Shallow water equations with a complete Coriolis force and topography

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This paper derives a set of two dimensional equations describing a thin inviscid fluid layer flowing over topography in a frame rotating about an arbitrary axis. These equations retain various terms involving the locally horizontal components of the angular velocity vector that are discarded in the usual shallow water equations. The obliquely rotating shallow water equations are derived both by averaging the three dimensional equations, and from an averaged Lagrangian describing columnar motion using Hamilton's principle. They share the same conservation properties as the usual shallow water equations, for the same energy and modified forms of the momentum and potential vorticity. They may also be expressed in noncanonical Hamiltonian form using the usual shallow water Hamiltonian and Poisson bracket. The conserved potential vorticity takes the standard shallow water form, but with the vertical component of the rotation vector replaced by the component locally normal to the surface midway between the upper and lower boundaries.

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I. INTRODUCTION

The shallow water equations describe a thin layer of inviscid fluid with a free surface. They are widely used as a prototype to study phenomena like wave-vortex interactions that occur in more complicated models of large scale atmosphere/ocean dynamics, such as the meteorological primitive equations. The meteorological primitive equations are themselves a simplified version of the full compressible gas dynamics equations in rotating spherical geometry. The main simplifications arise from the atmosphere itself being shallow, or of small aspect ratio, and are together known as the traditional and hydrostatic approximations. The traditional approximation[1] involves the neglect of the locally horizontal components of the rotation vector, as well as various so-called metric terms associated with spherical geometry. The hydrostatic approximation involves the neglect of all terms in the vertical momentum equation except the pressure gradient and buoyancy force. Although both approximations are formally valid in the small aspect ratio limit, recent work on "deep" atmospheres has relaxed these approximations in the hope of achieving more accurate depictions of the real atmosphere.[2–7] Similar developments have also taken place in oceanography.[8–10] Both cases are driven partly by the ability of numerical models to resolve shorter and shorter horizontal scales, for which the validity of approximations based on a small aspect ratio becomes increasingly questionable.

Most oceanic and atmospheric phenomena occur on lengthscales much larger than those directly affected by molecular viscosity or diffusity, so it is common to use ideal fluid dynamics. Probably the most important qualitative property is the existence of a materially conserved scalar called potential vorticity, because momentum and energy may be transported over large distances by waves, while potential vorticity is tied to material fluid elements. When deriving approximate models it seems beneficial to ensure that the approximate model satisfies some analogous conservation properties to the underlying equations. For instance, the shallow water equations also possess a potential vorticity conservation law. This may be accomplished most easily using Lagrangian and Hamiltonian formulations,[11–14] in which conservation laws are related to symmetries by Noether's theorem. While energy and momentum conservation arise from the usual translation symmetries in space and time, potential vorticity conservation arises from a more subtle particle relabeling symmetry (see Appendix A).

When rescaled for a vertical lengthscale H much smaller than the horizontal lengthscale L, the three dimensional Euler equations contain factors of the aspect ratio $\delta = H/L \ll 1$, as in Sec. III below. The Coriolis terms involving the horizontal components of the rotation vector appear at $O(\delta)$, while the vertical acceleration appears at $O(\delta^2)$. These scalings justify the traditional and hydrostatic approximations as $\delta \to 0$. Dropping all terms involving δ gives the meteorological primitive equations. White and Bromley[15] derived a set of "quasi-hydrostatic" equations that retain just the $O(\delta)$ terms, giving a complete treatment of the Coriolis force while still neglecting vertical acceleration.

In this paper we derive a shallow water analog of the quasi-hydrostatic equations to describe the vertically averaged behavior of a fluid layer of small aspect ratio flowing over fixed topography in a frame rotating about an arbitrary axis. We thus extend

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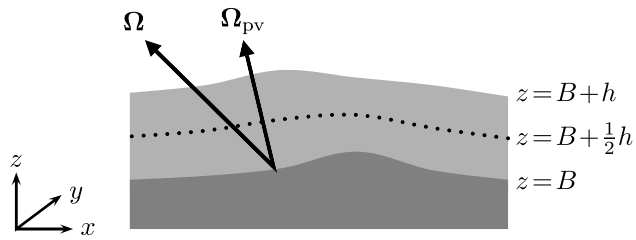

FIG. 1: Geometry of the layer, and the true and apparent angular velocity vectors. To obtain the potential vorticity (1) one replaces the vertical component of $\mathbf{\Omega}$ by the component $\mathbf{\Omega}_{\rm pv}$ normal to the (dotted) surface $z = B(x,y) + \frac{1}{2}h(x,y,t)$ midway between the upper and lower boundaries.

TABLE I: Correspondence between layer-averaged and three dimensional models

Hydrostatic primitive equations	Shallow water with vertical rotation
Quasi-hydrostatic equations[15]	Shallow water with oblique rotation
Tanguay *et al.* regional forecast model[24]	Green–Naghdi with vertical rotation
Rotating Euler equations	Green–Naghdi with oblique rotation

the traditional approximation shallow water equations by retaining various $O(\delta)$ terms due to the horizontal components of the rotation vector. We shall refer to these two models as the "traditional" and "obliquely rotating" shallow water equations.

We derive our obliquely rotating shallow water equations both by vertically averaging the three dimensional equations (Sec. III), and from a variational principle using a vertically averaged Lagrangian (Sec. V). The resulting equations are hyperbolic for sufficiently small velocities, and share the same energy, momentum, and potential vorticity conservation properties of the usual shallow water equations, albeit for modified forms of the momentum and potential vorticity. They may be formulated as a Hamiltonian system using the modified momentum and potential vorticity, and the usual shallow water Hamiltonian and Poisson bracket, just as Roulstone and Brice[16] showed that the quasi-hydrostatic equations, with a suitably redefined momentum and potential vorticity, share the Hamiltonian and Poisson bracket found by Holm and Long[17] for the meteorological primitive equations.

One of the key results of this paper is the derivation of the conserved potential vorticity

$$q = \frac{1}{h}\left[2\left(\Omega_z - \mathbf{\Omega}\cdot\nabla\left(B+\frac{h}{2}\right)\right) - h\nabla\cdot\mathbf{\Omega} + \frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right], \tag{1}$$

where h is the layer depth, $z = B(x,y)$ the lower boundary, and $\mathbf{u} = (u_x, u_y)$ the horizontal velocity. This expression differs from the usual shallow water potential vorticity by the extra terms $\mathbf{\Omega}\cdot\nabla(B+h/2)$ and $h\nabla\cdot\mathbf{\Omega}$ involving the horizontal components of $\mathbf{\Omega}$. The horizontal divergence term $h\nabla\cdot\mathbf{\Omega}$ is included only for completeness, and will usually vanish. One then just replaces the vertical component Ω_z of the rotation vector in the usual formula by the combination $\Omega_z - \mathbf{\Omega}\cdot\nabla(B+h/2)$, the component of the rotation vector that is locally normal to the surface midway between the upper and lower boundaries, as indicated in figure 1. The quantity (1) may also be derived by averaging a suitable Ertel potential vorticity across the layer (see Sec. X).

Extended shallow water equations with nonlinear dispersive terms arising from the $O(\delta^2)$ vertical acceleration have been obtained previously for nonrotating systems. They are usually called the Green–Nagdhi equations,[18] after their derivation using Cosserat surfaces from energy conservation and invariance under rigid-body motions, but their one-dimensional version had been derived previously using vertical averaging by Su and Gardner.[19] The derivation by averaging was extended to two horizontal dimensions by Bazdenkov *et al.*[20] Miles and Salmon[21] obtained the Green–Nagdhi equations from Hamilton's principle using the assumption of columnar motion, and thus derived a potential vorticity conservation law from the particle-relabeling symmetry in their Lagrangian (see Appendix A).

The Green–Naghdi dispersive terms may be included, along with the terms arising from oblique rotation, to obtain what should be a more accurate system of equations for layers with a small but finite aspect ratio. This system may also be thought of as the vertically-averaged analog of the unapproximated rotating Euler equations. In view of the many previous derivations of the Green–Naghdi equations,[11,12,19–23] we concentrate on the obliquely rotating shallow water equations, and only briefly indicate the necessary modifications for the Green–Naghdi version. Bazdenkov *et al.*[20] previously considered an oblique rotation vector in their rederivation of the Green–Naghdi equations by vertical averaging, but they omitted one term in the pressure gradient, the topographic term proportional to $\nabla B(x,y)$ in (11) below. This omission caused their equations to violate energy and potential vorticity conservation in the presence of topography.

We should emphasise that while the resulting obliquely rotating Green–Naghdi equations include the first corrections to columnar motion from both oblique rotation and vertical acceleration, they are not a complete $O(\delta^2)$ approximation to the original three dimensional equations unless the angular velocity vector is either small, or nearly vertical, so that the horizontal Coriolis terms become $O(\delta^2)$ instead of $O(\delta)$. However, Kasahara[7] found that omitting the vertical acceleration has a much larger effect than omitting the horizontal Coriolis terms on the frequencies of normal modes in a realistic stratified atmosphere.

In other words, the $O(\delta^2)$ effect is numerically larger than the $O(\delta)$ effect for realistic values of δ. This might be because neglect of the vertical acceleration is a singular perturbation, in the sense that the quasi-hydrostatic or primitive equations omit the time derivative of the vertical velocity. Similarly, the Green–Naghdi equations add higher derivatives multiplied by small coefficients to the shallow water equations. In this sense, omission of the horizontal Coriolis terms is a regular perturbation in both cases.

Moreover, equations retaining the vertical acceleration but not the (asymptotically larger) horizontal part of the rotation vector were adopted in a regional atmospherical model by Tanguay *et al.*[24] Layer-averaging these equations gives the Green–Naghdi equations with vertical rotation. Thus by retaining the Green–Naghdi terms we may obtain layer-averaged analogs of two addition equation sets: the full rotating Euler equations, and the Tanguay *et al.* model. Table I summarises the correspondence between various sets of three dimensional and layer-averaged equations.

We only consider Cartesian geometry in this paper, so the various other geometrical approximations that form part of the traditional approximation in spherical geometry do not arise. This Cartesian geometry is sometimes called an f-F plane,[4] where f and F are twice the normal and tangential components of $\boldsymbol{\Omega}$, in contrast to the usual f plane that rotates about a normal axis. In Cartesian geometry it is natural to think of conserved linear momenta arising via Noether's theorem from invariance of the Lagrangian under spatial translations.[13,25] However, the conserved zonal component of linear momentum is closely related[26] to the conserved angular momentum one would find from rotational invariance of a Lagrangian in spherical geometry, and is thus sometimes called "angular momentum" even in Cartesian geometry.[27]

The derivation of the obliquely rotating shallow water equations by averaging in Sec. III permits $\boldsymbol{\Omega}$ to vary spatially, so our equations may also be used on a β-plane analog where f and F vary with latitude. We exploit this freedom to study trapped waves on an equatorial β-plane in Sec. XII. The variational derivation in Sec. V assumes a constant $\boldsymbol{\Omega}$ for simplicity, but the calculations may easily be repeated for spatially varying $\boldsymbol{\Omega}$. By considering only an equatorial β-plane we avoid various issues involving approximation of the metric coefficients that arise in midlatitude β-planes.[2,27,28] The extension to spherical geometry is likely to be most easily accomplished using the variational principle developed in Sec. V, after expressing the vertically-averaged Lagrangian as an integral over a spherical surface.

