma$-algebra, and every $\frac{\omega}{2}$ has a unique extension to $F^\Theta$.

**Definition 4.** A set $N \in F^\Theta$ is called a $(\Theta)$-polar set if $\theta(N) = 0$ for all $\theta \in \Theta$.

**Remark 1.** A natural alternative to the use of $F^\Theta$ is to simply complete $F$ by adding the polar sets. That is, if $N$ denotes the polar sets, functions which are $F \_ N$-measurable are the main objects of study. By considering the set $F^\Theta$, we allow a far richer class of functions, as is made clear by the following easy proposition. The $\sigma$-algebra $F^\Theta$ is also used in [16] and [13], where it is called the universal completion of $F$.

**Proposition 1.** For $\Theta$ a family of probability measures on $(\Omega, F)$, where $N$ denotes the $\Theta$-polar sets and $F^\Theta$ the completion of $F$ under $\Theta$,

$$F \subseteq F \vee N \subseteq F^\Theta \subseteq F^\Theta$$

for any $\theta \in \Theta$.

**Example 1.** Let $\Omega = [0,1]$, $F = \mathcal{B}(\Omega)$ and $\Theta = \{\delta_x\}_{x \in [0,1]}$, the set of discrete point-mass measures on $\Omega$. Then $N = \{\emptyset\}$, so $F \vee N = \mathcal{B}(\Omega)$. However, $F^\Theta = 2^\Omega$ for all $\theta$, so $F^\Theta = 2^\Omega$. This is perfectly reasonable, as one can take the expectation of any function under $\delta_x$ for any $x$, so there is no need to insist on any stronger concepts of measurability.

**Definition 5.** Let $\Theta$ be a collection of probability measures on $(\Omega, F)$. We say that a function $X : \Omega \rightarrow \mathbb{R}$ is

- in $mF^\Theta$ if it is $F^\Theta$-measurable,
- in $H^\Theta_F$ if $X \in mF^\Theta$ and at least one of $E_\theta[X_0^+]$ and $E_\theta[X_0^-]$ is finite, and
- in $L^1(\mathcal{E}; F)$ if $X \in mF^\Theta$ and $\sup_\theta E_\theta[|X|]$ is finite

We can now extend $E$ to the larger space $H^\Theta_F$.

**Definition 6.** We define the operator

$$\tilde{E} : H^\Theta_F \rightarrow \mathbb{R}, X \mapsto \sup_{\theta \in \Theta} E_\theta[X],$$

It is easy to verify that $\tilde{E}$ satisfies properties (i-iv) and (vi) of Definition 2 with $H$ replaced by $H^\Theta_F$, as a map $H^\Theta_F \rightarrow \mathbb{R} \cup \{-\infty\}$. It also satisfies property (v) provided all terms are well defined (in particular, this is satisfied on $L^1(\mathcal{E})$). Furthermore, comparing with Definition 2 and Theorem 1 we have $H \subseteq H^\Theta_F$ and $\tilde{E}|_{H} = E$.

Hereafter, we shall take $\Theta$ as fixed, and simply write $H_F$ for $H^\Theta_F$ and $E$ for $\tilde{E}$, whenever this does not lead to confusion. To prevent confusion, we shall still distinguish between $F$ and $F^\Theta$.

**Definition 7.** We say that a statement holds quasi-surely (q.s.) if it holds except on a polar set.
2.1 Conditional sublinear expectations

Suppose now that we have a sub-$\sigma$-algebra $\mathcal{G} \subseteq \mathcal{F}$. In exactly the same way as before (Definition 5), we can define the space $\mathcal{H}^0_{\mathcal{G}}$, and it is easy to verify that $\mathcal{H}^0_{\mathcal{G}} \subseteq \mathcal{H}^0_{\mathcal{F}}$ and $\mathcal{G}^0 \subseteq \mathcal{F}^0$. As before, we shall simply write $\mathcal{H}_G$ for $\mathcal{H}^0_{\mathcal{G}}$.

We wish to consider the sublinear expectation conditional on $\mathcal{G}$. This is an operator satisfying the following properties.

**Definition 8.** A pair of maps
\[
\mathcal{E} : \mathcal{H}_G \to \mathbb{R} \\
\mathcal{E}_G : L^1(\mathcal{E};\mathcal{F}) \to L^1(\mathcal{E};\mathcal{G})
\]
is called a $\mathcal{G}$-consistent coherent sublinear expectation if for any $X,Y \in L^1(\mathcal{E};\mathcal{F})$

(i) $\mathcal{E}$ is a coherent sublinear expectation

(ii) (Recursivity) $\mathcal{E} \circ \mathcal{E}_G = \mathcal{E}$ on $L^1(\mathcal{E};\mathcal{F})$, that is, $\mathcal{E}(\mathcal{E}_G(X)) = \mathcal{E}(X)$,

(iii) ($\mathcal{G}$-Regularity) $\mathcal{E}_G(I_A Y) = I_A \mathcal{E}_G(Y)$ q.s. for all $A \in \mathcal{G}^0$.

(iv) $\mathcal{E}_G$ satisfies the requirements of a coherent sublinear expectation $\mathcal{G}^0$-conditionally, that is

(a) ($\mathcal{G}$-monotonicity) $X \geq Y$ implies $\mathcal{E}_G(X) \geq \mathcal{E}_G(Y)$ q.s.

(b) ($\mathcal{G}$-triviality) $\mathcal{E}_G(Y) = Y$ q.s. for all $Y \in L^1(\mathcal{E};\mathcal{G})$.

(c) ($\mathcal{G}$-cash additivity) $\mathcal{E}_G(X+Y) = \mathcal{E}_G(X)+\mathcal{E}_G(Y)$ q.s. for all $Y \in L^1(\mathcal{E};\mathcal{G})$.

