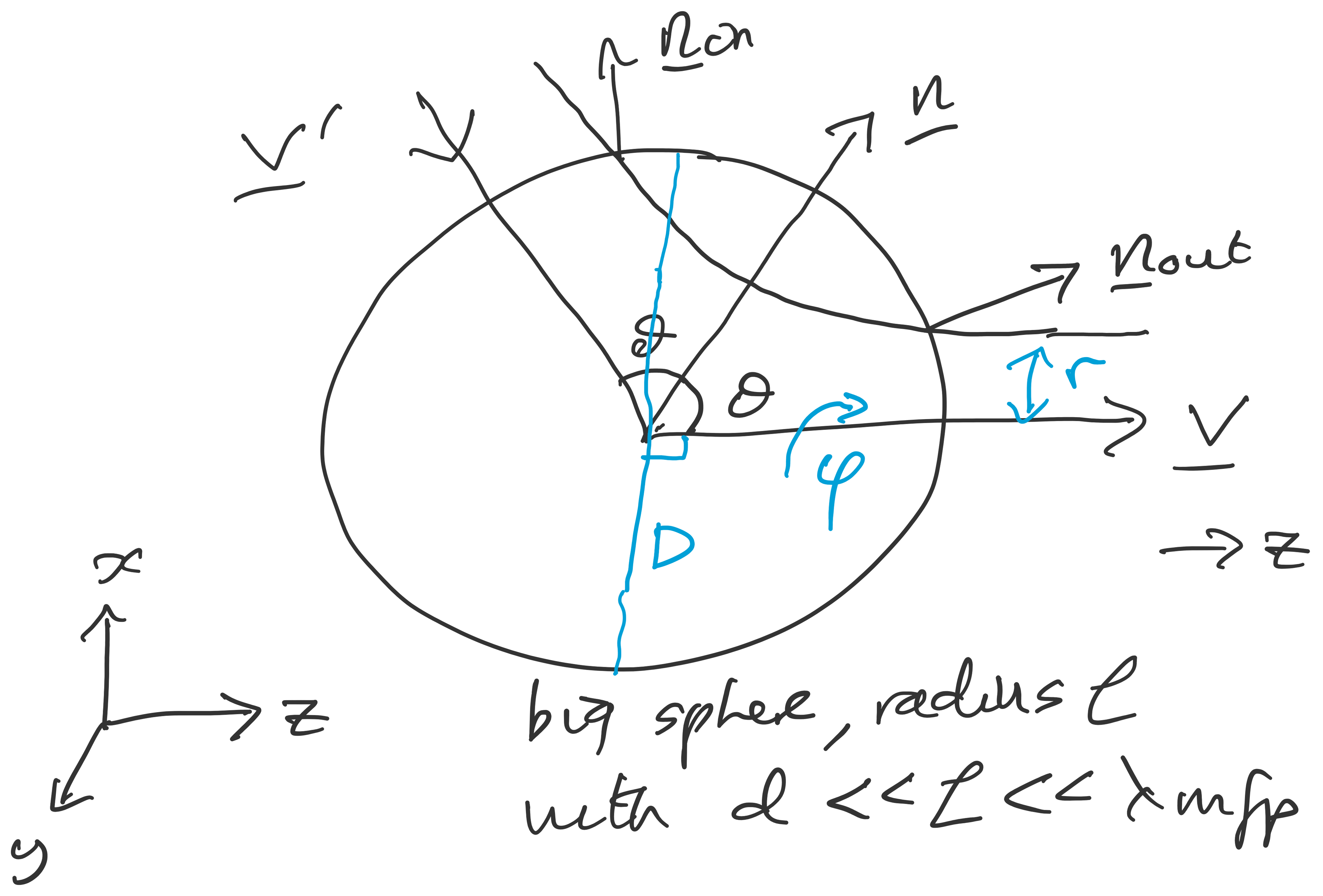
Last time: Boltzmann collision operator

$$\left.\frac{df_1}{dt}\right|_{coll} = G - L$$

$$= \int d\underline{p}_2 \int_{S_+} dS\, |\underline{n}_S \cdot \underline{V}|\, f_1(\underline{p}_1', \underline{q}_1', t) f_1(\underline{p}_2', \underline{q}_2', t)$$

$$- \int d\underline{p}_2 \int_{S_-} dS\, |\underline{n}_S \cdot \underline{V}|\, f_1(\underline{p}_1, \underline{q}_1, t) f_1(\underline{p}_2, \underline{q}_2, t)$$



big sphere, radius ℓ with $d \ll \ell \ll \lambda_{mfp}$

We can absorb details of binary collisions by parametrising each hemisphere $S_\pm$ using r, φ coordinates on the disc $D \perp \underline{V}$. r is the impact parameter, or distance of closest approach with no interaction.

