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Last time:

$$\int d\underline{v}\; C[f,f]\,\psi(\underline{v})$$
$$= \frac{1}{4}\int d\underline{v}\int d\underline{v}_*\int d\theta\, d\phi\; B(\theta,V)\,[f'f'_* - ff_*]\,[\psi + \psi_* - \psi' - \psi'_*]$$

where $f' = f(\underline{x}, \underline{v}', t)$ etc.

This vanishes if ψ is a linear combination of the 5 collision invariants: $1, v_x, v_y, v_z, |\underline{v}|^2$

VIII Boltzmann's inequality & the Maxwell-Boltzmann distribution

Which positive functions f satisfy $C[f,f]=0$?

Putting $\psi = \log f$ gives Boltzmann's inequality:

$$\int d\underline{v}\; C[f,f]\log f$$
$$= \frac{1}{4}\int d\underline{v}\int d\underline{v}_*\int d\theta\, d\phi\; B(\theta,V)\,[f'f'_* - ff_*]\,\log\left(ff_*/f'f'_*\right)$$
$$\leq 0$$

because $(x-y)(\log y - \log x) \leq 0$ for all $x, y \in \mathbb{R}^+$ with equality iff $x = y$.

Put $x = f'f'_*$ and $y = ff_*$.

Equality occurs iff $ff_* = f'f'_*$

$\Rightarrow \log f + \log f_* = \log f' + \log f'_*$

$\therefore \log f$ is a collision invariant, so $\log f = a + \underline{b}\cdot\underline{v} + c|\underline{v}|^2$, for constants a & c, constant vector $\underline{b}$.

We need $c<0$ so that $f \to 0$ as $|\underline{v}| \to \infty$.

Looking ahead to deriving fluid dynamics, it's more common to write this f as

$$f^{(0)}(\underline{v}) = \frac{n}{(2\pi\Theta)^{3/2}}\exp\left(-\frac{|\underline{v}-\underline{u}|^2}{2\Theta}\right)$$

This is a Maxwell-Boltzmann distribution, denoted by $^{(0)}$, and it's a property of $C[f,f]$.

The new constants $n, \underline{u}, \Theta$ satisfy

$$\int d\underline{v}\; f^{(0)} = n$$
$$\int d\underline{v}\; \underline{v} f^{(0)} = n\underline{u}$$
$$\frac{1}{3}\int d\underline{v}\; |\underline{v}-\underline{u}|^2 f^{(0)} = n\Theta$$

It turns out that $n, \underline{u}, \Theta$ are the number density, velocity and temperature in fluid equations that describe slowly varying solutions of the Boltzmann equation.

Θ is the temperature in so-called energy units, so $\Theta^{1/2}$ is the isothermal (Newtonian) sound speed. In more common units $\Theta = RT$ with gas constant R, T in Kelvin, $R = k_B/m$.

The energy density is

$$\mathcal{E} = m\int d\underline{v}\; \frac{1}{2}|\underline{v}|^2 f^{(0)}$$
$$= \frac{1}{2}\rho|\underline{u}|^2 + \frac{3}{2}\rho\Theta$$

where $\rho = mn$ is mass density

A spatially homogeneous $f(\underline{v}, t)$ satisfies $\partial_t f = C[f, f]$.

Multiply by $1 + \log f$ and $\int d\underline{v}$:

$$\int d\underline{v}\,(1+\log f)\,\partial_t f = \int d\underline{v}\,(1+\log f)\,C[f,f]$$

$$\int d\underline{v}\,\partial_t (f \log f) = \int d\underline{v}\,C[f,f] \log f$$
$$\leq 0$$

because 1 is a collision invariant

Boltzmann's H-function $H[f] = \int f \log f \,d\underline{v}$ is non-increasing in time.

$H[f]$ is a mathematician's convex entropy density for the Boltzmann equation.

$H[f]$ is non-increasing in time, and evolution under $\partial_t f = C[f, f]$ preserves the collision invariants.

It's natural to ask how small $H[f]$ can become while preserving the collision invariants $n, \underline{u}, \Theta$.