(d) ($\mathcal{G}$-sublinearity) $\mathcal{E}_G(X+Y) \leq \mathcal{E}_G(X) + \mathcal{E}_G(Y)$ q.s.

(e) ($\mathcal{G}$-coherence) $\mathcal{E}_G(\lambda Y) = \lambda^+ \mathcal{E}_G(Y) + \lambda^- \mathcal{E}_G(-Y)$ q.s. for all $\lambda \in m\mathcal{G}^0$

with $(\lambda Y) \in L^1(\mathcal{E};\mathcal{G})$.

The following simple lemma gives uniqueness of the conditional expectation.

**Lemma 1.** For a given coherent sublinear expectation $\mathcal{E}$, a given $\mathcal{G} \subseteq \mathcal{F}$, there exists at most one conditional coherent sublinear expectation $\mathcal{E}_G$, up to equality q.s.

**Proof.** For a given $X$, suppose $\mathcal{E}_G$ and $\mathcal{E}_G$ are two versions of the conditional expectation. By the $\mathcal{G}$-triviality and cash additivity properties, we can see that $\mathcal{E}_G(X - \mathcal{E}_G(X)) = 0$ q.s., and hence by regularity, for any $A \in \mathcal{G}^0$ we have $\mathcal{E}(I_A(X - \mathcal{E}_G(X))) = 0$. Similarly we see that

$\mathcal{E}(I_A(\mathcal{E}_G(X) - \mathcal{E}_G(X))) = \mathcal{E}(\mathcal{E}_G(I_A(X - \mathcal{E}_G(X)))) = \mathcal{E}(I_A(X - \mathcal{E}_G(X))) = 0$.

Therefore, taking $A_n = \{\omega : \mathcal{E}_G(X) > \mathcal{E}_G(X) + n^{-1}\} \in \mathcal{G}^0$, we have

$0 \leq \mathcal{E}(I_{A_n} n^{-1}) \leq \mathcal{E}(I_{A_n}(\mathcal{E}_G(X) - \mathcal{E}_G(X))) = 0$

and hence $\mathcal{E}(I_{A_n})$ is polar. Therefore $\cup_n A_n$ is polar, that is, $\mathcal{E}_G(X) \leq \mathcal{E}_G(X)$ q.s. Reversing the roles of $\mathcal{E}_G$ and $\mathcal{E}_G$ yields the reverse inequality. \(\square\)

Note that, as in the classical case, we shall only require in the definition that $\mathcal{E}_G$ is well defined on $L^1(\mathcal{E};\mathcal{F})$. However, it will often be the case (cf Remark 2) that the conditional expectation is well defined on a space of functions with significantly less integrability.
3 Representing the conditional expectation

For a given $\mathcal{G}$-consistent sublinear expectation $\mathcal{E}$, we wish to have a representation of the conditional expectation $\mathcal{E}_{\mathcal{G}}$ similar to that in Theorem 1. That is, we wish to write

$$\mathcal{E}_{\mathcal{G}}(X) = \sup_{\theta \in \Theta} \mathcal{E}_\theta[X|\mathcal{G}], \quad (2)$$

This statement has two key problems. First, the conditional expectation $\mathcal{E}_\theta[X|\mathcal{F}_t]$ is only defined $\theta$-a.s. rather than $\mathcal{E}$-q.s. When $\Theta$ consists of uncountably many possibly singular probability measures, this causes a significant problem. Second, if $\Theta$ is uncountable, the pointwise supremum may be an inappropriate choice, as it is unclear whether it is even in $m\mathcal{G}^\Theta$.

To deal with these issues, we shall first assume that our set of measures satisfies a certain decomposition property, which is a generalisation of the separability assumed in Soner et al. [16]. Under this assumption, we shall be able to give a consistent definition of the conditional expectation under $\theta$, in a quasi-sure sense. We then follow Detlefsen and Scandolo [7] in replacing the supremum in (2) with an essential supremum, which we construct quasi-surely. Hence, we show that the representation is valid. It is worth also noting the work of Bion-Nadal [1], where a similar representation is obtained (for the larger class of convex risk measures under uncertainty, that is, without the assumption of coherence) however no consideration is given to the construction of the conditional expectation in a quasi-sure sense.

3.1 Defining linear conditional expectations

Our key tool for the definition of the conditional expectation, in a sufficiently strong sense, will be the assumption that the following properties hold.

**Definition 9 (Hahn property).** We shall say that $\Theta$ has the Hahn property on $\mathcal{G}$ if for every $\theta \in \Theta$ there exists a set $S_{(\theta, \mathcal{G})} \in \mathcal{G}^\Theta$, such that

(i) $S_{(\theta, \mathcal{G})}$ supports $\theta$, that is,

$$\theta(S_{(\theta, \mathcal{G})}) = 1,$$

(ii) for every $\phi, \theta \in \Theta$, $\phi|\mathcal{G}$ and $\theta|\mathcal{G}$ are equivalent on $S_{(\phi, \mathcal{G})} \cap S_{(\theta, \mathcal{G})}$, that is, for any $N \subset S_{(\phi, \mathcal{G})} \cap S_{(\theta, \mathcal{G})} \cap S_{(\theta, \mathcal{G})}$ and $\mathcal{G}^{\Theta}$ with $\theta(N) = 0$ we know $\phi(N) = 0$,

(iii) $S_{(\theta, \mathcal{G})} = S_{(\phi, \mathcal{G})}$ if $\phi|\mathcal{G}$ is equivalent to $\theta|\mathcal{G}$.

The collection $\{S_{(\theta, \mathcal{G})}\}_{\theta \in \Theta}$ will be called a $(\Theta/\mathcal{G})$-Hahn decomposition (of $\Omega$). We say that $\Theta$ has the separable Hahn property on $\mathcal{G}$ if it has the Hahn property, and furthermore there exists another set of probability measures $\Phi$, also with the Hahn property, such that

(iv) the sets $\{S_{(\phi, \mathcal{G})}\}_{\phi \in \Phi}$ are disjoint, and

(v) $\Phi$ and $\Theta$ generate the same polar sets and $m\mathcal{G}^\Theta = m\mathcal{G}^\Phi$.