$$\int_{S_\pm} dS\, |\underline{n}_S \cdot \underline{V}| \ldots = V \int dr\, d\varphi\, r \ldots$$

$$= \int_D V r(\theta, V) \left| \frac{\partial r(\theta, V)}{\partial \theta} \right| d\theta\, d\varphi \ldots$$

$$= \int_D d\theta\, d\varphi\, B(\theta, V) \ldots$$

This defines a collisional kernel $B(\theta, V)$ by

$$|\underline{V} \cdot \underline{n}^{(S)}|\, dS = V r\, dr\, d\varphi$$
$$= B(\theta, V)\, d\theta\, d\varphi$$
$$= V \sigma(\theta, V) \underbrace{\sin\theta\, d\theta\, d\varphi}_{dS}$$

where dS is the area element on the unit sphere in spherical polars, and $\sigma(\theta, V)$ is the differential cross-section.

$$B(\theta, V) = V r \left| \frac{\partial r}{\partial \theta} \right| \quad \text{while}$$

$$\sigma(\theta, V) = \frac{1}{\sin\theta}\, r \left| \frac{\partial r}{\partial \theta} \right|$$

σ has dimensions of area.

With this machinery, the Boltzmann equation for a general 2-particle interaction is

$$\frac{\partial f_1}{\partial t} + \frac{\underline{p}_1}{m} \cdot \frac{\partial f_1}{\partial \underline{q}_1} = \left.\frac{df_1}{dt}\right|_{coll}$$

$$= \int d\underline{p}_2 \int_S d\theta\, d\varphi\, B(\theta, V) \big[f_1(\underline{p}_1', \underline{q}_1', t) f_1(\underline{p}_2', \underline{q}_2', t) - f_1(\underline{p}_1, \underline{q}_1, t) f_1(\underline{p}_2, \underline{q}_2, t) \big]$$

LHS is a function of $\underline{p}_1, \underline{q}_1, t$

In the RHS, $V = |\underline{p}_1 - \underline{p}_2|/m$

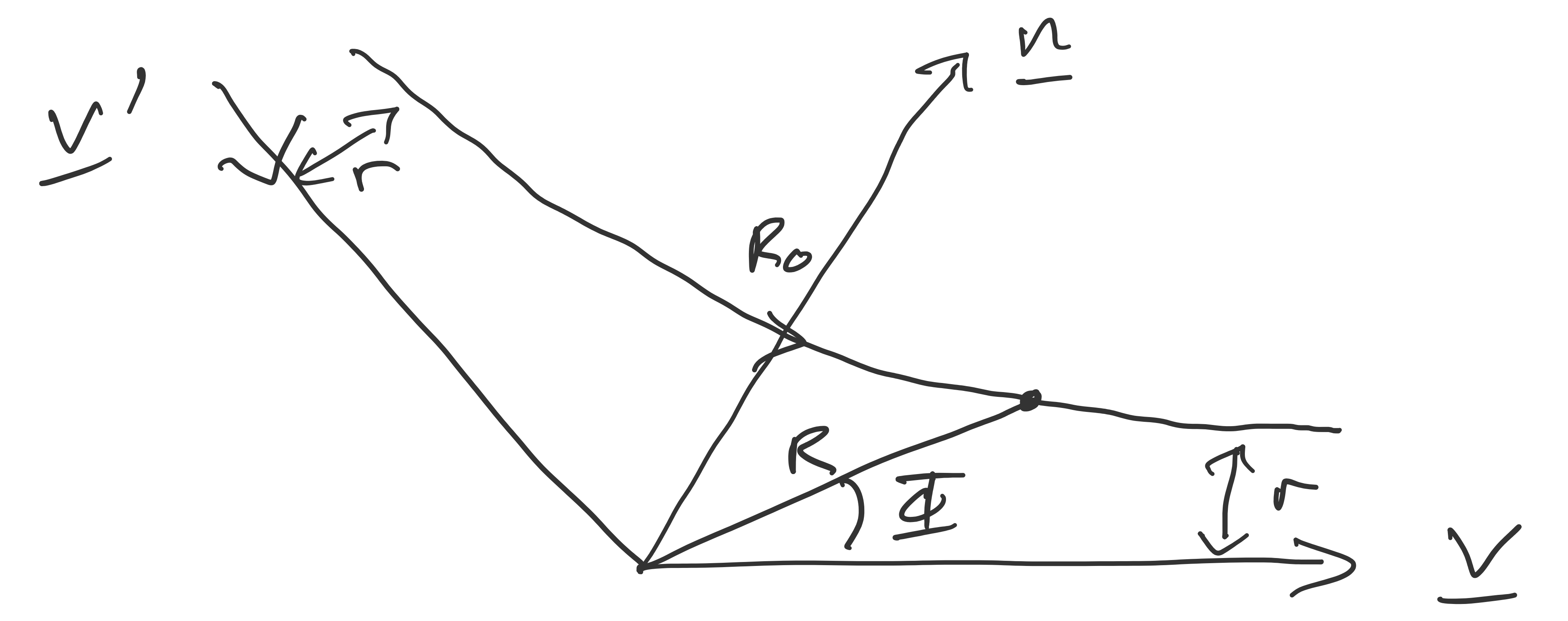
$\underline{n} = (\sin\theta \cos\varphi, \sin\theta \sin\varphi, \cos\theta)$