Using Lagrange multipliers, minimise

$$F = \int d\underline{v} \left\{ f \log f - (a' + \underline{b}\cdot\underline{v} + c|\underline{v}|^2) f \right\}$$
$$= H[f] - (a'n + \underline{b}\cdot\underline{u}\,n + 2c\,\mathcal{E})$$

$$0 = \delta F = \int d\underline{v} \left\{ (1+\log f) - (a' + \underline{b}\cdot\underline{v} + c|\underline{v}|^2 \right\} \delta f$$

This gives

$$\log f = \underbrace{(a'-1)}_{=a} + \underline{b}\cdot\underline{v} + c|\underline{v}|^2$$

so we're back with a Maxwell-Boltzmann distribution.

One can show that $H[f] \to H[f^{(0)}]$ implies $\int d\underline{v}\,|f - f^{(0)}| \to 0$ using a quantity called relative entropy.

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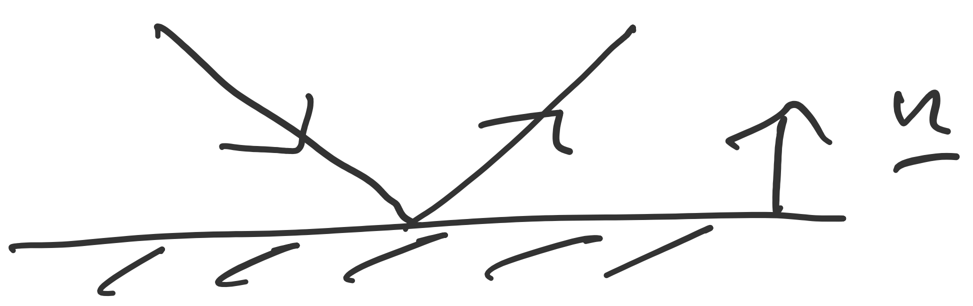
Boltzmann's H-theorem

Suppose $f(\underline{x},\underline{v},t)$ solves Boltzmann's equation
$$\partial_t f + \underline{v}\cdot\nabla f = C[f,f]$$
in a spatial domain Ω.

Multiply by $1+\log f$ and $\int d\underline{v}$ to get
$$\partial_t H + \nabla\cdot\underline{J} = S \le 0,$$
using Boltzmann's inequality, where
$$H(\underline{x},t) = \int d\underline{v}\, f\log f$$
$$\underline{J} = \int d\underline{v}\, \underline{v} f\log f$$
$$S = \int d\underline{v}\, \log f \, C[f,f] \le 0.$$

Integrating over the spatial domain Ω gives Boltzmann's H-theorem:
$$\frac{d\mathcal{H}}{dt} \le \int_{\partial\Omega} \underline{J}\cdot\underline{n}\, dS$$
where $\mathcal{H} = \int_\Omega H(\underline{x},t)\, d\underline{x}$ is the volume integral of H and $\underline{n}$ is the inward pointing normal on Ω.

The surface integral vanishes for various special cases: infinite domains, periodic domains, specularly reflecting boundaries. Like a mirror, $\underline{v}\cdot\underline{n}$ reverses when particles collide with the boundary, while $\underline{v}\times\underline{n}$ is preserved.



More formally,
$$f(\underline{x},\underline{v},t) = f(\underline{x},\underline{v}-2\underline{n}\,\underline{n}\cdot\underline{v},t)$$
for $\underline{x}\in\partial\Omega$ and $\underline{v}\cdot\underline{n} > 0$

The linearized Boltzmann collision operator

If $f = f^{(0)} + \varepsilon f^{(1)}$ is close to a Maxwell-Boltzmann distribution ($\varepsilon \ll 1$) then symmetry and bilinearity of $C[f,f]$ give

$$C[f^{(0)} + \varepsilon f^{(1)}, f^{(0)} + \varepsilon f^{(1)}]$$
$$= \underbrace{C[f^{(0)}, f^{(0)}]}_{=0} + \underbrace{2\varepsilon C[f^{(0)}, f^{(1)}]}_{\text{linear part}} + \varepsilon^2 C[f^{(1)}, f^{(1)}]$$