The collection $\{S_{(\phi, \mathcal{G})}\}_{\phi \in \Phi}$, with its associated $\Phi$, will be called a $\Theta/\mathcal{G}$-dominating partition of $\Omega$. (Note that $\{S_{(\phi, \mathcal{G})}\}$ is a $\mathcal{G}^{\Theta}$-measurable partition of $\Omega$ minus a polar set.)
Note that property (iii) is trivial, as one can use the axiom of choice to select a candidate measure for each class of equivalent measures, and then take the set associated with this candidate as representative of the whole class.

The following example shows that the existence of a Hahn decomposition is not trivial in general.

**Example 2.** Consider the space \( \Omega = [0, 1]^2 \) with its Borel \( \sigma \)-algebra. For simplicity, we take \( \mathcal{G} = \mathcal{B}(\omega_1) \), the Borel \( \sigma \)-algebra generated by the first component of \( \Omega \). Let \( \Theta = \{ \delta_{(x,y)} : (x, y) \in [0, 1]^2 \} \), the family of single-point measures on \( \Omega \). Then \( \Theta \) has the separable Hahn property, with \( \mathcal{S}_{\{(\delta_{(x,y)}, \mathcal{G}) = \{x \}. \} \) The set of measures obtained by taking all countable mixtures of elements of \( \Theta \) will also have the separable Hahn property, a \( \Theta / \mathcal{G} \)-dominating partition being the sets \( \{x\}_{x \in [0, 1]} \).

Conversely, if \( \Theta' = \Theta \cup \{\lambda\} \), where \( \lambda \) is Lebesgue measure on \( [0, 1]^2 \), then \( \Theta' \) does not have the Hahn property. This is clear as any set \( S \) with \( \lambda(S) = 1 \) is nonempty, but for any \( (x, y) \in S \) we have \( \delta_{(x,y)}(S) = 1 > 0 \). Note that this holds even though these measures are mutually orthogonal.

**Example 3.** Suppose there exists a ‘dominating’ measure \( \phi \) on \( \mathcal{G}^\Theta \), that is, \( \theta|_\mathcal{G} \) is absolutely continuous with respect to \( \phi \) for all \( \theta \in \Theta \). Then \( \Theta \) has the separable Hahn property with \( \Phi = \{\phi\} \).

To show \( \Theta \) has the Hahn property, take the Lebesgue decomposition of \( \phi \) with respect to \( \theta|_\mathcal{G} \), so \( \phi = \phi^\theta + \phi^\perp \), where \( \phi^\perp \) is orthogonal to \( \theta|_\mathcal{G} \) and \( \phi^\theta \ll \theta|_\mathcal{G} \), then take the (classical) Hahn decomposition of the signed measure \( \phi^* = \phi^\theta - \phi^\perp \). Finally, define \( \mathcal{S}_{\varnothing, \mathcal{G}} \) to be the support of the positive sets under \( \phi^* \). The separable Hahn property is easily verified.

The usefulness of the Hahn property is due to the following simple lemma.

**Lemma 2.** Let \( \Theta \) have the Hahn property on \( \mathcal{G} \) and let \( A \in \mathcal{G}^\Theta \) with \( A \subseteq \mathcal{S}_{\varnothing, \mathcal{G}} \) for some \( \theta \). Then \( A \) is polar if and only if \( A \) is \( \theta \)-null.

**Proof.** Note that for any \( \phi \), by property (ii) of the Hahn decomposition we know that \( \phi|_\mathcal{G} \) is absolutely continuous with respect to \( \theta|_\mathcal{G} \) on \( \mathcal{S}_{\varnothing, \mathcal{G}} \), and so if \( A \) is \( \theta \)-null, it must be \( \phi \)-null also. The converse is trivial.

In some cases, the separable Hahn property may be most easily verified using the following lemma.

**Lemma 3.** Suppose \( \Theta \) has the Hahn property on \( \mathcal{G} \), and there exists a subset \( \Phi \subseteq \Theta \) such that the sets \( \{\mathcal{S}_{(\phi, \mathcal{G})} : \phi \in \Phi \} \) are disjoint, and for any \( \theta \), there exists a countable set \( \{\phi^\theta_n\} \subseteq \Phi \) such that \( \mathcal{S}_{(\theta, \mathcal{G})} \subseteq \bigcup_n \mathcal{S}_{(\phi^\theta_n, \mathcal{G})} \). Then \( \Phi \) is a \( \Theta / \mathcal{G} \)-dominating partition.

**Proof.** We only need to show that \( \Phi \) and \( \Theta \) generate the same polar sets and \( m\mathcal{G}^\Theta = m\mathcal{G}^\Phi \). As \( \Phi \subseteq \Theta \), any \( \Theta \)-polar set is clearly \( \Phi \)-polar and \( m\mathcal{G}^\Theta \supseteq m\mathcal{G}^\Phi \).