$\underline{p}_1'$ & $\underline{p}_2'$ are known functions of $\underline{p}_1, \underline{p}_2, \theta$ & φ by

$$\underline{p}_1' = \underline{p}_1 + m\, \underline{n}\, \underline{n} \cdot \underline{V}$$
$$\underline{p}_2' = \underline{p}_2 - m\, \underline{n}\, \underline{n} \cdot \underline{V}$$

Tuesday, 20 October 2020 11:42

Calculating $B(\theta, V)$:



Plane polars (R, Φ) in the plane of the trajectory.

Central force problem for the displacement $\underline{q} = \underline{q}_2 - \underline{q}_1$ with reduced mass $\mu = \frac{1}{2}m$.

Conservation of energy & angular momentum:

$$\tfrac{1}{2}\mu\left(\dot{R}^2 + R^2\dot{\Phi}^2\right) + \phi(R) = \tfrac{1}{2}\mu V^2 + \phi(R_{cut})$$

$$R^2\dot{\Phi} = rV$$

cut-off the potential to be constant for $R \geqslant R_{cut}$.

Eliminate t to get the trajectory $\Phi(R)$ and $\Phi \to 0$ or 2θ as $R \to \infty$

closest approach R_0 satisfies

$$\frac{\mu}{2}V^2\left(1 - \frac{r^2}{R_0^2}\right) = \phi(R_0) - \phi(R_{cut})$$

$$\theta = \sqrt{\frac{\mu}{2}}\, rV \int_{R_0}^{R_{cut}} \left[\frac{\mu}{2}\left(1 - \frac{r^2}{R^2}\right) - \phi(R) + \phi(R_{cut})\right]^{-1/2} + \mathrm{sch}^{-1}(r/d).$$

For hard spheres, $R_{cut} = R_0$, we just get $r = d\,\mathrm{sch}\,\theta$ ($\underline{n}$ and $\underline{n}^S$ are the same)

In principle this defines $\theta(r)$. For suitable potentials, say purely repulsive, we can invert to get $r(\theta)$ and calculate

$$B(\theta, V) = Vr\left|\frac{\partial r}{\partial \theta}\right|$$

For power law potentials (no lengthscale)

$$\phi(|\underline{q}|) = k|\underline{q}|^{1-n}$$

and $n \notin \{2, 3\}$ some horrible substitutions (Cercignani '88 p.70) lead to the separable form

$$B(\theta, V) = V^{\alpha}\beta(\theta)$$

with exponent $\alpha = \frac{n-5}{n-1}$.

For "Maxwell molecules" with $n = 5$, $\alpha = 0$ so $B(\theta, V) = \beta(\theta)$ is independent of V.

$\beta(\theta)$ behaves like

$$\beta(\theta) = O(\theta) \quad \text{as } \theta \to 0$$

$$\beta(\theta) = O\left[\left(\frac{\pi}{2} - \theta\right)^{\frac{n+1}{n-1}}\right] \quad \text{as } \theta \to \frac{\pi}{2}$$

β diverges for small angle deflections

Expanding the Boltzmann collision integral for small-angle deflections & Coulomb interactions $(n=2)$ leads to the Landau collision operator in plasma physics.