II. ORDERS OF MAGNITUDE

Working in spherical geometry, White and Bromley[15] introduced the velocity scale

$$U_\Omega = 2\Omega H \cos\phi \tag{2}$$

to represent the effects of the horizontal part $\Omega\cos\phi$ of the rotation vector at latitude ϕ. Here H is the layer depth, and U_Ω may be understood as the change in zonal velocity u due to a fluid parcel rising by distance H while conserving its total zonal angular momentum $(u + \Omega r\cos\phi) r\cos\phi$ per unit mass, r being distance from the parcel to the planet's center. With $H = 15\,\mathrm{km}$ being a typical tropopause height, $U_\Omega \sim 2\,\mathrm{m\,s^{-1}}$ for a parcel at the equator rising from surface to tropopause. This is smaller than, but not enormously smaller than, typical eddy velocities of $20\,\mathrm{m\,s^{-1}}$, or typical baroclinic wave speeds of 20 to $80\,\mathrm{m\,s^{-1}}$.[29]

In oceanic applications, the shallow water equations usually arise as a reduced gravity or equivalent barotropic[30] approximation to the two layer equations, in which the lower layer is taken to be very deep and quiescent relative to the upper layer.[11,31] The upper layer then evolves according to the shallow water equations, although in fact the free surface is approximately flat (due to the much greater density difference between air and water than between water masses) and it is the internal interface position that evolves according to the continuity equation. For baroclinic ocean waves the phase speed $c = \sqrt{g'H}$ is typically in the range $0.5\ \mathrm{m\ s^{-1}}$ to $3\ \mathrm{m\ s^{-1}}$, where $g' = g\Delta\rho/\rho$ is the reduced gravity, and the layer depth H is typically $500\ \mathrm{m}$.

Our analyses of linear waves on an f-F plane and an equatorial β-plane both focus attention on the dimensionless parameter δ given by

$$\delta = \frac{2\Omega H}{c}, \quad \delta\cos\phi = \frac{U_\Omega}{c}. \tag{3}$$

We thus find that $0.02 \lesssim \delta \lesssim 0.14$, with slower waves corresponding to larger values of δ. One may think of δ as a reduced Lamb parameter measuring non-traditional effects (the usual Lamb parameter $\Omega R/c$ being based on the planetary radius R instead of the layer depth H). Alternatively, δ coincides with the aspect ratio based on a deformation radius R_{d},

$$\delta = \frac{H}{R_{\mathrm{d}}}, \quad R_{\mathrm{d}} = \frac{c}{2\Omega}. \tag{4}$$

This differs from the usual definition of a deformation radius based on the vertical component $\Omega_z = \Omega\sin\phi_0$ of the rotation vector, and so remains valid at the equator where $\Omega\sin\phi_0 = 0$. As we show subsequently, the aspect ratio based on the equatorial deformation radius $R_{\mathrm{ed}} = \sqrt{c/2\beta}$ is *not* the correct scaling for non-traditional effects in waves on an equatorial β-plane.

Moreover, one might suppose from (3) that Rossby waves on an equatorial β-plane, which have phase speeds much smaller than c, might be more sensitive to non-traditional effects. However, we find subsequently that this is not the case. The Rossby waves are in fact much less affected than the inertial-gravity waves. This perhaps counter-intuitive result agrees with Kasahara's[7] analysis of normal modes in a stratified atmosphere: that faster, shorter wavelength inertia-gravity waves are more sensitive to non-traditional effects than longer and slower waves. Kasahara's analysis used an f-F plane that does not support Rossby waves, so there were no slow modes in the sense of modes that may be captured by a balanced model.

III. DERIVATION BY AVERAGING

Camassa *et al.*[32,33] derived their weakly nonlinear (or rigid lid) "great lake" version of the Green–Naghdi equations by vertically averaging an approximate solution to the rescaled three dimensional Euler equations, found as an asymptotic expansion in a small aspect ratio. We follow a fully nonlinear version of this approach that permits $O(1)$ displacements of the free surface, like that used by Choi and Camassa[23] to derive one dimensional equations describing internal waves in a two layer system, and by Dellar[22] to derive a magnetohydrodynamic analog of the Green–Naghdi equations. More *ad hoc* averaging procedures were applied previously to the unscaled Euler equations by Su and Gardner[19] and Bazdenkov *et al.*[20] to derive the one- and two dimensional Green–Naghdi equations respectively.

The governing equations for a layer of incompressible fluid of unit density, between a rigid base at $z = B(x, y)$ and a free surface at $z = h(x, y, t) + B(x, y)$, are

$$\begin{aligned} \partial_t \mathbf{u}_3 + \mathbf{u}_3 \cdot \nabla_3 \mathbf{u}_3 + 2\boldsymbol{\Omega}_3 \times \mathbf{u}_3 &= -\nabla_3 p - g\hat{\mathbf{z}}, && \text{(5a)} \\ \nabla_3 \cdot \mathbf{u}_3 &= 0. && \text{(5b)} \end{aligned}$$

In this section a subscript 3 is used to indicate a three-component vector, *e.g.* $\mathbf{u}_3 = (u_x, u_y, u_z)$, while unsubscripted vectors like $\mathbf{u} = (u_x, u_y)$ are taken to be purely horizontal. In this section we allow the rotation vector $\boldsymbol{\Omega}_3$ to vary horizontally, in preparation for studying waves on an equatorial β-plane[11,29,31] in Sec. XII.

Equations (5) are subject to the lower boundary condition that $u_z = \mathbf{u}\cdot\nabla B$ on $z = B(x, y)$. The kinematic free surface condition is $\partial_t h = \mathbf{u}_3 \cdot \mathbf{n}_3$ on $z = h(x, y, t) + B(x, y)$, where the (unnormalised) normal vector $\mathbf{n}_3 = (-\partial_x(h + B), -\partial_y(h + B), 1)$ points upwards out of the fluid. We work with the true pressure p that vanishes on the free surface, whereas Camassa *et al.*[32] preferred the modified pressure $p^\star = p + gz$ that absorbs the gravitational term in (5a).

After introducing a typical horizontal lengthscale L, and vertical lengthscales H, the important step in shallow water theory is to scale the vertical coordinate z with a small parameter $\delta = H/L$, the aspect ratio. The reader wishing for a fully dimensionless treatment may take L to be the deformation radius R_d defined in (4), and then adopt the velocity scale $U = 2\Omega R_\mathrm{d}$ giving unit Rossby number. Otherwise, one should think of the the Rossby number $\mathrm{Ro} = U/(2\Omega L)$, and Burger number $\mathrm{Bu} = gH/(4\Omega^2 L^2)$, as both remaining $O(1)$ as $\delta \to 0$.

The incompressibility condition $\nabla_3 \cdot \mathbf{u}_3 = 0$ suggests scaling the vertical velocity u_z to be $O(\delta)$, so we set $\mathbf{u}_3 = (\mathbf{u}, \delta w)$. However, we leave the rotation vector unscaled as $\boldsymbol{\Omega}_3 = (\Omega_x, \Omega_y, \Omega_z)$. Equations (5) then become

$$\begin{aligned} \partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + w\partial_z \mathbf{u} + 2\Omega_z \hat{\mathbf{z}}\times\mathbf{u} + 2\delta\boldsymbol{\Omega}\times\hat{\mathbf{z}}w + \nabla p &= 0, && \text{(6a)} \\ \delta^2(\partial_t w + \mathbf{u}\cdot\nabla w + w\partial_z w) + 2\delta(u_y\Omega_x - u_x\Omega_y) + \partial_z p + g &= 0, && \text{(6b)} \\ \nabla\cdot\mathbf{u} + \partial_z w &= 0, && \text{(6c)} \end{aligned}$$

where ∇, $\boldsymbol{\Omega}$, and $\mathbf{u}$ denote the horizontal (x and y) components of the three dimensional objects ∇_3, $\boldsymbol{\Omega}_3$, and $\mathbf{u}_3$ respectively. The vertical momentum equation (6b) becomes just $\partial_z p + g = 0$ in the $\delta \to 0$ limit. The pressure is thus purely hydrostatic, leading to the usual (non-dispersive) shallow water equations with a purely vertical rotation vector.[11,31]

To improve upon these shallow water equations we seek solutions of (6) as asymptotic expansions in the small aspect ratio δ,

$$\begin{aligned} \mathbf{u} &= \mathbf{u}^{(0)} + \delta\mathbf{u}^{(1)} + \cdots, \qquad p = p^{(0)} + \delta p^{(1)} + \delta^2 p^{(2)} + \cdots, \\ w &= w^{(0)} + \delta w^{(1)} + \cdots. \end{aligned}$$

The $O(1)$ terms in (6b) imply that $p^{(0)}$ is the hydrostatic pressure, $p^{(0)} = g\,(h(x, y, t) + B(x, y) - z)$, with the property that $\nabla p^{(0)} = g\nabla(h + B)$ is independent of z. The horizontal momentum equations are thus satisfied at leading order by a z-independent velocity $\mathbf{u}^{(0)} = \mathbf{u}^{(0)}(x, y, t)$. The continuity equation (6c), and the lower boundary condition $w = \mathbf{u}^{(0)} \cdot \nabla B$ on $z = B(x, y)$, together determine the vertical velocity as

$$w^{(0)} = \nabla\cdot(\mathbf{u}^{(0)} B) - z\nabla\cdot\mathbf{u}^{(0)}. \tag{7}$$

Having determined $\mathbf{u}^{(0)}$ and $w^{(0)}$, the vertical momentum equation (6b) gives

$$\partial_z p^{(1)} = [2u_x^{(0)}\Omega_y - 2u_y^{(0)}\Omega_x], \tag{8}$$

at $O(\delta)$. Since the term in square brackets $[\cdot]$ is independent of z, (8) integrates to give

$$p^{(1)} = (z - h(x, y, t) - B(x, y))[2u_x^{(0)}\Omega_y - 2u_y^{(0)}\Omega_x], \tag{9}$$

using the free surface condition $p = 0$ on $z = h(x, y, t) + B(x, y)$. Moreover,

$$\nabla p^{(1)} = (z - h - B)\nabla[\cdot] - (\nabla(h + B))[\cdot], \tag{10}$$

and

$$\int_B^{h+B} \nabla p^{(1)} dz = -\frac{1}{2}\nabla(h^2[\cdot]) - h(\nabla B)[\cdot]. \tag{11}$$

Horizontal differentiation does not commute with the z integral (or with layer averaging) because the layer depth h and lower boundary B are themselves functions of x and y.

In principle, the $O(\delta)$ corrections $\mathbf{u}^{(1)}$ and $w^{(1)}$ may now be computed from the $O(\delta)$ terms in (6). However, it is simpler to derive equations for the layer mean velocity $\overline{\mathbf{u}}$ given by

$$\overline{\mathbf{u}}(x,y,t) = \frac{1}{h(x,y,t)} \int_{B(x,y)}^{h(x,y,t)+B(x,y)} \mathbf{u}(x,y,z,t)dz, \tag{12}$$

where an overbar denotes a layer-averaged quantity. Wu[34] showed that

$$h\,(\overline{\partial_t F + \mathbf{u}_3 \cdot \nabla_3 F}) = \partial_t(h\overline{F}) + \nabla\cdot(h\,\overline{\mathbf{u}F}), \tag{13}$$

for general F, by integrating by parts in z and using the kinematic boundary conditions for u_z at the two material surfaces $z = B(x,y)$ and $z = h(x,y,t) + B(x,y)$.