For the converse, for any \( \theta \in \Theta \) by assumption there is a countable set \( \{\phi^\theta_n\} \) in \( \Phi \) such that \( \mathcal{S}_{(\theta, \mathcal{G})} \subseteq \bigcup_n \mathcal{S}_{(\phi^\theta_n, \mathcal{G})} \). For any \( \Phi \)-polar \( A \), we then have that

\[
\theta(A) = \theta(A \cap \mathcal{S}_{(\theta, \mathcal{G})}) \leq \sum_n \theta(A \cap \mathcal{S}_{(\phi^\theta_n, \mathcal{G})})
\]

However \( \Phi \subseteq \Theta \), so \( \theta \) is equivalent to \( \phi^\theta_n \) on \( \mathcal{S}_{(\phi^\theta_n, \mathcal{G})} \). Hence \( \theta(A \cap \mathcal{S}_{(\phi^\theta_n, \mathcal{G})}) = 0 \), and we see \( \theta(A) = 0 \). As \( \theta \) was arbitrary we then know \( A \) is polar.
Similarly, if $X \in m\mathcal{G}_\theta^R$, then for any $\theta \in \Theta$, we see that for each $n$, $X$ differs from a $\mathcal{G}$-measurable function on a $\phi_n^\theta$-null set. On $S(\phi_n^\theta, \mathcal{G})$, we know $\phi_n^\theta$ and $\theta$ are equivalent, so there is a $\mathcal{G}$-measurable function $\bar{X}$ such that $\{X \neq \bar{X}\} \cap S(\phi_n^\theta, \mathcal{G})$ is $\theta$-null. Writing $X = \sum_n I_{S(\phi_n^\theta, \mathcal{G})}X$, we see that $X \in \mathcal{G}^\theta$ for all $\theta$, so $X \in m\mathcal{G}_\theta^R$. 

We can now see that the setting of Soner et al. [16] has the separable Hahn property.

Example 4. Let $\Omega$ be the classical Wiener space, with canonical process $B$ starting at zero. Let $\mathcal{F}_t = \sigma\{B_s\}_{0 \leq s \leq t}$, and $\mathcal{G} = \mathcal{F}_t$ for some $t$. Let $\langle B \rangle$ be the quadratic variation, which is a progressively measurable continuous function and can be universally defined for all local martingale measures on $B$, as in Karandikar [10] (this is a scalar version of the setting of [16], see also Nutz [13]).

Consider the set of orthogonal measures $\theta$ parameterised by some subset of the $\mathcal{F}$-predictable absolutely continuous nonnegative functions, where under $\theta_e$, $\mathcal{F}$ is a local martingale with quadratic variation $v$. Then we can take $S(\phi_n^\theta, \mathcal{G}) = \{\omega : \langle B \rangle_s = v_s \text{ for all } s \leq t\}$, which is a $\mathcal{G}$-measurable set. Soner et al. [16] take $v$ of the form

$$
\frac{dv}{dt} = \sum_{n=0}^{\infty} \sum_{i=0}^{\infty} a_{n,i} I_{E_n^i}(\tau_n, \tau_{n+1}],
$$

where the $(a_{n,i})$ come from a generating class (for example, the class of deterministic processes), the $(\tau_n)$ is an increasing sequence of stopping times taking countably many values and q.s. reaching $\infty$ for finite $n$, and $\{E_n^i\} \subset \mathcal{F}_{\tau_n}$ is a family of partitions of $\Omega$. Such processes $v$ are said to satisfy the separability condition.

Lemma 5.2 of [16] then proves, under this separability condition, the equivalence (in fact, the equality) of any two measures in $\Theta$ on the intersection of their supports (there denoted $\Omega_{a,b}^R$). Hence properties (i) and (ii) of the Hahn decomposition are satisfied, and property (iii) is trivial. The measures associated with the generating class, restricted to $\mathcal{G}$, have either identical or disjoint supports, and so by our Lemma 3 a subset of these form a $\Theta/\mathcal{G}$-dominating partition of $\Omega$, satisfying properties (iv) and (v).

### 3.2 The essential supremum

It is useful to be able to combine families of random variables in a quasi-surely consistent manner. A key tool for doing this is the essential supremum, which we now construct in a quasi-sure sense. To begin, we cite the following result on the existence of the essential supremum in a classical setting.

Theorem 2 (Föllmer and Schied [8] (Thm A.18)). Let $\mathcal{X}$ be any set of $\mathcal{G}$-measurable random variables on a (complete) probability space $(\Omega, \mathcal{G}, \theta)$.

(i) Then there exists a random variable $X^*$ such that $X^* \geq X$ $\theta$-a.s. for all $X \in \mathcal{X}$. Moreover $X^*$ is $\theta$-a.s. unique in the sense that any other random variable $Y$ with this property satisfies $Y \geq X$ $\theta$-a.s. We call $X^*$ the $\theta$-essential supremum of $\mathcal{X}$, and write $X^* = \theta$-ess sup $\mathcal{X}$. 

(ii) Suppose that \( \mathcal{X} \) is upward directed, that is, for \( \mathcal{X}, \mathcal{X}' \in \mathcal{X} \) there is \( \mathcal{X}'' \in \mathcal{X} \) with \( \mathcal{X}'' \supseteq \mathcal{X} \wedge \mathcal{X}' \). Then there exists an increasing sequence \( X_1 \leq X_2 \ldots \) in \( \mathcal{X} \) such that \( X^* = \lim_n X_n \) \( \theta \)-a.s.

We can extend the first half of this result to our setting, using the separable Hahn property.

**Theorem 3.** Suppose \( \Theta \) is a collection of measures with the separable Hahn property on \( \mathcal{G} \). Then for any set \( \mathcal{X} \subset m\mathcal{G}^\Theta \), the result of Theorem 2(i) holds, where all random variables are taken to be in \( m\mathcal{G}^\Theta \), and inequalities are taken to hold \( \Theta \)-q.s. For clarity, we denote the \( \Theta \)-q.s. essential supremum by \( \Theta \)-ess sup.