Better to write instead $f = f^{(0)}(1+\varepsilon h)$ and define the linearized collision operator L by

$$Lh = \frac{2}{f^{(0)}} C[f^{(0)}, f^{(0)} h]$$
$$= \frac{1}{f^{(0)}} \iiint d\underline{v}_* \, d\theta \, d\varphi \, B(\theta, V) \{ f^{(0)\prime} f_*^{(0)\prime} (h' + h'_*) - f^{(0)} f_*^{(0)} (h + h_*) \}$$

$$= \iiint d\underline{v}_* \, d\theta \, d\varphi \, B(\theta, V) f_*^{(0)} (h' + h'_* - h - h_*)$$

because $f^{(0)}(\underline{x}, \underline{v}, t)$ is independent of the integration variables, and

$$f^{(0)\prime} f_*^{(0)\prime} = f^{(0)} f^{(0)\prime}$$

because
$$\underline{v}' + \underline{v}'_* = \underline{v} + \underline{v}'$$
$$|\underline{v}'|^2 + |\underline{v}'_*|^2 = |\underline{v}|^2 + |\underline{v}'|^2$$

Moreover, using the symmetrized form of $C[f,k]$

$$\int d\underline{v} \, f^{(0)} g L h$$
$$= -\frac{1}{4} \iiiint d\underline{v} \, d\underline{v}_* \, d\theta \, d\varphi \, B(\theta, V) f^{(0)} f_*^{(0)} (h' + h'_* - h - h_*)(g' + g'_* - g - g_*)$$

$$= \int d\underline{v} \, f^{(0)} h L g$$ by symmetry under $h \Leftrightarrow g$.

Hence L is a self-adjoint operator

$$\langle g, Lh \rangle = \langle Lg, h \rangle$$

for the weighted inner product

$$\langle g, h \rangle = \int d\underline{v} \, f^{(0)}(\underline{v}) g(\underline{v}) h(\underline{v})$$

with a Maxwell-Boltzmann distribution as the weight function.

Moreover,

$$\langle h, Lh \rangle = -\frac{1}{4} \iiiint d\underline{v} \, d\underline{v}_* \, d\theta \, d\varphi \, B(\theta, V) f^{(0)} f_*^{(0)} (h' + h'_* - h - h_*)^2$$
$$\leq 0$$

with equality iff h is a collision invariant.

Spectrum of the linearized collision operator

If we assume either that the inter-particle potential is constant at long distances (i.e. $R \geq R_{cut}$) or we use Grad's angular cut-off that sets $B(\theta, V) = 0$ for $\theta \geq \theta_{cut}$ near $\pi/2$, we can decompose

$$L h = K h - \nu(\underline{v}) h,$$

where
$$\nu(\underline{v}) = 2\pi \iint d\underline{v}_* d\theta \, B(\theta, V) f_*^{(0)}$$
is a purely multiplicative operator, often called the collision frequency, and

$$K h = \iiint d\underline{v}_* d\theta \, d\varphi \, B(\theta, V) f_*^{(0)} (h' + h'_*)$$
$$- 2\pi \underbrace{\int d\underline{v}_* d\theta \, B(\theta, V) f_*^{(0)}}\, h_* .$$

We can always do this formally, but without the extra assumptions the separate integrals wouldn't converge.

This term is a standard linear integral operator with kernel
$$2\pi f^{(0)}(\underline{v}_*) \int_0^{\pi/2} d\theta \, B(\theta, |\underline{v} - \underline{v}_*|).$$

One can also transform the other term in K into this form, but it's a long calculation.

The operator K is bounded and completely continuous: for any bounded sequence $h_1, h_2, \ldots$ the sequence $K h_1, K h_2, \ldots$ contains a convergent subsequence. A theorem by Weyl establishes that the continuous part of the spectrum of L is determined by $\nu(\underline{v})$, which is multiplicative (and hence self-adjoint).

The operator K can only change the discrete spectrum of L.

The collision frequency $\nu(\underline{v})$ for power-law potentials with angular cut-offs.

$$B(\theta, V) = V^{\alpha}\beta(\theta) \quad \text{with } \alpha = \frac{n-5}{n-1}.$$

We can define $\beta_0 = 2\pi\int_0^{\pi/2}\beta(\theta)\,d\theta$, which is only finite with the cut-off, so

$$\nu(\underline{v}) = \beta_0\int d\underline{v}_* f_*^{(0)}\,|\underline{v}_* - \underline{v}|^{\alpha}$$

If we write $\underline{v} = \underline{u} + \underline{c}$ and $\underline{v}_* = \underline{u} + \underline{c}_*$, where $\underline{c}$ and $\underline{c}_*$ are "peculiar" velocities relative to the fluid velocity $\underline{u}$ in the Maxwell-Boltzmann distribution we're linearising around, $\nu(\underline{v}) = \nu(c)$ for $c = |\underline{c}|$;

$$\nu(c) = \beta\frac{\rho}{(2\pi\theta)^{3/2}}\int d\underline{c}_*\exp\left(-\frac{|\underline{c}_*|^2}{2\theta}\right)|\underline{c}_* - \underline{c}|^{\alpha}$$

