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The multiple scales Chapman–Enskog expansion.

A systematic approach to finding closed evolution equations for $\rho, \underline{u}, \Theta$, the conserved moments.

Take the Boltzmann–BGKW equation and put a small parameter ε in the collision term. Think of $\varepsilon = \tau/T \ll 1$ as the Knudsen number.

$$\partial_t f + \underline{v}\cdot\nabla f = -\frac{1}{\varepsilon\tau}\left(f - f^{(0)}\right).$$

Expand f as a series in ε:

$$f = f^{(0)} + \varepsilon f^{(1)} + \varepsilon^2 f^{(2)} + \cdots$$

Multiplying through by ε:

$$\varepsilon\left(\partial_t f + \underline{v}\cdot\nabla f\right) = -\frac{1}{\tau}\left(f - f^{(0)}\right)$$

ε multiplies all the derivatives

This is called the Hilbert expansion. It becomes disordered after long times when $t \sim 1/\varepsilon$ since $\varepsilon f^{(1)}$ becomes comparable to $f^{(0)}$.

This is the exactly the timescale on which we'd expect viscous & thermal conductive effects to appear.

Also, we never get the Navier–Stokes equations. At each order we get the Euler equations, or some linearized version, with forcing terms from lower orders in the expansion.

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To avoid this disordering at long times we also expand the time derivative as

$$\partial_t = \partial_{t_0} + \varepsilon \partial_{t_1} + \varepsilon^2 \partial_{t_2} + \dots$$

The different t_n represent evolution on different timescales:

t_0 advective

t_1 viscous diffusive

This is equivalent to considering solutions of the form

$f(\underline{x}, \underline{v}, t_0, t_1, t_2, \dots)$

with $t_0, t_1, \dots$ treated as independent variables.

There is now a problem. A function, say $f = \varepsilon t$, can be expanded as both

$f_0(t_1) = t_1$ $\qquad (t_1 = \varepsilon t)$

$f_1(t_0) = t_0$

To make the expansion unique we impose the solvability condition

$$\int d\underline{v}\, f^{(n)} = 0, \quad \int d\underline{v}\, f^{(n)} \underline{v} = 0$$

$$\int d\underline{v}\, \tfrac{1}{2}|\underline{v}|^2 f^{(n)} = 0$$

for $n = 1, 2, \dots$

The higher terms $f^{(1)}, f^{(2)}, \dots$ do not contribute to the conserved moments.

[Cf leaving the conserved moments unexpanded before]

These are the right conditions to avoid the appearance of "secular terms" that would cause the expansion to become disordered at long times. Normally in the method of multiple scales one would consider the general solution, then choose solvability conditions to stop the expansion becoming disordered. There is no general solution to the Euler equations.

substituting these two expansions

$$f = f^{(0)} + \varepsilon f^{(1)} + \dots$$

$$\partial_t = \partial_{t_0} + \varepsilon \partial_{t_1} + \dots$$

into $(\partial_t + \underline{v}\cdot\nabla) f = -\frac{1}{\tau\varepsilon}(f - f^{(0)})$

gives

$$(\partial_{t_0} + \varepsilon\partial_{t_1} + \dots)(f^{(0)} + \varepsilon f^{(1)} + \dots)$$

$$+ \underline{v}\cdot\nabla(f^{(0)} + \varepsilon f^{(1)} + \dots) = -\frac{1}{\tau}(f^{(1)} + \varepsilon f^{(2)} + \dots)$$

This balances at $O(1/\varepsilon)$ because the first term is $f^{(0)}$.

At $O(1)$: $\partial_{t_0} f^{(0)} + \underline{v}\cdot\nabla f^{(0)} = -\frac{1}{\tau} f^{(1)}$

At $O(\varepsilon)$: $\partial_{t_1} f^{(0)} + \partial_{t_0} f^{(1)} + \underline{v}\cdot\nabla f^{(1)} = -\frac{1}{\tau} f^{(2)}$

If we had the linearized Boltzmann collision operator we'd get

$$(\partial_{t_0} + \underline{v}\cdot\nabla) f^{(0)} = f^{(0)} L h^{(1)}$$

where $f^{(1)} = f^{(0)} h^{(1)}$.

$$\circledast \begin{cases} L h^{(1)} = \frac{1}{f^{(0)}}(\partial_{t_0} + \underline{v}\cdot\nabla) f^{(0)} \\ \quad = (\partial_{t_0} + \underline{v}\cdot\nabla)\log f^{(0)} \end{cases}$$

The operator L is self-adjoint, with a kernel spanned by the five collision invariants.

$\circledast$ has a solution iff the RHS is perpendicular to the kernel. This gives exactly the same set of solvability conditions.

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At $O(1)$: $(\partial_{t_0} + \underline{v}\cdot\nabla) f^{(0)} = -\frac{1}{\tau} f^{(1)}$

Taking the five conserved moments gives the Euler equations:

$$\partial_{t_0}\rho + \nabla\cdot(\rho\underline{u}) = 0,$$

$$\partial_{t_0}(\rho\underline{u}) + \nabla\cdot\underline{\underline{\Pi}}^{(0)} = 0,$$

$$\partial_{t_0}\Theta + \underline{u}\cdot\nabla\Theta + \frac{2}{3}\Theta\nabla\cdot\underline{u} = 0.$$

The RHS all vanish by the solvability conditions on $f^{(1)}$.