**Proof.** Let \( \{S^\phi\} \) be a \( \Theta/\mathcal{G} \)-dominating partition of \( \Omega \). As \( m\mathcal{G}^\Theta = m\mathcal{G}^\Phi \), we know that \( X \in \mathcal{X} \) is \( \mathcal{G}^\Phi \)-measurable for all \( \phi \). Hence we can use Theorem 2(i) to construct the essential supremum \( X^*_\phi = \phi \)-ess sup\( \{X\} \), and then define the universal essential supremum by the disjoint sum

\[ X^* := \sum_{\phi \in \Phi} I_{S^\phi} X^*_\phi. \]

Clearly for any \( X \in \mathcal{X} \) we have \( X^* \geq X \) \( \Theta \)-q.s. on \( S^\phi \) for all \( \phi \), hence by Lemma 3, \( X^* \geq X \) \( \Theta \)-q.s. on \( \Omega \). It is easy to verify that \( X^* \) is unique and \( X^* \in m\mathcal{G}^\Phi = m\mathcal{G}^\Theta \). \( \square \)

We can now reproduce an aggregation result similar to that of Touzi, Soner and Zhang [16].

**Theorem 4.** Suppose \( \Theta \) has the separable Hahn property on \( \mathcal{G} \). Let \( \{X^\theta\}_{\theta \in \Theta} \) be any family of functions such that for all \( \theta, \psi \in \Theta \)

- \( X^\theta \) is \( \mathcal{G}^\theta \)-measurable (where \( \mathcal{G}^\theta \) is the completion of \( \mathcal{G} \) under \( \theta \)) and
- \( X^\theta = X^\psi \) (\( \theta \)-a.s.) on \( S_{(\theta, \mathcal{G})} \cap S_{(\psi, \mathcal{G})} \).

Then there exists an aggregation function \( Y \) which is \( \mathcal{G}^\Theta \)-measurable, such that \( Y = X^\theta \) \( \theta \)-a.s. for all \( \theta \).

**Proof.** Simply take

\[ Y = \Theta \text{-ess sup}_{\theta \in \Theta} X^\theta. \]

For any \( \theta \in \Theta \), by our second assumption we see that \( Y = X^\theta \) \( \theta \)-a.s. on \( S_{(\theta, \mathcal{G})} \), and as \( S_{(\theta, \mathcal{G})} \) supports \( \theta \), \( Y = X^\theta \) \( \theta \)-a.s. \( \square \)

As shown in [16], many of the results of stochastic analysis can be obtained as soon as we have a result of this kind.

### 3.3 A dual representation

We now prove that a modified version of the representation (2) is valid.

**Lemma 4.** Let \( \mathcal{E} \) be a \( \mathcal{G} \)-consistent sublinear expectation, with representation \( \mathcal{E}(\cdot) = \sup_{\theta \in \Theta} \mathbb{E}_\theta[X] \). Then for any \( \theta \in \Theta \), any \( X \) such that all terms are \( \theta \)-a.s. finite, any \( t < \infty \),

\[ -\mathcal{E}(-X|\mathcal{G}) \leq \mathbb{E}_\theta[X|\mathcal{G}] \leq \mathcal{E}(X|\mathcal{G}) \quad \theta \text{-a.s.} \]
Proof. For any $A \in \mathcal{G}^\Theta$, any $X$ we have

$$E_{\mathcal{G}}[I_A(X - E_i(X))] = E_{\mathcal{G}}(I_A X) - I_A E_{\mathcal{G}}(X) = 0$$

and so by time consistency $E_{\mathcal{G}}[I_A(X - E_i(X))] = 0$. Hence

$$E_{\mathcal{G}}[I_A(X - E_{\mathcal{G}}(X))] = 0$$

and rearrangement gives $E_{\mathcal{G}}[I_A X] \leq E_{\mathcal{G}}[I_A E_{\mathcal{G}}(X)]$, which is equivalent to the upper bound $E_{\mathcal{G}}[X | \mathcal{G}] \leq E_{\mathcal{G}}(X)$. For the lower bound, applying this result to $-X$ gives

$$E_{\mathcal{G}}[X | \mathcal{G}] = -E_{\mathcal{G}}[-X | \mathcal{G}] \geq -E_{\mathcal{G}}(-X) \quad \text{q.s.}$$

Using the separable Hahn property, we can consistently define our conditional expectations $E_{\mathcal{G}}[X | \mathcal{G}]$ up to equality $\mathcal{E}$-q.s.

**Definition 10.** Suppose $\Theta$ has the Hahn property on $\mathcal{G}$. For each $\theta \in \Theta$, we define

$$E_{\theta | \Theta}[X | \mathcal{G}] = \begin{cases} Y & \omega \in S(\theta, \mathcal{G}) \\ -\infty & \omega \notin S(\theta, \mathcal{G}) \end{cases}$$

where $Y$ is any version of the classical conditional expectation $E_{\theta}[X | \mathcal{G}]$. By Lemma 2, this definition is unique up to a polar set (as it is unique up to a $\theta$-null subset of $S(\theta, \mathcal{G})$).

Note that $E_{\theta | \Theta}[X | \mathcal{G}]$ satisfies the usual properties of the conditional expectation, i.e. linearity, recursivity, monotonicity, etc., but does so $\mathcal{E}$-q.s., rather than simply $\theta$-a.s. (The exception to this is that for $A \in \mathcal{G}^\Theta$, $E_{\theta | \Theta}[I_A X | \mathcal{G}] = I_A E_{\theta | \Theta}[X | \mathcal{G}]$ only on $S(\theta, \mathcal{G})$. The reason for setting the expectation to $-\infty$ off $S(\theta, \mathcal{G})$ is simply so that we can take the supremum in a simple manner.)

We can now prove our general representation.

**Theorem 5.** Let $\mathcal{E}$ be a $\mathcal{G}$-consistent sublinear expectation, with a representation $\Theta$ having the separable Hahn property on $\mathcal{G}$. Then the conditional expectation has a representation

$$E_{\mathcal{G}}(X) = \Theta \text{-ess sup}_{\theta \in \Theta} \{E_{\theta | \Theta}[X | \mathcal{G}]\}$$

up to equality q.s.