Tuesday, 20 October 2020 11:42

VII Properties of Boltzmann's collision operator

Change notation to $\underline{v} = \underline{p}/m$ and drop the 1 suffix on f_1 since we no longer need f_2.

Write
$$f = f(\underline{x}, \underline{v}, t)$$
$$f_* = f(\underline{x}, \underline{v}_*, t)$$
$$f' = f(\underline{x}, \underline{v}', t)$$
$$f'_* = f(\underline{x}, \underline{v}'_*, t)$$

Absorb m into B, leaving

$$\frac{\partial f}{\partial t} + \underline{v}\cdot\nabla f = C[f,f] \quad \text{bilinear in } f$$
$$= \int d\underline{v}_* \int d\theta\, d\phi\, B(\theta, V)\left(f'f'_* - ff_*\right)$$

where
$$\underline{v}' = \underline{v} - \underline{n}\,\underline{n}\cdot(\underline{v}_* - \underline{v})$$
$$\underline{v}'_* = \underline{v}_* + \underline{n}\,\underline{n}\cdot(\underline{v}_* - \underline{v})$$

Consider the general bilinear expression
$$C[f,g] = \tfrac{1}{2}\int d\underline{v}_* \int d\theta\, d\phi\, B(\theta,V)\left(f'g'_* + g'f'_* - fg_* - gf_*\right)$$

and its integral
$$\int d\underline{v}\, C[f,g]\,\psi(\underline{v})$$
$$= \tfrac{1}{2}\int d\underline{v} \int d\underline{v}_* \int d\theta\, d\phi\, \psi(\underline{v}) B(\theta,V)\left(f'g'_* + g'f'_* - fg_* - gf_*\right)$$

Swapping the $*$ and unstarred variables gives
$$\int d\underline{v}\, C[f,g]\,\psi(\underline{v})$$
$$= \tfrac{1}{2}\int d\underline{v} \int d\underline{v}_* \int d\theta\, d\phi\, \psi(\underline{v}_*)\, B(\theta,V)\left(f'g'_* + g'f'_* - fg_* - gf_*\right)$$

This is now $\psi(\underline{v}_*)$.

Can also swap $\underline{v}, \underline{v}_*$ with $\underline{v}', \underline{v}'_*$ and $d\underline{v}'\,d\underline{v}'_* = d\underline{v}\,d\underline{v}_*$ since this transformation has unit Jacobian, and $|\underline{n}\cdot\underline{V}| = |\underline{n}'\cdot\underline{V}|$.

Now we can swap starred & unstarred variables again.

Combining these four equivalent expressions for the LHS gives

$$\int d\underline{v}\, C[f,g]\,\psi(\underline{v}) \quad (1)$$
$$= \frac{1}{8}\int d\underline{v} \int d\underline{v}_* \int d\theta\, d\phi\, B(\theta,V)\left(f'g'_* + f'_*g' - fg_* - f_*g\right)\left(\psi + \psi_* - \psi' - \psi'_*\right)$$

and finally
$$\int d\underline{v}\, C[f,f]\,\psi(\underline{v}) \quad (2)$$
$$= \frac{1}{4}\int d\underline{v} \int d\underline{v}_* \int d\theta\, d\phi\, B(\theta,V)\left(f'f'_* - ff_*\right)\left(\psi + \psi_* - \psi' - \psi'_*\right).$$

The moments (1) and (2) of $C[f,g]$ or $C[f,f]$ with respect to $\psi(\underline{v})$ vanish if

$$\underbrace{\psi(\underline{v}) + \psi(\underline{v}_*)}_{\text{pre-collisional velocities}} = \underbrace{\psi(\underline{v}') + \psi(\underline{v}'_*)}_{\text{post-collisional velocities}}.$$

We constructed the map
$$(\underline{v}, \underline{v}_*) \to (\underline{v}', \underline{v}'_*)$$
to conserve energy and momentum, the only 5 continuous functions ψ that satisfy $\psi(\underline{v}) + \psi(\underline{v}_*) = \psi(\underline{v}') + \psi(\underline{v}'_*)$ are $\psi = 1$, $\psi = \underline{v}$, $\psi = |\underline{v}|^2$, and their linear combinations. These will imply macroscopic mass, momentum & energy conservation.