The layer-averaged continuity equation

$$\partial_t h + \nabla\cdot(h\overline{\mathbf{u}}) = 0, \tag{14}$$

is given by (13) with $F = 1$, for which the left hand side vanishes. Equation (6a) may be integrated using (13) with $F = u_x$ and $F = u_y$ to give

$$\partial_t(h\overline{\mathbf{u}}) + \nabla\cdot(h\overline{\mathbf{u}\mathbf{u}}) + 2\Omega_z\hat{\mathbf{z}}\times h\overline{\mathbf{u}} + 2\,\delta\,\mathbf{\Omega}\times\hat{\mathbf{z}}\int_B^{h+B} w\;dz + \int_B^{h+B}\nabla p\;dz = 0. \tag{15}$$

The layer-averaged Reynolds stress factorizes as $\overline{\mathbf{u}\mathbf{u}} = \overline{\mathbf{u}}\,\overline{\mathbf{u}} + O(\delta^2)$, because the cross term $\overline{\mathbf{u}^{(1)}\mathbf{u}^{(1)}}$ in the z-integration is $O(\delta^2)$[19,32]. On replacing w by $w^{(0)}$ from (7), and p by $p^{(0)} + \delta p^{(1)}$, (15) becomes

$$\begin{aligned}\partial_t(h\overline{\mathbf{u}}) + \nabla\cdot(h\overline{\mathbf{u}}\,\overline{\mathbf{u}}) + 2h\Omega_z\hat{\mathbf{z}}\times\overline{\mathbf{u}} \;&+\; gh\nabla(h+B) + 2\,\delta\,\mathbf{\Omega}\times\hat{\mathbf{z}}\left(h\overline{\mathbf{u}}\cdot\nabla B - \frac{1}{2}h^2\nabla\cdot\overline{\mathbf{u}}\right) \\ &-\; \delta\nabla\big(h^2(\Omega_y\overline{u}_x - \Omega_x\overline{u}_y)\big) - 2\,\delta\,h(\nabla B)(\Omega_y\overline{u}_x - \Omega_x\overline{u}_y) = O(\delta^2),\end{aligned} \tag{16}$$

where, to close the system, the vertically integrated terms have been evaluated using $\overline{\mathbf{u}}$ instead of $\mathbf{u}^{(0)}$ by incurring a further error of $O(\delta^2)$. Since it is unnecessary to compute $\mathbf{u}^{(1)}$ explicitly, the structure of $\mathbf{u}^{(1)}$ in z need not be specified. However, it would be natural to seek a $\mathbf{u}^{(1)}$ involving a term proportional to z plus a second z-independent term. Equation (16) may be further simplified into

$$\partial_t(h\overline{\mathbf{u}}) + \nabla\cdot(h\overline{\mathbf{u}}\,\overline{\mathbf{u}}) + 2h\,(\Omega_z - \mathbf{\Omega}\cdot\nabla B)\,\hat{\mathbf{z}}\times\overline{\mathbf{u}} + gh\nabla(h+B) - \mathbf{\Omega}\times\hat{\mathbf{z}}\,h^2\nabla\cdot\overline{\mathbf{u}} - \nabla\left(h^2(\Omega_y\overline{u}_x - \Omega_x\overline{u}_y)\right) = 0, \tag{17}$$

after discarding the $O(\delta^2)$ terms and formally setting the expansion parameter $\delta = 1$.

The $O(\delta)$ terms in (16) would be absent without rotation, or with rotation about a vertical axis. However, an $O(\delta^2)$ correction $p^{(2)}$ to the pressure still arises from the $O(\delta^2)$ acceleration term in the vertical momentum equation (6b) with $w^{(0)}$ given by (7) as above. Averaging this correction leads to the Green–Naghdi equations,[18,19,21]

$$\partial_t h + \nabla\cdot(h\overline{\mathbf{u}}) \;=\; 0, \tag{18a}$$

$$\partial_t(h\overline{\mathbf{u}}) + \nabla\cdot(h\overline{\mathbf{u}}\,\overline{\mathbf{u}}) + gh\nabla(h+B) \;=\; -\frac{1}{3}\nabla\left(h^2\mathcal{D}^2\left(h + \frac{3}{2}B\right)\right) - h(\nabla B)\mathcal{D}^2\left(\frac{1}{2}h + B\right), \tag{18b}$$

where $\mathcal{D} = \partial_t + \overline{\mathbf{u}}\cdot\nabla$ is the Lagrangian or material time derivative. The dispersive terms on the right hand side of (18b) may be added to the right hand side of the obliquely rotating shallow water momentum equation (17) too, and should improve the accuracy of the approximation. However, the resulting equations are not a consistent treatment to $O(\delta^2)$ unless $\Omega_{x,y} = O(\delta)$. In principle we should continue the expansion consistently to $O(\delta^2)$, but this would require the determination of the vertical structure of the $O(\delta)$ correction $\mathbf{u}^{(1)}$ in order to evaluate the horizontal Coriolis terms to $O(\delta^2)$, and also the $\overline{\mathbf{u}^{(1)}\mathbf{u}^{(1)}}$ Reynolds stress at $O(\delta^2)$. This may not be possible within the confines of a set of three evolution equations for a single velocity vector and a height field.

IV. OBLIQUELY ROTATING SHALLOW WATER EQUATIONS

Dropping the overbar to write $\overline{\mathbf{u}} = \mathbf{u} = (u, v)$ for simplicity, the acceleration, or primitive variable, form of equations (17) may be written after some rearrangement as

$$\partial_t u + \mathbf{u}\cdot\nabla u - 2\left(\Omega_z - \mathbf{\Omega}\cdot\nabla\left(B+\frac{h}{2}\right)\right)v + \partial_x\Big[g(h+B) + h(v\Omega_x - u\Omega_y)\Big] - \Omega_y\nabla\cdot(h\mathbf{u}) \;=\; 0, \tag{19a}$$

$$\partial_t v + \mathbf{u}\cdot\nabla v + 2\left(\Omega_z - \mathbf{\Omega}\cdot\nabla\left(B+\frac{h}{2}\right)\right)u + \partial_y\Big[g(h+B) + h(v\Omega_x - u\Omega_y)\Big] + \Omega_x\nabla\cdot(h\mathbf{u}) \;=\; 0. \tag{19b}$$

This particular form may be motivated using the noncanonical Hamiltonian (Sec. VIII B) or Euler–Poincaré (Sec. IX) formulations that yield evolution equations for the specific momentum $\mathbf{m}/h$ instead of the fluid particle velocity. The usual Coriolis term appears, but with the vertical component Ω_z of the rotation vector replaced by the component $\Omega_z - \mathbf{\Omega}\cdot\nabla(B + h/2)$ that is locally normal to the surface $z = B + h/2$ midway between the upper and lower boundaries. The horizontal part of the rotation vector also contributes to the pressure-like quantity $[g(h+B) + h(v\Omega_x - u\Omega_y)]$ above, which represents the up- and down-welling driven by the extra term in the vertical momentum equation (6b). The last terms in (19a,b) arise from the extra Coriolis term involving the vertical velocity w in (6a). Equations (19) also yield the energy equation

$$\frac{\partial}{\partial t}\left(\frac{1}{2}h|\mathbf{u}|^2 + \frac{1}{2}gh(h+2B)\right) + \nabla\cdot\left(h\mathbf{u}\Big[|\mathbf{u}|^2 + g(h+B) + h(v\Omega_x - u\Omega_y)\Big]\right) = 0. \tag{20}$$

The energy density is completely unchanged by rotation, and may be derived by integrating the three dimensional energy density $\frac{1}{2}|\mathbf{u}|^2 + gz$ across the layer, but the energy flux acquires a contribution from the horizontal part of the rotation vector.

Using the continuity equation in the form $\nabla\cdot(h\mathbf{u}) = -h_t$ to rewrite the last terms in each of (19a,b) and cross-differentiating leads to a conservation law $\partial_t q + \mathbf{u}\cdot\nabla q = 0$ for the potential vorticity given previously by (1),

$$q = \frac{1}{h}\left[2\left(\Omega_z - \mathbf{\Omega}\cdot\nabla\left(B+\frac{h}{2}\right)\right) - h\nabla\cdot\mathbf{\Omega} + \frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right].$$

This definition of the potential vorticity differs from the usual shallow water form by extra terms involving the horizontal components of the rotation vector. Two systematic derivations of the same potential vorticity conservation law are given below, one from a particle relabeling symmetry in a variational formulation (Sec. V), and a second from the Casimir invariants of a noncanonical Hamiltonian formulation (Sec. VIII).

Equations (19) may also be rewritten in matrix form as

$$\begin{aligned}\frac{\partial}{\partial t}\begin{pmatrix}h\\u\\v\end{pmatrix} \;+\; &\begin{pmatrix}u & h & 0\\ g+2v\Omega_x-2u\Omega_y & u-2h\Omega_y & h\Omega_x\\ 0 & h\Omega_x & u\end{pmatrix}\frac{\partial}{\partial x}\begin{pmatrix}h\\u\\v\end{pmatrix}\\ +\; &\begin{pmatrix}v & 0 & h\\ 0 & v & -h\Omega_y\\ g+2v\Omega_x-2u\Omega_y & -h\Omega_y & v+2h\Omega_x\end{pmatrix}\frac{\partial}{\partial y}\begin{pmatrix}h\\u\\v\end{pmatrix} = 2(\Omega_z - \mathbf{\Omega}\cdot\nabla B)\begin{pmatrix}0\\v\\-u\end{pmatrix} - g\begin{pmatrix}0\\ \partial_x B\\ \partial_y B\end{pmatrix}.\end{aligned} \tag{21}$$

A conservation form for the left hand sides of (21) is derived in Sec. VIII below using the momenta $h\mathbf{u} + h^2\mathbf{\Omega}\times\hat{\mathbf{z}}$ instead of u and v. However, a conservation form is only necessary to find the speeds of finite amplitude shocks. Equation (21) suffices to give the speeds of small amplitude discontinuities (weak shocks) propagating in the x direction as the eigenvalues of the matrix multiplying the x-derivatives,

$$c_0 = u, \quad c_\pm = u + h\Omega_y \pm \sqrt{gh + h^2(\Omega_x^2 + \Omega_y^2) + 2h(v\Omega_x - u\Omega_y)}. \tag{22}$$

These eigenvalues are real, so the obliquely rotating shallow water equations are hyperbolic, provided $2g^{-1}\,|\mathbf{\Omega}_2||\mathbf{u}| < 1$. Physically, this constraint requires the hydrostatic component of the pressure to exceed the contribution from fluid up- and down-welling driven by the horizontal components $\mathbf{\Omega}_2 = (\Omega_x, \Omega_y)$ of the rotation vector. Since the latter is supposed to be a small correction to the hydrostatic pressure, this constraint is not overly restrictive. The two-layer shallow water equations are also only hyperbolic when the difference in velocity between the two layers is not too large.[35]

V. DERIVATION FROM A VARIATIONAL PRINCIPLE

The same set of equations may be derived from a variational principle, Hamilton's principle of least action, applied to an two dimensional vertically-averaged Lagrangian. The exact Lagrangian for a three dimensional, incompressible fluid of unit density in a frame rotating with angular velocity $\boldsymbol{\Omega}$ may be written as[36,37]

$$\mathcal{L}_{3\mathrm{D}} = \int da\, db\, dc\, \frac{1}{2}\left|\frac{\partial \mathbf{x}}{\partial \tau} + \boldsymbol{\Omega}\times\mathbf{x}\right|^2 - \frac{1}{2}\left|\boldsymbol{\Omega}\times\mathbf{x}\right|^2 - gz + p(\mathbf{a},\tau)\left(\frac{\partial(x,y,z)}{\partial(a,b,c)} - 1\right). \tag{23}$$

The integral is expressed over Lagrangian particle labels $\mathbf{a} = (a,b,c)$, and the particle positions $\mathbf{x} = (x,y,z)$ should be treated as functions of $\mathbf{a}$ and time τ. The variable τ is used to emphasise that partial time derivatives $\partial/\partial\tau$ are taken at fixed particle labels $\mathbf{a}$, instead of at fixed spatial coordinates $\mathbf{x}$. Thus $\partial/\partial\tau = \partial/\partial t + \mathbf{u}_3\cdot\nabla_3$ in Eulerian variables, where $\mathbf{u}_3 = \partial\mathbf{x}/\partial\tau$ is the Eulerian fluid velocity as seen in the rotating frame.