**Proof.** First note that for any $A \in \mathcal{G}^\Theta$,

$$E(I_A E_{\mathcal{G}}(X)) = E(I_A X) = \sup_{\theta \in \Theta} E_{\theta}[I_A X] = \sup_{\theta \in \Theta} E_{\theta}[I_A E_{\theta | \Theta}[X | \mathcal{G}]]$$

$$\leq \sup_{\theta \in \Theta} E_{\theta}[I_A (\Theta \text{-ess sup}_{\phi \in \Theta} \{E_{\phi | \Theta}[X | \mathcal{G}]\})]$$

$$= E(I_A (\Theta \text{-ess sup}_{\phi \in \Theta} \{E_{\phi | \Theta}[X | \mathcal{G}]\}))$$

from which we see

$$E_{\mathcal{G}}(X) \leq \Theta \text{-ess sup}_{\phi \in \Theta} \{E_{\phi | \Theta}[X | \mathcal{G}]\} \quad \text{q.s.}$$
Conversely, by Lemma 4, we know that for every $\phi \in \Theta$, $E_\phi[X|G] \leq E_\phi(X)$ $\phi$-a.s. As $E_\phi[X|G] = -\infty$ except on $S(\phi, G)$, by Lemma 2 we know $E_\phi[X|G] \leq E_G(X)$ $\text{q.s.}$ Therefore, by Theorem 3,

$$\Theta\text{-ess sup}_{\phi \in \Theta} \{E_\phi[X|G]\} \leq E_G(X) \text{ q.s.}$$

giving the desired equality. \hfill \square

As mentioned earlier, Bion-Nadal [1] gives a similar result to this, however without a quasi-sure construction of the conditional expectation. Therefore, her result presents only the $\theta$-a.s. equality of the conditional sublinear expectation and the $\theta$-essential supremum. Our result is strictly stronger, as both the equality and the essential supremum are taken in a quasi sure sense.

**Remark 2.** We note that this result immediately allows us to consistently extend $E_G$ to the larger space $H_F$, using a generalised conditional expectation, as in [9, p2]. That is, we no longer require substantial integrability conditions on $X$ to define $E_G(X)$. This will, however, lead to somewhat different statements of the properties of the conditional expectation (as finiteness is no longer guaranteed).

### 3.4 $G$-consistency and pasting of measures

Using this result, we can give a type of ‘pasting stability’ of the measures related to $G$-consistency. This is closely related to the $m$-stability of Delbaen [5].

**Definition 11.** For $\Theta$ with the Hahn property, we say $\Theta$ is stable under $G$-pasting if for any $\theta, \phi \in \Theta$, any $A \subseteq S(\theta, G) \cap S(\phi, G)$, $A \in G^\theta$ we have $\psi \in \Theta$, where $\psi$ is the measure on $\Omega$ with

$$\psi(B) := E_\theta[I_A E_\phi[I_B|G] + I_A^c I_B].$$

For a set $\Theta$, we can define $\Theta^G$, the finite $G$-stabilisation of $\Theta$, as the set of all measures obtained from $\Theta$ through finitely many combinations of this form. Clearly if $\Theta$ has the (separable) Hahn property on $G$ then so will $\Theta^G$.

Note that this pasting only needs to hold for $A$ in the intersection of the supports of the two measures, where $\theta$ and $\phi$ are equivalent, so the conditional expectation is defined simultaneously without complication. In some applications the analogous stabilisation where countably many combinations are permitted may be of interest, however the finite case will be sufficient for our result.

**Theorem 6.** Let $E$ be a sublinear expectation with representation $\Theta$. Suppose $\Theta$ has the separable Hahn property on $G$. Then

(i) If $E$ is $G$-consistent, then $E$ has an equivalent representation $E(X) = \sup_{\theta \in \Theta^G} E_\theta[X]$.

(ii) If $\Theta = \Theta^G$ then $E$ is $G$-consistent.

**Proof.** (i) Suppose $E$ is $G$-consistent. Clearly $\Theta \subseteq \Theta^G$, and so

$$E(X) \leq \sup_{\theta \in \Theta^G} E_\theta[X].$$
Conversely, for any $\psi \in \Theta^G$ we know $\psi$ is of the form

$$\psi(B) = E_\theta \left[ \sum_n I_{A_n} E_{\phi_n}[I_B | G] \right]$$

for some finite partition $\{ A_n \}$ of $\Omega$, and some measures $\theta$ and $\phi_n$ in $\Theta$. Then

$$E_\theta[X] = E_\theta \left[ \sum_n I_{A_n} E_{\phi_n}[I_B | G] \right] \leq \sup_{\theta} E_\theta[\Theta\text{-ess sup}_{G}E_{\phi}[X | G]] = \mathcal{E}(X)$$

and so

$$\mathcal{E}(X) \geq \sup_{\theta \in \Theta^G} E_\theta[X].$$

(ii) As $\Theta = \Theta^G$ has the separable Hahn property, for each fixed $X \in L^1(\mathcal{E}, \mathcal{F})$ we can define the putative sublinear conditional expectation

$$\tilde{\mathcal{E}}_G(X) := \Theta\text{-ess sup}_{G}E_{\phi}[X | G].$$

All the properties of a $G$-consistent sublinear expectation are trivial to verify except recursivity.