Now we can use the equation itself to determine

$$f^{(1)} = -\tau\left(\partial_{t_0} f^{(0)} + \underline{v}\cdot\nabla f^{(0)}\right).$$

We can evaluate the RHS in terms of ρ, $\underline{u}$, Θ and their spatial derivatives using the multiple scales expansion in time.

$$f^{(0)} = \frac{\rho/m}{(2\pi\Theta)^{3/2}}\exp\left(-\frac{|\underline{v}-\underline{u}|^2}{2\Theta}\right)$$

$$h^{(1)} = \frac{f^{(1)}}{f^{(0)}} = -\tau(\partial_{t_0} + \underline{v}\cdot\nabla)\log f^{(0)}$$

$$= -\tau(\partial_{t_0} + \underline{v}\cdot\nabla)\left(\log\rho - \frac{3}{2}\log\Theta - \frac{|\underline{v}-\underline{u}|^2}{2\Theta} + \text{constant}\right)$$

$$= -\tau\left(\frac{1}{\rho}(\partial_{t_0} + \underline{v}\cdot\nabla)\rho - \frac{3}{2\Theta}(\partial_{t_0} + \underline{v}\cdot\nabla)\Theta + \frac{1}{2\Theta^2}|\underline{v}-\underline{u}|^2(\partial_{t_0} + \underline{v}\cdot\nabla)\Theta - \frac{1}{\Theta}(\underline{v}-\underline{u})\cdot(\partial_{t_0} + \underline{v}\cdot\nabla)(\underline{v}-\underline{u})\right)$$

$$= -\tau\Bigg(\frac{1}{\rho}\left(-\nabla\cdot(\rho\underline{u}) + \underline{v}\cdot\nabla\rho\right) \left(\frac{1}{2\Theta^2}|\underline{v}-\underline{u}|^2 - \frac{3}{2\Theta}\right) \times\left((\underline{v}-\underline{u})\cdot\nabla\Theta - \frac{2}{3}\Theta\nabla\cdot\underline{u}\right) + \frac{1}{\Theta}(\underline{v}-\underline{u})\cdot\left((\underline{v}-\underline{u})\cdot\nabla\underline{u} - \frac{1}{\rho}\nabla(\rho\Theta)\right)\Bigg)$$

This only depends on ρ, $\underline{u}$, Θ and their spatial derivatives

$$f^{(1)} = -\tau f^{(0)}\left(\frac{1}{\Theta}(w_i w_j - \Theta\delta_{ij})E_{ij} - \frac{1}{2\Theta^2}(|\underline{w}|^2 - \Theta)\,\underline{w}\cdot\nabla\Theta\right)$$

where $\underline{w} = \underline{v} - \underline{u}$ is the peculiar velocity,

$$E_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}\frac{\partial u_k}{\partial x_k}\delta_{ij}\right)$$

$$f^{(0)} = \frac{\rho/m}{(2\pi\Theta)^{3/2}}\exp\left(-\frac{|\underline{w}|^2}{2\Theta}\right)$$

Now we know $f^{(1)}$, taking the conserved moments of

$$\partial_{t_0} f^{(1)} + \partial_{t_1} f^{(0)} + \underline{v}\cdot\nabla f^{(1)} = -\frac{1}{\tau} f^{(2)}$$

gives

$$\partial_{t_1}\rho = 0$$

$$\partial_{t_1}(\rho\underline{u}) + \nabla\cdot\underline{\underline{\Pi}}^{(1)} = 0,$$

$$\partial_{t_1}\Theta + \frac{2}{3}\frac{1}{\rho}\nabla\cdot\underline{q}^{(1)} = 0,$$

where $\underline{\underline{\Pi}}^{(1)} = \int d\underline{v}\, \underline{v}\,\underline{v}\, f^{(1)}$

$$= -\tau\rho\Theta\, 2\underline{\underline{E}}$$

$$\underline{q}^{(1)} = -\frac{5}{2}\tau\rho\Theta\nabla\Theta$$

Putting the expansion back together:

$$\partial_t(\rho\underline{u}) = (\partial_{t_0} + \varepsilon\partial_{t_1} + \dots)(\rho\underline{u})$$

$$= -\nabla\cdot\left(\underline{\underline{\Pi}}^{(0)} + \varepsilon\underline{\underline{\Pi}}^{(1)} + \dots\right)$$

$$\partial_t\Theta = (\partial_{t_0} + \varepsilon\partial_{t_1} + \dots)\Theta$$

$$= -\underline{u}\cdot\nabla\Theta - \frac{2}{3}\Theta\nabla\cdot\underline{u} - \frac{2}{3}\frac{1}{\rho}\nabla\cdot\underline{q}^{(1)} + \dots$$

Again, we've derived the compressible Navier-Stokes-Fourier for an ideal monatomic gas with $\gamma = 5/3$, and Prandtl number 1 from the BGKW collision operator.

A better collision operator would give give the correct coefficients

$$\frac{f^{(1)}}{f^{(0)}} = -\tau_q\frac{1}{2\Theta^2}(|\underline{w}|^2 - \Theta)\,\underline{w}\cdot\nabla\Theta + \tau_\pi\frac{1}{\Theta}(w_i w_j - \Theta\delta_{ij})E_{ij}$$

from two eigenvalues of L.