The derivative of ν with respect to c is

$$\frac{d\nu}{dc} = \frac{\underline{c}}{c}\cdot\frac{d\nu}{d\underline{c}}$$

$$= -\alpha\frac{\beta_0}{c}\frac{\rho}{(2\pi\theta)^{3/2}}\int d\underline{c}_*\exp\left(-\frac{|\underline{c}_*|^2}{2\theta}\right)\underline{c}\cdot(\underline{c}_* - \underline{c})\,|\underline{c}_* - \underline{c}|^{\alpha-2}.$$

This is all negative as

$$|\underline{c}_*|^2 = |\underline{c}|^2 + |\underline{c}_* - \underline{c}|^2 + 2\underline{c}\cdot(\underline{c}_* - \underline{c})$$

The positive contribution from the half-space with $\underline{c}\cdot(\underline{c}_* - \underline{c}) > 0$ is smaller in modulus than the negative contribution from the half-space with $\underline{c}\cdot(\underline{c}_* - \underline{c}) < 0$.

$$\frac{d\nu}{dc} = -\alpha\,(\text{something negative})$$

and $\alpha = \dfrac{n-5}{n-1}$.

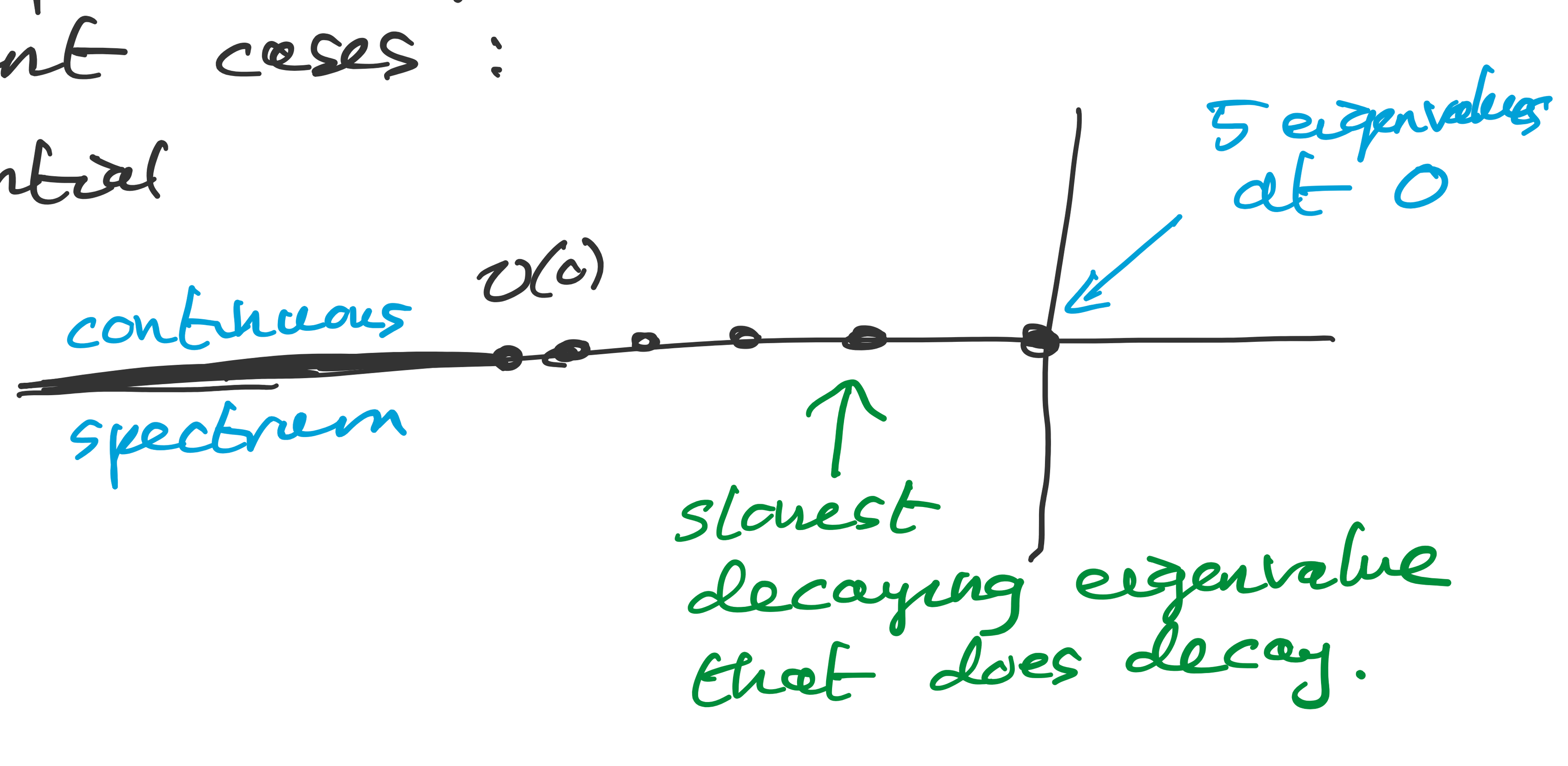
For $n > 5$, hard potentials, $\dfrac{d\nu}{dc} > 0$, so ν is monotonic increasing from $\nu(0)$.