The first term $\frac{1}{2}\left|\frac{\partial \mathbf{x}}{\partial \tau} + \boldsymbol{\Omega}\times\mathbf{x}\right|^2$ in the Lagrangian is the kinetic energy as seen in a nonrotating inertial frame. It arises from applying the well-known relation[25]

$$\left.\frac{\partial}{\partial\tau}\right|_{\mathrm{inertial}} = \left.\frac{\partial}{\partial\tau}\right|_{\mathrm{rotating}} + \boldsymbol{\Omega}\times \tag{24}$$

to the velocity vector inside the usual expression $\frac{1}{2}\left|\frac{\partial \mathbf{x}}{\partial \tau}\right|^2$ for kinetic energy in an inertial frame. Alternatively, one may think of $\frac{\partial \mathbf{x}}{\partial \tau} + \Omega\times\mathbf{x}$ as being the velocity with respect to an inertial frame. The second term $-\frac{1}{2}\left|\boldsymbol{\Omega}\times\mathbf{x}\right|^2$ in (23) is used to subtract out the contribution from the kinetic energy that gives rise to the centrifugal force, because in geophysical fluid dynamics the centrifugal force is conventionally incorporated into the gravitational acceleration g appearing in the third term, the gravitational potential energy. We shall neglect spatial variations in the combined g. The variations due to the spatial dependence of the centrifugal force are in fact smaller than the variations in the true gravitational acceleration with height due to the inverse square law.[3] In the final term, the pressure $p(\mathbf{a},\tau)$ appears as a Lagrange multiplier enforcing incompressibility, represented as the map from $\mathbf{a}$ to $\mathbf{x}(\mathbf{a})$ having unit scalar Jacobian, $\partial(x,y,z)/\partial(a,b,c) = 1$.

According to Hamilton's principle of least action, the equations of motion are those that render the action $\mathcal{S}$ stationary,

$$\delta\mathcal{S} = \delta\int d\tau\,\mathcal{L}_{3\mathrm{D}} = 0. \tag{25}$$

Taking variations separately with respect to $\mathbf{x}$ and p yields the three dimensional incompressible Euler equations with Coriolis force and gravity,

$$\partial_t\mathbf{u}_3 + \mathbf{u}_3\cdot\nabla_3\mathbf{u}_3 + 2\boldsymbol{\Omega}\times\mathbf{u}_3 + \nabla_3 p \;=\; -g\hat{\mathbf{z}}, \quad \nabla_3\cdot\mathbf{u}_3 = 0, \tag{26}$$

where $\mathbf{u}_3 = \partial\mathbf{x}/\partial\tau$. The variations $\boldsymbol{\delta}\mathbf{x}$ and δp should vanish at the endpoints of the τ integration, as in classical particle mechanics,[25] to allow integrations by parts with respect to τ in the action. The variations δp enforce the constraint $\nabla_3\cdot\mathbf{u}_3 = 0$.

A. Restriction to columnar motion

Two dimensional approximations including the shallow water[38] and Green–Naghdi[21] equations, both with purely vertical rotation, have been derived from the above three dimensional Lagrangian by restricting the fluid to move in columns. In other words, we approximate the horizontal particle positions by

$$x = x(a,b,\tau), \quad y = y(a,b,\tau), \tag{27}$$

with no dependence on the third Lagrangian label c. The incompressibility constraint enforced by the pressure then factorizes into[21]

$$\frac{\partial(x,y,z)}{\partial(a,b,c)} = \frac{\partial(x,y)}{\partial(a,b)}\frac{\partial z}{\partial c} = 1. \tag{28}$$

The labels c may be assigned so that $c = 0$ on the rigid bottom $z = B(x,y)$, and $c = 1$ on the free surface $z = B(x,y) + h(x,y,t)$. Thus (28) becomes

$$z = \frac{\partial(a,b)}{\partial(x,y)}c + B(x,y) = h(x,y,t)c + B(x,y). \tag{29}$$

These formulas allow the c integration in (23) to be completed. The incompressibility constraint is now automatically satisfied, so the term multiplied by the pressure $p(\mathbf{x}, t)$ in (23) may be discarded. The remaining terms give the reduced two dimensional Lagrangian

$$\begin{aligned}
\mathcal{L}_{\mathrm{GN}}[x(a,b,\tau), y(a,b,\tau)] \; &= \; \int dadb \int_0^1 dc\, \frac{1}{2}\left|\frac{\partial \mathbf{x}}{\partial \tau}\right|^2 + \frac{\partial \mathbf{x}}{\partial \tau}\cdot(\boldsymbol{\Omega}\times\mathbf{x}) - gz, \\
&= \int dadb\, \frac{1}{2}\left(\frac{\partial x}{\partial \tau}\right)^2 + \frac{1}{2}\left(\frac{\partial y}{\partial \tau}\right)^2 \\
&\quad + \frac{1}{6}\left(\frac{\partial h}{\partial \tau}\right)^2 + \frac{1}{2}\left(\frac{\partial h}{\partial \tau}\right)\left(\frac{\partial B}{\partial \tau}\right) + \frac{1}{2}\left(\frac{\partial B}{\partial \tau}\right)^2 \\
&\quad - \frac{1}{2} g(h+2B) + \Omega_z\left(x\frac{\partial y}{\partial \tau} - y\frac{\partial x}{\partial \tau}\right) + (h+2B)\left(\frac{\partial x}{\partial \tau}\Omega_y - \frac{\partial y}{\partial \tau}\Omega_x\right),
\end{aligned} \tag{30}$$

which depends only on the horizontal positions $x(a, b, \tau)$ and $y(a, b, \tau)$ of the fluid columns. An integration by parts on τ has been used to simplify the final term involving the horizontal components of $\boldsymbol{\Omega}$. As shown in the next section, this is equivalent to replacing $\boldsymbol{\Omega}\times\mathbf{x}$ by some other vector field $\mathbf{R}$, given explicitly in (42) below, satisfying $\nabla\times\mathbf{R} = 2\boldsymbol{\Omega}$ and $\hat{\mathbf{z}}\cdot\mathbf{R} = 0$. These manipulations change the Lagrangian by an exact time derivative, and thus make no contribution to the action, or to the evolution equations obtained from the action via Hamilton's principle.

The terms in (30) that are quadratic in the time derivatives arise from the kinetic energy. In particular, the three terms that are quadratic in $\partial h/\partial\tau$ and $\partial B/\partial\tau$ arise from the vertical velocity's contribution $\frac{1}{2}(\partial z/\partial\tau)^2$ to the kinetic energy, once z has been rewritten in terms of h and B using (29). These three terms give rise to the nonlinear dispersive corrections in the Green–Naghdi equations. Omitting these terms, on the grounds that they are $O(\delta^2)$ smaller than the contributions from the horizontal velocities, leads to the simpler shallow water Lagrangian

$$\begin{aligned}
\mathcal{L}_{\mathrm{SW}}[x(a,b,\tau), y(a,b,\tau)] = \int dadb\, \frac{1}{2}\left(\frac{\partial x}{\partial \tau}\right)^2 \; &+ \; \frac{1}{2}\left(\frac{\partial y}{\partial \tau}\right)^2 + \Omega_z\left(x\frac{\partial y}{\partial \tau} - y\frac{\partial x}{\partial \tau}\right) \\
&+ \; (h+2B)\left(\frac{\partial x}{\partial \tau}\Omega_y - \frac{\partial y}{\partial \tau}\Omega_x\right) - \frac{1}{2} g(h+2B).
\end{aligned} \tag{31}$$

B. Free surface boundary condition

The other boundary condition, that the pressure should vanish on the free surface $z = h(x, y, t) + B(x, y)$, has not yet appeared explicitly. In fact it is implicit in the form of the Lagrangians (23) and (30). The three dimensional Lagrangian (23) applies either to fluid of infinite extent, or to fluid in a bounded domain with boundary conditions that do no work on the fluid. The latter includes both no flux rigid boundaries ($\mathbf{u}\cdot\mathbf{n} = 0$) and free surfaces with zero pressure boundary conditions. In both cases the work done, being the force multiplied by the displacement, vanishes because either the force or the displacement vanishes. An imposed external pressure variation may be included via an additional surface integral representing the work done by the external pressure.[21] Similarly, Lewis *et al.*[39] obtained a Hamiltonian for a fluid with surface tension and zero ambient pressure by adding an extra surface integral to account for the pressure just inside the fluid being proportional to the curvature of the free surface.

C. Equations of motion

The most direct route to the equations of motion is via Hamilton's principle: the variations of the action integral with respect to $\mathbf{x}$ must vanish. The Lagrangian density in (31) depends on $\mathbf{x}$ not only explicitly, but also implicitly via $B(x, y)$ and $h(x, y, t)$. The expression in (29) for h,

$$h(x,y,t) = \frac{\partial(a,b)}{\partial(x,y)}, \tag{32}$$

leads to $\delta h = -h\nabla\cdot(\boldsymbol{\delta}\mathbf{x})$, while $\delta B = \boldsymbol{\delta}\mathbf{x}\cdot\nabla B$ since the topography $B(x, y)$ is assumed to be a prescribed function of x and y. This formula for δh leads to the useful result[21]

$$\int dadb\, F\delta h = -\int dadb\, Fh\nabla\cdot(\boldsymbol{\delta}\mathbf{x}) = \int dadb\, \frac{1}{h}\nabla(h^2 F)\cdot\boldsymbol{\delta}\mathbf{x}, \tag{33}$$

that may be derived by transforming the left hand side into an integral with respect to $dxdy$, integrating by parts, and transforming back to an integral with respect to $dadb$. The extra factors of h arise because $dadb = hdxdy$ from the definition of h as a Jacobian in (32).