To show recursivity, first select some $\theta \in \Theta$. The quasi-sure essential supremum given by Theorem 3 must also be a ($\theta$-a.s.) version of the $\theta$-a.s. essential supremum given by Theorem 2. As $E_{\phi}[X | G] = -\infty$ except on $S(\phi, G)$, and $\Theta = \Theta^G$, we see that

$$\tilde{\mathcal{E}}_G(X) = \Theta\text{-ess sup}_{G}E_{\phi}[X | G] \quad \theta - a.s.$$

Furthermore, the family $\{ I_A E_{\phi}[X | G] + I_{A^c} E_{\phi}[X | G] \}$ is upward directed (up to equality $\theta$-a.s.). By Theorem 2(ii), we can then find appropriate sequences $\phi_n^\theta, A_n^\theta$ such that

$$\{ I_A E_{\phi_n^\theta}[X | G] + I_{A^c} E_{\phi_n^\theta}[X | G] \} \uparrow \tilde{\mathcal{E}}_G(X) \quad \theta - a.s.$$

We now relax our selection of $\theta$, and consider the equation

$$\mathcal{E}(\tilde{\mathcal{E}}_G(X)) = \sup_{\theta} E_\theta[\tilde{\mathcal{E}}_G(X)]$$

$$= \sup_{\theta} E_\theta[\lim_{n \to \infty} I_A E_{\phi_n^\theta}[X | G] + I_{A^c} E_{\phi_n^\theta}[X | G]]$$

$$= \sup_{\theta} E_\theta[ I_A E_{\phi_n^\theta}[X | G] + I_{A^c} E_{\phi_n^\theta}[X | G]]$$

$$= \sup_{\theta} E_\theta[\psi_n(X)]$$

where

$$\psi_n(B) := E_\theta[I_A E_{\phi_n^\theta}[I_B | G] + I_{A^c} E_{\phi_n^\theta}[I_B | G]].$$

As we know $\Theta = \Theta^G$, all the induced measures $\psi_n$ are in $\Theta$. Therefore we have

$$\mathcal{E}(\tilde{\mathcal{E}}_G(X)) = \sup_{\theta} E_\theta[X] = \mathcal{E}(X)$$

and so $\tilde{\mathcal{E}}_G(X)$ satisfies the recursivity assumption. $\square$
4 Integrability and convergence

We now seek to look at some consequences of this representation. In particular, we shall use the representation of Theorem 5 to show that, if we have a filtration \( \mathcal{F}_t \) and we can consistently define our sublinear expectation for \( \mathcal{F}_t \)-measurable random variables for any \( t < \infty \), then we can consistently define our sublinear expectation for all \( \mathcal{F}_\infty \)-measurable random variables.

For simplicity, we shall assume that time is discrete. However, as we shall make no significant assumptions on the structure of the filtration (beyond the Hahn property), this allows consideration of the ‘skeleton’ of a continuous filtration (possibly at stopping times) with no difficulties.

**Definition 12.** Let \( \{\mathcal{F}_t\} \) be a discrete-time filtration on a measurable space \((\Omega, \mathcal{F})\). A family of maps

\[
\mathcal{E} : \mathcal{H}_\mathcal{F} \to \mathbb{R} \\
\mathcal{E}_t : L^1(\mathcal{E}; \mathcal{F}) \to L^1(\mathcal{E}; \mathcal{F}_t)
\]

is called a \( \{\mathcal{F}_t\} \)-consistent coherent sublinear expectation (on \( \mathcal{F} \)) if

(i) \( \mathcal{E} \) is a coherent sublinear expectation and \( \mathcal{E}(X) = \mathcal{E}_0(X) \) q.s.

(ii) (Recursivity) For \( s \leq t \) we have \( \mathcal{E}_s \circ \mathcal{E}_t = \mathcal{E}_t \) on \( L^1(\mathcal{E}; \mathcal{F}) \), that is, \( \mathcal{E}_t(\mathcal{E}_s(X)) = \mathcal{E}_s(X) \) for all \( X \in \mathcal{H}_\mathcal{F} \).

(iii) (\( \mathcal{F}_t \)-Regularity) \( \mathcal{E}_t(I_A X) = I_A \mathcal{E}_t(X) \) q.s. for all \( A \in \mathcal{F}_\infty \), \( X \in \mathcal{H}_\mathcal{F} \).

(iv) For all \( t \), \( \mathcal{E}_t \) satisfies the requirements of a coherent sublinear expectation \( \mathcal{F}_\infty \)-conditionally (as in Definition 8).

A \( \{\mathcal{F}_t\} \)-consistent sublinear expectation (on \( \mathcal{F} \)) will, for simplicity, be called an \( \text{SL} \)-expectation (on \( \mathcal{F} \)).

**Definition 13.** We say that \( \Theta \) has the (separable) Hahn property on \( \{\mathcal{F}_t\} \) if it has the (separable) Hahn property on \( \mathcal{F}_t \) for all \( t \).

To obtain convergence results, the following concepts are useful, and can be found in [3].

**Definition 14.** Consider \( K \subset L^1 \). \( K \) is said to be uniformly integrable (u.i.) if \( \mathcal{E}(I_{|X| \geq c}|X|) \) converges to 0 uniformly in \( X \in K \) as \( c \to \infty \).

**Definition 15.** Let \( L^p_b \) be the completion of the set of bounded functions \( X \in \mathcal{H} \), under the norm \( \| \cdot \|_p = \mathcal{E}(| \cdot |^p)^{1/p} \). Note that \( L^p_b \subset L^p \).

**Lemma 5.** For each \( p \geq 1 \),

\[
L^p_b = \{ X \in L^p : \lim_{n \to \infty} \mathcal{E}(|X|^p I_{|X| > n}) = 0 \}.
\]

**Definition 16.** We say \( X_n \) is a uniformly integrable \( \mathcal{E} \)-submartingale if \( X_n \) is \( \mathcal{F}^\Theta_n \)-measurable for all \( n \), \( \{X_n\} \) is uniformly integrable and \( X_n \leq \mathcal{E}_n(X_{n+1}) \) q.s. for all \( n \). Similarly we define \( \mathcal{E} \)-supermartingales and \( \mathcal{E} \)-martingales.