For $n < 5$, soft potentials, $\dfrac{d\nu}{dc} < 0$, so ν is monotonic decreasing from $\nu(0)$ down to 0.

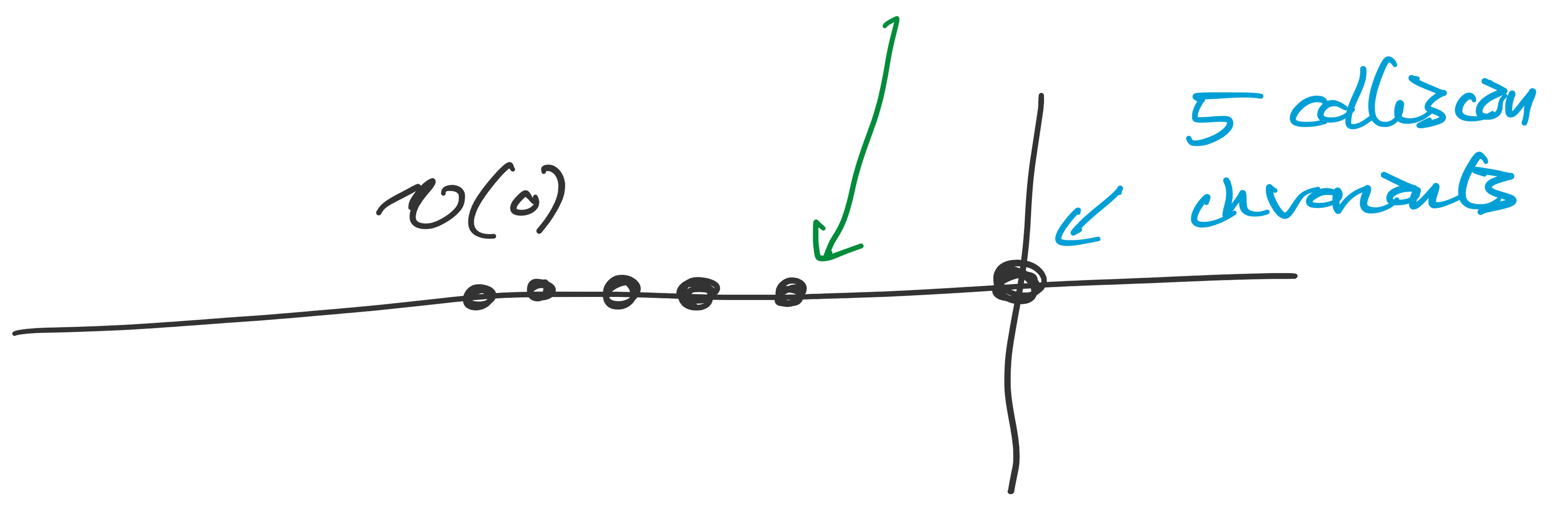
For $n = 5$, $\dfrac{d\nu}{dc} = 0$ so $\nu(c) = \nu(0)$ is constant.

The spectra of L for the different cases:

hard potential
$n > 5$



$n = 5$



$n < 5$
soft potentials