Taking variations of the action for the shallow water Lagrangian (31) using these formulas gives

$$\delta\int d\tau\mathcal{L}_{\rm SW} \;=\; \int dadbd\tau\left\{-\left(\frac{\partial^2 x}{\partial\tau^2},\frac{\partial^2 y}{\partial\tau^2}\right)+2\Omega_z\left(\frac{\partial y}{\partial\tau},-\frac{\partial x}{\partial\tau}\right)+(-\Omega_y,\Omega_x)\left(\frac{\partial h}{\partial\tau}+2\frac{\partial B}{\partial\tau}\right)\right. \tag{34}$$
$$\left. +\;2\nabla B\left(\Omega_y\frac{\partial x}{\partial\tau}-\Omega_x\frac{\partial y}{\partial\tau}\right)-g\nabla B+\frac{1}{h}\nabla\left(h^2\left(-\frac{1}{2}g+\Omega_y\frac{\partial x}{\partial\tau}-\Omega_x\frac{\partial y}{\partial\tau}\right)\right)\right\}\cdot\boldsymbol{\delta}\mathbf{x},$$

from which we may read off the obliquely rotating shallow water equations (19), on recalling that $\partial_\tau=\partial_t+\mathbf{u}\cdot\nabla$ is a Lagrangian time derivative following a fluid particle. The Green–Nagdhi analog may be written as[21]

$$\delta\int d\tau\mathcal{L}_{\rm GN}=\delta\int d\tau\mathcal{L}_{\rm SW}-\int dadbd\tau\,\left\{\frac{1}{h}\nabla\left[h^2\frac{\partial^2}{\partial\tau^2}\left(\frac{1}{3}h+\frac{1}{2}B\right)\right]+\nabla B\left(\frac{1}{2}\frac{\partial^2 h}{\partial\tau^2}+\frac{\partial^2 B}{\partial\tau^2}\right)\right\}\cdot\boldsymbol{\delta}\mathbf{x}. \tag{35}$$

VI. CHANGING GAUGE IN THE CORIOLIS FORCE

More generally, the Coriolis force (but not the centrifugal force) may be included in Hamilton's principle by replacing $\mathbf{u}$ by $\mathbf{u}+\mathbf{R}$, where $\mathbf{R}$ is any vector potential for the angular velocity satisfying $\nabla\times\mathbf{R}=2\boldsymbol{\Omega}$.[40] This includes $\mathbf{R}=\boldsymbol{\Omega}\times\mathbf{x}$ as a special case when $\boldsymbol{\Omega}$ is constant. The kinetic energy in the Lagrangian then becomes

$$\frac{1}{2}\int dV|\mathbf{u}+\mathbf{R}|^2-|\mathbf{R}|^2=\int dV\,\frac{1}{2}|\mathbf{u}|^2+\mathbf{u}\cdot\mathbf{R}, \tag{36}$$

where the second term leads to the Coriolis force. An equivalent expression $q\mathbf{A}\cdot\mathbf{u}$ occurs in the Lagrangian for a particle with charge q in the magnetic field given by $\mathbf{B}=\nabla\times\mathbf{A}$ in terms of a magnetic vector potential $\mathbf{A}$.[25] The Lorentz force $q\mathbf{B}\times\mathbf{u}$ exerted on the particle is mathematically equivalent to the Coriolis force, and $\mathbf{u}+\mathbf{R}$ is equivalent to the canonical momentum $m\mathbf{u}+q\mathbf{A}$ for a charged particle with mass m.[26]

We may replace $\mathbf{R}$ by $\mathbf{R}'=\mathbf{R}+\nabla\varphi$ for any scalar field φ while still satisfying $\nabla\times\mathbf{R}=\nabla\times\mathbf{R}'=2\boldsymbol{\Omega}$. We call this a change of gauge, by analogy with equivalent transformations of magnetic vector potentials. Since $\mathbf{R}$ only appears within the integral

$$\int_V dV\,\mathbf{u}\cdot\mathbf{R}, \tag{37}$$

a change of gauge from $\mathbf{R}$ to $\mathbf{R}+\nabla\varphi$ in (36) changes the Lagrangian by

$$\mathcal{L}_\varphi=\int_V dV\,\mathbf{u}\cdot\nabla\varphi=\int_{\partial V}dS\,\varphi\mathbf{u}\cdot\mathbf{n}, \tag{38}$$

because $\nabla\cdot\mathbf{u}=0$. In most three dimensional calculations the fluid velocity is supposed to either decay at infinity, or to satisfy no flux ($\mathbf{u}\cdot\mathbf{n}=0$) boundary conditions on rigid boundaries. In both cases the surface integral in (38) vanishes.

However, when applied to a fluid layer with a free surface, the surface integral in (38) turns into an integral over the free surface,

$$\mathcal{L}_\varphi=\int_{\partial V}dS\,\varphi\mathbf{u}\cdot\mathbf{n}=\int dxdy\,\varphi(x,y,h(x,y,t))\frac{\partial h}{\partial t}, \tag{39}$$

using the free surface condition that $\partial_t h+\mathbf{u}\cdot\nabla h=u_z$ on $z=h(x,y,t)$. The surface integral over the bottom vanishes because $\mathbf{u}\cdot\mathbf{n}=0$, even with variable topography, and we assume either that $\mathbf{u}\cdot\mathbf{n}=0$ on horizontal boundaries, or that the flow decays at infinity. Defining a second function $\Phi(x,y,z)$ by

$$\Phi(x,y,z)=\int_0^z\varphi(x,y,z')dz',\quad \varphi(x,y,z)=\frac{\partial\Phi}{\partial z}, \tag{40}$$

this contribution to the Lagrangian may be rewritten as

$$\mathcal{L}_\varphi=\int_{\partial V}dS\,\varphi\mathbf{u}\cdot\mathbf{n}=\int dxdy\,\partial_t\Phi(x,y,h(x,y,t))=\frac{d}{dt}\int dxdy\,\Phi(x,y,h(x,y,t)). \tag{41}$$

Being a total time derivative, this term leaves the action unchanged, and thus the equations of motion obtained from Hamilton's principle unchanged. In other words, all choices of $\mathbf{R}$ satisfying $\nabla\times\mathbf{R}=2\boldsymbol{\Omega}$ lead to the same equations of motion. There is no analog of the charge conservation law one deduces from requiring invariance under changes of gauge for magnetic vector potentials.

The manipulation leading to the two dimensional Lagrangian in (30) above is equivalent to choosing the vector potential

$$\mathbf{R} = \boldsymbol{\Omega}\times\mathbf{x} + \nabla(xz\Omega_y - yz\Omega_x) = (2z\Omega_y - y\Omega_z, x\Omega_z - 2z\Omega_x, 0). \tag{42}$$

Adding the gradient of $\varphi = xz\Omega_y - yz\Omega_x$ removes the vertical component of $\mathbf{R}$. Since we seek a two dimensional reduced system of equations it is natural to make $\mathbf{R}$ purely horizontal too. However, it is still possible to make further changes of gauge while keeping $\mathbf{R}$ purely horizontal using a potential $\varphi_\perp(x, y)$ that has no z dependence. We exploit this freedom in the next section to find conserved components of momentum.

VII. CANONICAL MOMENTA AND GAUGE TRANSFORMATIONS

The equations of motion may also be obtained from the Euler–Lagrange equations for Hamilton's principle,

$$\frac{d}{d\tau}\left(\frac{\delta\mathcal{L}}{\delta\mathbf{x}_\tau}\right) - \frac{\delta\mathcal{L}}{\delta\mathbf{x}} = 0, \tag{43}$$

which are evolution equations for the canonical momenta

$$p_x = \frac{\delta\mathcal{L}}{\delta x_\tau} = \frac{\partial x}{\partial\tau} - y\,\Omega_z + (h+2B)\Omega_y, \quad p_y = \frac{\delta\mathcal{L}}{\delta y_\tau} = \frac{\partial y}{\partial\tau} + x\,\Omega_z - (h+2B)\Omega_x. \tag{44}$$

The variational derivatives are defined by

$$\delta\mathcal{L} = \int dadb\,\frac{\delta\mathcal{L}}{\delta\mathbf{x}}\cdot\delta\mathbf{x} + \frac{\delta\mathcal{L}}{\delta\mathbf{x}_\tau}\cdot\delta\mathbf{x}_\tau. \tag{45}$$

This integral with respect to $dadb$ corresponds to using a *mass-weighted* inner product to define the variational derivatives. Moreover, we interpret (33) to mean that, for example,

$$\begin{aligned}\frac{\delta\mathcal{L}_{\mathrm{SW}}}{\delta x} &= \frac{1}{h}\frac{\partial}{\partial x}\left(h^2\frac{\delta\mathcal{L}_{\mathrm{SW}}}{\delta h}\right) + \frac{\delta\mathcal{L}_{\mathrm{SW}}}{\delta x}\Big|_h, \\ &= \frac{1}{h}\frac{\partial}{\partial x}\left(h^2\left(-\frac{1}{2}g + \frac{\partial x}{\partial\tau}\Omega_y - \frac{\partial y}{\partial\tau}\Omega_x\right)\right) - g\frac{\partial B}{\partial x} + \Omega_z\frac{\partial y}{\partial\tau} + 2\frac{\partial B}{\partial x}\left(\frac{\partial x}{\partial\tau}\Omega_y - \frac{\partial y}{\partial\tau}\Omega_x\right),\end{aligned} \tag{46}$$

where $|_h$ means the variational derivative with respect to x holding h fixed, in other words while ignoring the implicit dependence of h on x and y via (32).

Although canonical, the momenta (44) are not conserved, in the sense of Noether's theorem, because the Lagrangian depends explicitly on x and y through the Ω_z term. Moreover, they are not invariant under the changes of gauge in the vector potential $\mathbf{R}$ discussed in the last section. However, the combination of p_x and p_y in the general formula for the potential vorticity that arises from the relabeling symmetry,

$$q = \frac{1}{h}\left(\frac{\partial p_y}{\partial x} - \frac{\partial p_x}{\partial y}\right), \tag{47}$$

is invariant under gauge transformations. For more details see Appendix A. The canonical momenta may also be used to construct the conserved energy or Hamiltonian through a Legendre transform of the Lagrangian.[25]

We may choose a gauge for $\mathbf{R}$ to eliminate one spatial coordinate from the Lagrangian (as in Ref. 41). Choosing $\varphi_\perp = -xy\Omega_z$ leads to the vector potential

$$\mathbf{R}' = \boldsymbol{\Omega}\times\mathbf{x} + \nabla(xz\Omega_y - yz\Omega_x) + \nabla\varphi_\perp = (2z\Omega_y - 2y\Omega_z, -2z\Omega_x, 0), \tag{48}$$

which is still purely horizontal, but now has no explicit x dependence. The corresponding Lagrangian also has no explicit x dependence, because the Coriolis contribution becomes

$$\mathcal{L}_{\mathrm{Coriolis}} = \int dadb\,\Omega_z\left(-2y\frac{\partial x}{\partial\tau}\right) + (h+2B)\left(\frac{\partial x}{\partial\tau}\Omega_y - \frac{\partial y}{\partial\tau}\Omega_x\right). \tag{49}$$

Noether's theorem now applies for translation invariance in x, giving a conservation law for the modified canonical momentum

$$p_x' = \frac{\delta\mathcal{L}'}{\delta x_\tau} = \frac{\partial x}{\partial\tau} - 2y\,\Omega_z + (h+2B)\Omega_y. \tag{50}$$

Similarly, a different change of gauge removes the y dependence from the Lagrangian. Restoring the factor of h arising from the mass-weighted inner product, we find that the vector

$$\mathbf{M} = h(\mathbf{u} + 2\boldsymbol{\Omega}\times\mathbf{x}) + h(h+2B)\boldsymbol{\Omega}\times\hat{\mathbf{z}} \tag{51}$$

is conserved, in the sense that $\partial_t\mathbf{M} + \nabla\cdot\mathsf{T} = 0$ for some momentum flux or stress tensor T, but there is no choice of $\mathbf{R}$ making both components of $\mathbf{M}/h$ canonical simultaneously. The last terms in (50) and (51) are Cartesian approximations to the contributions to the total angular momentum that arise from the varying perpendicular distance between the free surface and the effective rotation axis. This connection is discussed further in Sec. XII.

VIII. NONCANONICAL HAMILTONIAN FORMULATION

Many of the advantages of the above variational formulation may be retained in a Hamiltonian formulation that avoids the introduction of particle labels. Although the canonical coordinates for an ideal fluid involve particle labels, there is a well developed theory of noncanonical Hamiltonian systems that may be formulated entirely in Eulerian variables.[11–14] The key elements in this theory are the Hamiltonian functional $\mathcal{H}$ and a Poisson bracket $\{\cdot,\cdot\}$. Together they determine the time evolution of any functional $\mathcal{F}$ via $\partial_t\mathcal{F} = \{\mathcal{F},\mathcal{H}\}$. Poisson bracket must be bilinear, antisymmetric, and satisfy the Jacobi identity $\{\mathcal{F},\{\mathcal{G},\mathcal{K}\}\}+\{\mathcal{G},\{\mathcal{K},\mathcal{F}\}\}+\{\mathcal{K},\{\mathcal{F},\mathcal{G}\}\}=0$ for all functionals $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{K}$.