In [3], we have obtained the following convergence result in this space.
Theorem 7. Let \( \{X_n\}_{n \in \mathbb{N}} \) be a uniformly integrable \( SL \)-submartingale. Then \( X_n \) converges quasi surely and in \( L^1(\mathcal{E}) \) to some random variable \( X_\infty \). Furthermore, the process \( \{X_n\}_{n \in \mathbb{N} \cup \{\infty\}} \) is also a uniformly integrable \( SL \)-submartingale. In particular, this implies that \( X_\infty \in L^1 \). The same result holds for \( \mathcal{E} \)-supremum limits and \( \mathcal{E} \)-martingales.

Lemma 6. For each \( T \in \mathbb{N} \), let \( \mathcal{E}_T^T(\cdot) \) be an \( SL \)-expectation on \( \mathcal{F}_T \), with the consistency property that

\[
\mathcal{E}_T^T(X) = \mathcal{E}_T^{T'}(X) \quad \text{for all } X \in \mathcal{H}_{\mathcal{F}_T'}, \quad \text{all } T' \leq T
\]

Then there exists a set of test probability measures \( \Theta \) such that \( \mathcal{E}_T(X) = \sup_{\theta \in \Theta} E_\theta[X] \) for all \( T \).

Proof. For each \( n \), let \( \Theta_n \) be a set of test measures for \( \mathcal{E}_n \). By the assumed consistency property, for \( m \leq n \), for any \( X \in \mathcal{H}_{\mathcal{F}_m} \),

\[
\sup_{\theta \in \Theta_n} E_\theta[X] = \mathcal{E}_n(X) = \mathcal{E}_m(X) = \sup_{\theta \in \Theta_m} E_\theta[X]
\]

that is, \( \Theta_n \) is also a valid set of test measures for \( \mathcal{E}_m \). It follows that, without loss of generality, we can take \( \Theta'_m = \bigcup_{n \geq m} \Theta_n \), as a set of test measures for \( \mathcal{E}_m \). Furthermore, as \( \mathcal{E}_n(X) = \sup_{\theta \in \Theta_n} E_\theta[X] \) does not vary with \( n \geq m \) and \( \Theta'_n \) is nondecreasing, we have,

\[
\mathcal{E}_m(X) = \inf_{n \geq m} \sup_{\theta \in \Theta'_n} E_\theta[X] = \sup_{\theta \in \bigcap_{n \geq m} \Theta'_n} E_\theta[X],
\]

that is, without loss of generality, \( \Theta'_m = \lim \sup_{n \geq m} \Theta_n = \bigcap_{k \geq m} \bigcup_{n \geq k} \Theta_n \) is also a set of test measures for \( \mathcal{E}_m \). As the \( \lim \sup_{n \geq m} \) is independent of \( m \), it follows that there exists a single set of test measures \( \Theta \) such that \( \mathcal{E}_m(X) = \sup_{\theta \in \Theta} E_\theta[X] \) for all \( m \).

Remark 3. The set \( \Theta \) constructed in the previous Lemma is generally not unique, as the finite-time expectations \( \mathcal{E}_n \) can give us no information on the `correct' capacity \( \mathcal{E}(I_A) \) associated with an event in the tail \( \sigma \)-algebra.

Using our convergence result and our earlier representation, we can now show how to extend a finite-horizon \( SL \)-expectation to an infinite horizon.

Theorem 8. Let \( \mathcal{E}^T \) and \( \Theta \) be as in Lemma 6, and suppose \( \Theta \) has the separable Hahn property on \( \{\mathcal{F}_t\} \). Then the operator

\[
\mathcal{E}(X) := \sup_{\theta \in \Theta} E_\theta[X]
\]

is an \( SL \)-expectation on \( L^1(\mathcal{E}; \mathcal{F}_\infty) \). In particular, it is time consistent.

Proof. Let \( \mathcal{E}_s := \Theta - \text{ess sup}_{\theta \in \Theta} [E_\theta[X|\mathcal{F}_s]] \). Then it is easy to show that, for any \( r \leq s \leq t < \infty \), \( \mathcal{E}_r(X) = \mathcal{E}_s(X) \) for all \( X \in L^1(\mathcal{E}; \mathcal{F}_r) \). We need to show this holds for \( t = \infty \).

For \( X \in L^1(\mathcal{E}; \mathcal{F}_\infty) \), we know that \( \mathcal{E}_r(X) \) is an \( \mathcal{E} \)-martingale, hence \( \mathcal{E}_r(X) \to X \) q.s. and in \( L^1(\mathcal{E}) \), by Theorem 7. Define the operator

\[
\mathcal{E}_s^{(t)}(X) := \mathcal{E}_s(\mathcal{E}_t(X))
\]
and then by continuity of $E_s$ in $L^1(\mathcal{E})$ we have $E_s(X) = \lim_t E_s^t (X)$ where the limit holds q.s. and in $L^1(\mathcal{E})$. Hence we have

$$E_r(E_s(X)) = E_r(\lim_t E_s^t (X)) = \lim_t E_r(E_s(X))$$
$$= \lim_t E_r(E_t(X)) = \lim_t E_r^t (X) = E_r(X),$$

where the third inequality is because recursivity holds for finite-time-measurable variables.

\section{Conclusion}

We have considered sublinear expectations on general probability spaces, where the set of measures in the dual representation of the expectation are not necessarily absolutely continuous with respect to any dominating measure. In this context, we have shown that the assumption of a Hahn property provides a simple means to aggregate processes defined with respect to each measure, thereby giving a straightforward approach to quasi-sure analysis in this context.

Our methods generalise the approach of [16], as the Hahn property has a natural interpretation in a general setting. Consequently, this paper provides a quasi-sure construction of the conditional expectation under each test measure, and shows that a dual representation then holds for the conditional sublinear expectation. We have given a version of the aggregation result of [16].

For any specific problem, determining whether the Hahn property holds may be a difficult task, (as is made clear by the analysis in [16]). However, our approach shows that for any given problem, once the Hahn property has been shown, many of the results of stochastic analysis transfer simply into a quasi-sure setting.

\section*{References}