The Poisson bracket for shallow water systems may be written in terms of a momentum $\mathbf{m}$ and layer depth h as the spatial integral

$$\{\mathcal{F},\mathcal{G}\} = \int dxdy\ \left(\frac{\delta\mathcal{F}}{\delta m_i},\frac{\delta\mathcal{F}}{\delta h}\right)\mathsf{J}_{ij}\begin{pmatrix}\delta\mathcal{G}/\delta m_j\\ \delta\mathcal{G}/\delta h\end{pmatrix}, \tag{52}$$

involving the Poisson tensor

$$\mathsf{J}_{ij} = -\begin{pmatrix} m_j\partial_i+\partial_j m_i & h\partial_i\\ \partial_j h & 0\end{pmatrix}, \tag{53}$$

where partial derivatives act on everything to their right. Variational derivatives are now defined using the Euclidean inner product instead of the earlier mass-weighted inner product. The evolution equation $\partial_t\mathcal{F} = \{\mathcal{F},\mathcal{H}\}$ for all functionals $\mathcal{F}$ then corresponds to[11–14]

$$\frac{\partial}{\partial t}\begin{pmatrix}m_i\\ h\end{pmatrix} = \mathsf{J}_{ij}\begin{pmatrix}\delta\mathcal{H}/\delta m_j\\ \delta\mathcal{H}/\delta h\end{pmatrix}. \tag{54}$$

Assuming suitable boundary conditions, for instance solutions that decay to a rest state of uniform depth at infinity, (52) may be integrated by parts to obtain the antisymmetric form

$$\{\mathcal{F},\mathcal{G}\} = -\int dxdy\ \mathbf{m}\cdot\left(\frac{\delta\mathcal{F}}{\delta\mathbf{m}}\cdot\nabla\frac{\delta\mathcal{G}}{\delta\mathbf{m}}-\frac{\delta\mathcal{G}}{\delta\mathbf{m}}\cdot\nabla\frac{\delta\mathcal{F}}{\delta\mathbf{m}}\right)+h\left(\frac{\delta\mathcal{F}}{\delta\mathbf{m}}\cdot\nabla\frac{\delta\mathcal{G}}{\delta h}-\frac{\delta\mathcal{G}}{\delta\mathbf{m}}\cdot\nabla\frac{\delta\mathcal{F}}{\delta h}\right). \tag{55}$$

The integrand in (55) is the inner product of the field variables $\mathbf{m}$ and h with a certain Lie bracket of the variational derivatives of $\mathcal{F}$ and $\mathcal{G}$. This special structure, called a Lie–Poisson structure, enables the Jacobi identity for the Poisson bracket to be established easily using results due to Morrison.[42] The particular Lie algebra whose bracket appears in (55) is the so-called semidirect product Lie algebra for vector fields ($\mathbf{m}$) and densities (h). It is therefore not surprising that the same bracket (albeit with different $\mathbf{m}$ and $\mathcal{H}$) appears in both the shallow water and the Green–Naghdi equations; without rotation, with vertical rotation, and (as we show below) with oblique rotation.

The shallow water Hamiltonian is the spatial integral of the energy density found in (20),

$$\mathcal{H} = \frac{1}{2}\int dxdy\ h|\mathbf{u}|^2+gh(h+2B), \tag{56}$$

which does not involve the rotation vector $\mathbf{\Omega}$. The necessary momentum is given by the previous canonical momenta (44),

$$\mathbf{m} = h\mathbf{p} = h(\mathbf{u}+\Omega_z\hat{\mathbf{z}}\times\mathbf{x})+h(h+2B)\mathbf{\Omega}\times\hat{\mathbf{z}}. \tag{57}$$

The factor of h in $\mathbf{m} = h\mathbf{p}$ is due to using an Euclidean inner product, instead of the earlier mass-weighted inner product, to define variational derivatives. This momentum also arises from integrating $\mathbf{u}+\mathbf{R}$ across the layer for the special two dimensional vector potential in (42).

We show below that the combination of $\mathcal{H}$ from (56) and $\mathbf{m}$ from (57) yields the obliquely rotating shallow water equations. Retaining just the first term in (57), $\mathbf{m} = h(\mathbf{u}+\Omega_z\hat{\mathbf{z}}\times\mathbf{x})$, gives the traditional shallow water equations with rotation about a vertical axis,[13,17] while taking $\mathbf{m} = h\mathbf{u}$ gives the nonrotating shallow water equations. All three systems arise from the same Hamiltonian and Poisson bracket. Further modifications to the Hamiltonian and momentum give the nonrotating Green–Naghdi equations using the same Poisson bracket.[43]

A. Equations of motion

The variational derivatives of the Hamiltonian (56) expressed as a functional of $\mathbf{m}$ and h are

$$\frac{\delta\mathcal{H}}{\delta m_x}=u,\quad \frac{\delta\mathcal{H}}{\delta m_y}=v,\quad \frac{\delta\mathcal{H}}{\delta h}=g(h+B)-\frac{1}{2}(u^2+v^2)+\Omega_z(yu-xv)+2(h+B)(v\Omega_x-u\Omega_y). \tag{58}$$

The continuity equation follows directly from (54), while the momentum part of (54) may be rewritten as[44]

$$\partial_t m_i=-\partial_j\left(m_i\frac{\delta\mathcal{H}}{\delta m_j}\right)-\partial_i\left(m_j\frac{\delta\mathcal{H}}{\delta m_j}+h\frac{\delta\mathcal{H}}{\delta h}\right)+\left(\frac{\delta\mathcal{H}}{\delta m_j}\partial_i m_j+\frac{\delta\mathcal{H}}{\delta h}\partial_i h\right). \tag{59}$$

The first two terms are already in conservation form, while the final term simplifies to

$$\frac{\delta\mathcal{H}}{\delta m_j}\nabla m_j+\frac{\delta\mathcal{H}}{\delta h}\nabla h=\nabla\left(\frac{1}{2}h(u^2+v^2)+\frac{1}{2}gh(h+2B)\right)-h(g+2v\Omega_x-2u\Omega_y)\nabla B+h\Omega_z(\mathbf{u}\times\hat{\mathbf{z}}). \tag{60}$$

The gradient term on the right hand side of (60) is the gradient of the Hamiltonian (or energy) density H for which $\mathcal{H}=\int H\,dxdy$. Equation (59) thus becomes

$$\partial_t m_x+\partial_x\left(um_x+\frac{1}{2}gh^2+h^2(v\Omega_x-u\Omega_y)\right)+\partial_y(vm_x)=\Omega_z hu-h(g+2v\Omega_x-2u\Omega_y)\partial_x B, \tag{61a}$$

$$\partial_t m_y+\partial_x(um_y)+\partial_y\left(vm_y+\frac{1}{2}gh^2+h^2(v\Omega_x-u\Omega_y)\right)=-\Omega_z hv-h(g+2v\Omega_x-2u\Omega_y)\partial_y B, \tag{61b}$$

which coincide with the obliquely rotating shallow water equations. The terms on the right hand side of (61) are those that break translation invariance by involving x and y, either directly in the Ω_z Coriolis term or via the prescribed topography $B(x,y)$. When these are absent (61) takes the conservation form $\partial_t\mathbf{m}+\nabla\cdot\mathsf{T}=0$ for a stress tensor T, as required by Noether's theorem.

A related form that may be useful for numerical implementations is

$$\partial_t(hu+h^2\Omega_y)+\partial_x\left(hu^2+\frac{1}{2}gh^2+h^2v\Omega_x\right)+\partial_y\left(huv+h^2v\Omega_y\right)\;=\;2(\Omega_z-\boldsymbol{\Omega}\cdot\nabla B)hv-gh\partial_x B, \tag{62a}$$

$$\partial_t(hv+h^2\Omega_x)+\partial_x\left(huv-h^2u\Omega_x\right)+\partial_y\left(hv^2+\frac{1}{2}gh^2-h^2u\Omega_y\right)\;=\;-2(\Omega_z-\boldsymbol{\Omega}\cdot\nabla B)hu-gh\partial_y B. \tag{62b}$$

The only terms not in conservation form are those due to variable topography (∇B) and the vertical component Ω_z of the rotation vector. Since these terms are already present in the traditional shallow water equations it should be straightforward to modify existing numerical algorithms to solve the obliquely rotating shallow water equations in the form (62). However, the stress tensor in (62) is not symmetric, since the off-diagonal terms are $huv+h^2v\Omega_y$ and $huv-h^2u\Omega_x$. Angular momentum about an axis normal to the xy plane is not conserved by the obliquely rotating shallow water equations, because the horizontal projection of the rotation vector $\boldsymbol{\Omega}$ defines a preferred direction in the xy plane. This would complicate a lattice Boltzmann formulation analogous to Salmon's[45] formulation of the traditional shallow water equations.

B. Casimirs and potential vorticity conservation

In two dimensions, the evolution equations (54) imply the conservation equation $\partial_t q+\mathbf{u}\cdot\nabla q=0$ for a potential vorticity q given by

$$q=\frac{1}{h}\hat{\mathbf{z}}\cdot\nabla\times\left(\frac{\mathbf{m}}{h}\right). \tag{63}$$

With $\mathbf{m}$ given by (57) above, this definition of the potential vorticity coincides with the one obtained previously from the particle relabeling symmetry. In terms of the Poisson bracket (55), material conservation of q is a consequence of the existence of the so-called Casimir functionals

$$\mathcal{C}=\int dxdy\,h\,c(q). \tag{64}$$

For any function $c(q)$, the corresponding Casimir functional satisfies $\{\mathcal{F},\mathcal{C}\}=0$ for every functional $\mathcal{F}$. In particular $\mathcal{C}_t=\{\mathcal{C},\mathcal{H}\}=0$, so the Casimir functionals provide an infinite family of conserved integrals of q.

The rôle of the potential vorticity may be further highlighted by making a change of variables from $\mathbf{m}$ to $\mathbf{v} = \mathbf{m}/h$. Using the variational chain rule

$$\frac{\delta\mathcal{H}}{\delta\mathbf{v}} = h\frac{\delta\mathcal{H}}{\delta\mathbf{m}}, \quad \frac{\delta\mathcal{H}}{\delta h}\Big|_{\mathbf{v}} = \frac{\delta\mathcal{H}}{\delta h}\Big|_{\mathbf{m}} + \mathbf{v}\cdot\frac{\delta\mathcal{H}}{\delta\mathbf{m}}, \tag{65}$$

and $\delta\mathcal{H}/\delta\mathbf{m} = \mathbf{u}$, (54) transforms into

$$\partial_t h + \nabla\cdot(h\mathbf{u}) \;=\; 0, \tag{66a}$$

$$\partial_t\mathbf{v} - \mathbf{u}\times\nabla\times\mathbf{v} + \nabla\left(\frac{\delta\mathcal{H}}{\delta h}\Big|_{\mathbf{v}}\right) \;=\; 0. \tag{66b}$$

The quantity whose gradient appears in (66b) is the Bernoulli function,

$$\frac{\delta\mathcal{H}}{\delta h}\Big|_{\mathbf{v}} = \frac{1}{2}|\mathbf{u}|^2 + g(h+B) + h(v\Omega_x - u\Omega_y), \tag{67}$$

where the last term arises from the horizontal part of the rotation vector. Equation (66b) thus coincides with the form (19) given previously, after rewriting $\mathbf{u}\cdot\nabla\mathbf{u} = -\mathbf{u}\times\boldsymbol{\omega} + \nabla(\frac{1}{2}|\mathbf{u}|^2)$ and eliminating $\partial_t h$ using the continuity equation. Equation (66b) also leads immediately to a Kelvin circulation theorem for the obliquely rotating shallow water equations,

$$\frac{d}{dt}\oint_C \mathbf{v}\cdot d\mathbf{l} = 0, \tag{68}$$

for any closed curve C moving with the fluid velocity $\mathbf{u}$. This is the integral form of the potential vorticity conservation law.

The cross products in (66b) may be simplified to give

$$\frac{\partial}{\partial t}\begin{pmatrix} v_x \\ v_y \\ h \end{pmatrix} = -\begin{pmatrix} 0 & -q & \partial_x \\ q & 0 & \partial_y \\ \partial_x & \partial_y & 0 \end{pmatrix}\begin{pmatrix} \delta\mathcal{H}/\delta v_x \\ \delta\mathcal{H}/\delta v_y \\ \delta\mathcal{H}/\delta h \end{pmatrix}. \tag{69}$$

This corresponds to rewriting the Poisson bracket (52) and Hamiltonian (56) in terms of $\mathbf{v}$ and h instead of $\mathbf{m}$ and h. For the traditional shallow water equations, $\mathbf{v}$ may be replaced by $\mathbf{u}$ (as in Refs. 13,17) because $\mathbf{u}$ and $\mathbf{v}$ only differ by $\boldsymbol{\Omega}\times\mathbf{x}$. This cannot be done in the obliquely rotating case due to the extra $(h{+}2B)$ term in the relation between $\mathbf{u}$ and $\mathbf{v} = \mathbf{u}{+}\boldsymbol{\Omega}{\times}\mathbf{x}{+}(h{+}2B)\boldsymbol{\Omega}{\times}\hat{\mathbf{z}}$.

IX. EULER–POINCARÉ FORMULATION

The Euler–Poincaré formulation[46] returns to Hamilton's variational principle for the action as the key component, instead of the Hamiltonian and Poisson bracket introduced in the noncanonical Hamiltonian formulation above. However, the Euler–Poincaré formulation retains the use of Eulerian variables, avoiding the particle labels introduced in the previous variational principle, by minimising the action only with respect to a restricted class of variations. These variations are generated by a Lie algebra $[\cdot,\cdot]$ according to $\delta\mathbf{u} = \dot{\boldsymbol{\xi}} + [\boldsymbol{\xi},\mathbf{u}]$, where $\boldsymbol{\xi}$ is an arbitrary vector field vanishing at the endpoints of the τ integration in the action. In typical fluid applications, the Lie algebra is the same Lie algebra of vector fields whose bracket is contracted with $\mathbf{m}$ in the first term in the Lie–Poisson bracket (55).

The Lagrangian coincides with the Lagrangian calculated previously, on recalling that $dadb = hdxdy$,

$$\begin{aligned}\mathcal{L} \;&=\; \mathcal{L}_{\rm SW} + \int dxdy \int_B^{h+B} dz\,\mathbf{u}\cdot\mathbf{R} \\ &=\; \mathcal{L}_{\rm SW} + \int dxdy\; h(xu_y\Omega_z - yu_x\Omega_z) + h(h+2B)(u_x\Omega_y - u_y\Omega_x),\end{aligned} \tag{70}$$

where $\mathbf{R}$ is the vector potential with no z component given by (42), and $\mathcal{L}_{\rm SW}$ is the nonrotating shallow water Lagrangian,[46]

$$\mathcal{L}_{\rm SW} = \frac{1}{2}\int dxdy\, h(u_x^2+u_y^2) - g(h+B)^2. \tag{71}$$

Changing gauge in $\mathbf{R}$ changes the Lagrangian by an exact time derivative, as shown in Sec. VI. The Euler–Poincaré constrained variational principle then gives an equation equivalent to (66b),

$$\partial_t\mathbf{v} - \mathbf{u}\times\nabla\times\mathbf{v} + \nabla\left(\mathbf{u}\cdot\mathbf{v} - \frac{\delta\mathcal{L}}{\delta h}\right) = 0, \quad \text{where } \mathbf{v} = \frac{1}{h}\frac{\delta\mathcal{L}}{\delta\mathbf{u}} = \frac{\mathbf{m}}{h}, \tag{72}$$

and $\mathbf{m}$ and $\mathbf{v}$ coincide with those given previously, as in (57).

X. CONNECTION WITH THREE DIMENSIONAL VORTICITY CONSERVATION

The conserved potential vorticity q may also be found by averaging a suitable three dimensional Ertel potential vorticity, as found by Miles and Salmon[21] for the Green–Naghdi equations. The Ertel potential vorticity is defined by[11,28,31]

$$\Pi_3 = \frac{1}{\rho}(2\mathbf{\Omega}_3 + \nabla_3\times\mathbf{u}_3)\cdot\nabla_3\theta, \tag{73}$$

where ρ is the fluid density, and θ any materially conserved scalar (satisfying $\partial_t\theta + \mathbf{u}_3\cdot\nabla_3\theta = 0$). If the fluid's equation of state is homentropic, so that $\nabla p\times\nabla\rho = 0$, the Ertel potential vorticity is also a materially conserved scalar.

The particle labels a, b, c introduced in Sec. V are materially conserved scalars by definition. For a fluid of unit density moving in columns, as assumed in our earlier derivations, the particle label c may be expressed in terms of Eulerian quantities as $c = h^{-1}(z - B)$. Inserting this expression into (73) and layer-averaging, we obtain

$$\frac{1}{h}\int_B^{h+B} \Pi_3 dz = q. \tag{74}$$

Thus the conserved potential vorticity q of the layer-averaged equations coincides with the layer average of this particular Ertel potential vorticity.

The same relation may be reached by a different route. Averaging the three dimensional vorticity equation for an incompressible fluid, Camassa and Levermore derived the conservation law[47]

$$\partial_t\left(h\,\overline{\boldsymbol{\omega}_3\cdot\nabla_3\zeta}\right) + \nabla\cdot\Big(h\,\overline{\mathbf{u}\,\boldsymbol{\omega}_3\cdot\nabla_3\zeta - \boldsymbol{\omega}\left(\mathbf{u}_3\cdot\nabla_3\zeta - \zeta\partial_t h\right)}\Big) = 0. \tag{75}$$

Here $\mathbf{u}_3$ is a solution of the three dimensional non-rotating Euler equations in the domain $B(x,y)\le z\le h(x,y,t)+B(x,y)$, and $\boldsymbol{\omega}_3 = \nabla_3\times\mathbf{u}_3$ is the corresponding three dimensional vorticity. The horizontal components of these vectors are denoted by $\mathbf{u}$ and $\boldsymbol{\omega}$ in our notation. The scalar field ζ is given by

$$\zeta = \frac{z - B(x,y)}{h(x,y,t)}, \tag{76}$$

which varies linearly between $\zeta = 0$ at the lower boundary $z = B(x,y)$ and $\zeta = 1$ at the upper boundary $z = h(x,y,t) + B(x,y)$.

Equation (75) is an exact result for solutions to the three dimensional Euler equations, and we may include the Coriolis force by replacing $\boldsymbol{\omega}_3 = \nabla_3\times\mathbf{u}_3$ with $\boldsymbol{\omega}_3 = \nabla_3\times\mathbf{u}_3 + 2\mathbf{\Omega}_3$. However, in general the flux inside the divergence cannot be related to other layer-averaged quantities. For the special case of approximately columnar motion in a shallow layer, we recall that $\mathbf{u}(\mathbf{x}_3,t) = \mathbf{u}(x,y,t) + O(\delta)$, and $\nabla\times\mathbf{u}_3 = (\partial_x u_y - \partial_y u_x)\hat{\mathbf{z}} + O(\delta)$. These estimates differ from some given previously[33] because the non-traditional part of the Coriolis force generates deviations from columnar flow at $O(\delta)$, instead of at $O(\delta^2)$ like the vertical acceleration. Thus (75) becomes

$$\partial_t\left(h\,\overline{\boldsymbol{\omega}_3\cdot\nabla_3\zeta}\right) + \nabla\cdot\Big(h\mathbf{u}\,\overline{\boldsymbol{\omega}_3\cdot\nabla_3\zeta} + O(\delta)\Big) = 0, \tag{77}$$

while

$$\overline{\boldsymbol{\omega}_3\cdot\nabla_3\zeta} = \frac{1}{h}\int_B^{h+B} dz\;\frac{1}{h}\left(-\zeta\nabla h - \nabla B, 1\right)\cdot\Big(2\mathbf{\Omega}_3 + (\partial_x u_y - \partial_y u_x)\hat{\mathbf{z}} + O(\delta)\Big) = q + O(\delta). \tag{78}$$

In other words, while q is exactly materially conserved by our layer-averaged equations, in the underlying three dimensional Euler equations, the averaged quantity $\overline{\boldsymbol{\omega}_3\cdot\nabla_3\zeta}$ is close to q, exactly conserved, and nearly materially conserved.

This close relation arises because the interpolating scalar field ζ is precisely the Lagrangian particle label c defined by (29) for a fluid moving in columns. Moreover, the quantity $\boldsymbol{\omega}_3\cdot\nabla_3\zeta$ is then exactly the same as the Ertel potential vorticity used by Miles and Salmon,[21] so equations (74) and (78) coincide for columnar motion (or as $\delta\to 0$).

XI. LINEAR PLANE WAVES

As a first step towards investigating the properties of the obliquely rotating shallow water equations, we consider linear plane waves superimposed on a state of rest. We recall the simplest case with no bottom topography, $B(x,y) = 0$, no Green–Naghdi dispersion, and choose Cartesian axes in which x is eastward, y northward, and z axis radially outwards,

$$\partial_t h \;+\; \partial_x(hu) + \partial_y(hv) = 0, \tag{79a}$$
$$\partial_t u \;+\; \mathbf{u}\cdot\nabla u - (2\Omega_z - \Omega_y\partial_y h)v + \partial_x(gh - h\Omega_y u) - \Omega_y\nabla\cdot(h\mathbf{u}) = 0, \tag{79b}$$
$$\partial_t v \;+\; \mathbf{u}\cdot\nabla v + (2\Omega_z - \Omega_y\partial_y h)u + \partial_y(gh - h\Omega_y u) = 0. \tag{79c}$$

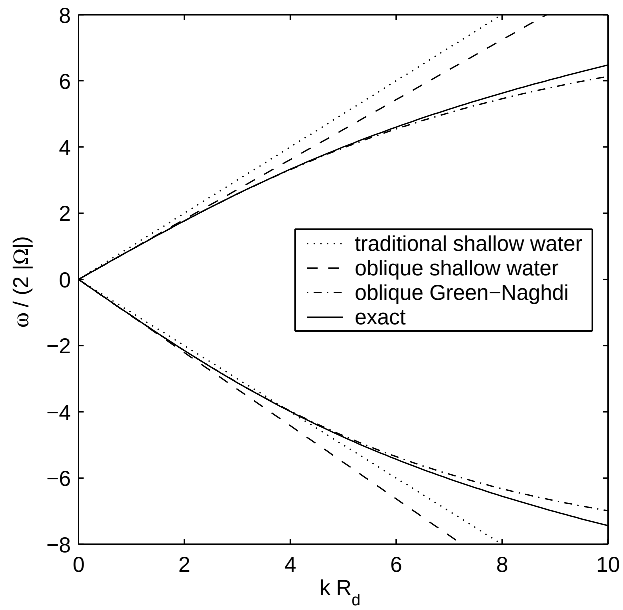

FIG. 2: Frequencies of east/westward propagating waves at the equator for aspect ratio $\delta = 0.2$.

We suppose these axes to be tangent to a sphere at the fixed latitude ϕ_0, so that $\Omega_x = 0$, $\Omega_y = \Omega\cos\phi_0$, and $\Omega_z = \Omega\sin\phi_0$. We ignore the variation of the true latitude ϕ with the y coordinate.

The dispersion relation for linear plane inertia-gravity waves, of the form $h(x,t) = H + h'\exp(i(kx + ly - \omega t))$, in the obliquely rotating shallow water equations may be written as

$$\frac{\omega}{2\Omega} = -\frac{\delta}{2}K\cos\phi_0 \pm \sqrt{\sin^2\phi_0 + (K^2+L^2)\left(1+\frac{1}{4}\delta^2\cos^2\phi_0\right)}, \tag{80}$$

where $K = kR_\mathrm{d}$ and $L = lR_\mathrm{d}$ are dimensionless wavenumbers based on the deformation radius $R_\mathrm{d} = c/(2\Omega)$ defined in (4). The traditional shallow water dispersion relation is modified by the $\delta\cos\phi_0$ terms arising from the horizontal component $\Omega_y = \Omega\cos\phi_0$ of the rotation vector. Defining $|\mathbf{K}|^2 = K^2+L^2$, the Green–Naghdi version of the inertia-gravity wave dispersion relation is

$$\frac{\omega}{2\Omega}\left(1+\frac{1}{3}\delta^2|\mathbf{K}|^2\right) = -\frac{\delta}{2}K\cos\phi_0 \pm \sqrt{(\sin^2\phi_0 + |\mathbf{K}|^2)\left(1+\frac{1}{3}\delta^2|\mathbf{K}|^2\right) + \frac{1}{4}\delta^2\cos^2\phi_0|\mathbf{K}|^2\left(1+\frac{1}{3}\delta^2L^2\right)}. \tag{81}$$

Numerical experiments[48] suggest that the Green–Naghdi version might be valid even when $kH = K\delta \sim 1/3$, for which the $\delta\cos\phi_0$ terms in (81) are definitely significant.

Figures 2 to 5 show the dispersion relations for inertia-gravity waves under the traditional shallow water equations, the oblique shallow water equations, the oblique Green–Naghdi equations, and finally the exact finite depth dispersion relation obtained in Appendix B. The four figures show east/westward and northeast/southwestward propagating waves at the equator, and at latitude $45°$. All four figures take $\delta = 1/5$, which is somewhat larger than the oceanic range $0.02 \lesssim \delta \lesssim 0.14$ estimated previously, but helps to show the effects of oblique rotation in these figures. This value is not unreasonable for the solar tachocline, where the deformation radius may be as small as four layer depths.[49,50]

As in the traditional shallow water equations, there is also a third type of mode with $\omega = 0$, corresponding to steady geostrophic flow. In these modes the velocity perturbations u' and v' are nondivergent, and related to the height perturbation h' by

$$\frac{u'}{\sqrt{gH}} = \frac{-iL}{2\sin\phi_0 - i\delta L\cos\phi_0}\frac{h'}{H}, \quad \frac{v'}{\sqrt{gH}} = \frac{iK}{2\sin\phi_0 - i\delta L\cos\phi_0}\frac{h'}{H}. \tag{82}$$

Again, the $\delta L\cos\phi_0$ terms represent modifications to the traditional shallow water equations due to the horizontal component of Ω. They cause a phase shift between the velocity and height perturbations compared with the traditional shallow water equations, as well as a reduction in the relative amplitude of the velocity perturbations. In particular, the streamfunction for the geostrophic flow is no longer simply proportional to the local height, due to the wavenumber L dependence of the denominators in (82)

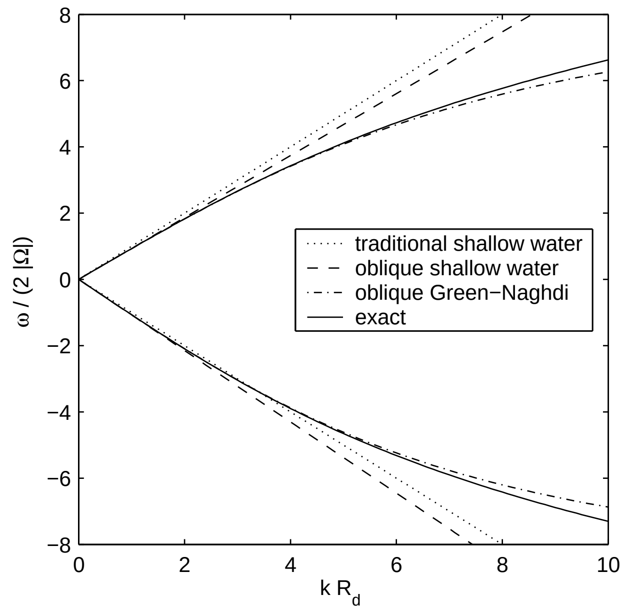

FIG. 3: Frequencies of northeast/southwestward propagating waves at the equator for aspect ratio $\delta = 0.2$.

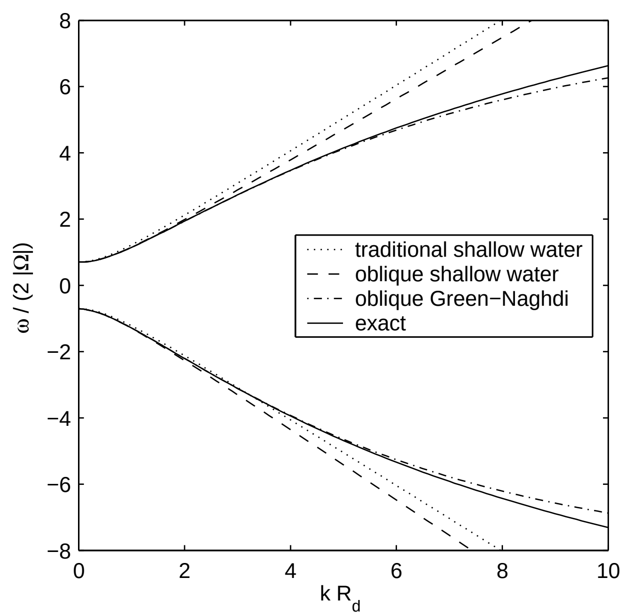

FIG. 4: Frequencies of east/westward propagating waves at latitude 45° for aspect ratio $\delta = 0.2$.

XII. TRAPPED WAVES ON AN EQUATORIAL β-PLANE

The analysis in the previous section applies precisely on the equator of a spherical planet. However, even baroclinic ocean waves will typically extend far enough in latitude to be affected by the variation with latitude of the the vertical component $\Omega \sin \phi$ of the rotation vector. This variation may be included in a Cartesian model using an approximation called the equatorial β-plane.[51,52] Our treatment follows chapter 11 of Gill.[29]

In the usual GFD axes, with x eastward, y northward, and z radially outwards, the rotation vector $\mathbf{\Omega}$ has components $\Omega_x = 0$, $\Omega_y = \Omega = |\mathbf{\Omega}_{\mathrm{Earth}}|$, while $\Omega_z = \frac{1}{2}\beta y$ is proportional to the latitude y. This approximation captures the first order effects of varying latitude, by linearising the earlier relations $\Omega_y = \Omega \cos \phi$ and $\Omega_z = \Omega \sin \phi$ for small ϕ. It may be usefully applied within 30° of the equator.[29] A horizontally varying $\mathbf{\Omega}$ was explicitly permitted in our earlier derivation by averaging. Alternatively, the equatorial β-plane equations may be derived from the variational principle in Sec. V using the vector potential $\mathbf{R} = (2\Omega z - \frac{1}{2}\beta y^2, 0, 0)$. This vector potential has no x dependence, so applying Noether's theorem to the two dimensional vertically-

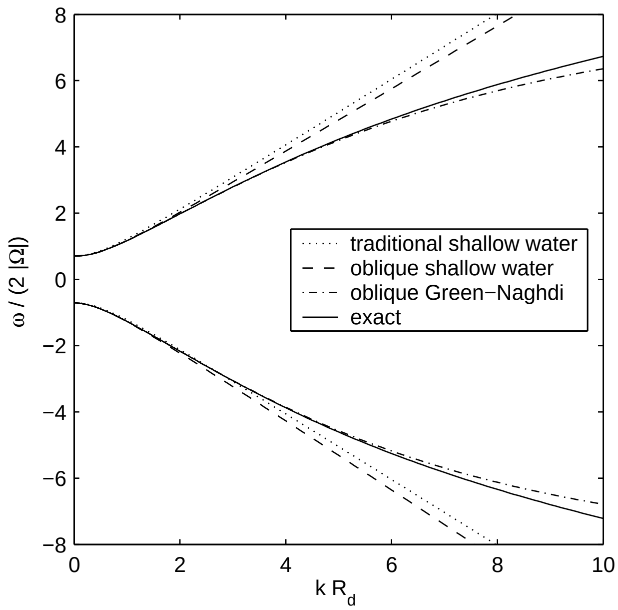

FIG. 5: Frequencies of northeast/southwestward propagating waves at latitude 45° for aspect ratio $\delta = 0.2$.

averaged Lagrangian gives the conserved zonal momentum

$$m_x = h\left(u - \frac{1}{2}\beta y^2 + \Omega(h + 2B)\right). \tag{83}$$

The last term in this expression modifies the usual zonal momentum[26] for the traditional shallow water equations on an equatorial β-plane. It represents the contribution due to variations in the perpendicular distance between the top of the fluid layer and the rotation axis.

Linearising the obliquely rotating shallow water equations, with $h(x, y, t) = H + h'(x, y, t)$ and so forth as before, we obtain

$$\begin{aligned} u'_t - \beta y v' + g h'_x - 2H\Omega\, u'_x - H\Omega\, v'_y &= 0, &(84a)\\ v'_t + \beta y u' + g h'_y \ - H\Omega\, u'_y &= 0, &(84b)\\ h'_t + H(u'_x + v'_y) &= 0. &(84c)\end{aligned}$$

The terms involving $H\Omega$ arise from the horizontal part of the rotation vector, and are not present in the traditional shallow water equations.

Motivated by the dispersion relations in Gill,[29] we nondimensionalise using the gravity wave speed $c = \sqrt{gH}$, and the equatorial deformation radius $R_{\mathrm{ed}} = \sqrt{c/2\beta}$, to obtain the system

$$\begin{aligned} \tilde{u}_t - \tfrac{1}{2} y\tilde{v} + \tilde{h}_x - \delta\left(\tilde{u}_x + \frac{1}{2}\tilde{v}_y\right) &= 0, &(85a)\\ \tilde{v}_t + \tfrac{1}{2} y\tilde{u} + \tilde{h}_y - \frac{1}{2}\delta\tilde{u}_y &= 0, &(85b)\\ \tilde{h}_t + \tilde{u}_x + \tilde{v}_y &= 0, &(85c)\end{aligned}$$

for the dimensionless perturbations $\tilde{h}$, $\tilde{u}$, and $\tilde{v}$. The remaining parameter $\delta = 2\Omega H/c$ is the reduced Lamb parameter defined previously in (3). Note that δ is *not* the aspect ratio based on the equatorial deformation radius R_{ed}, which is the geometrical mean of the earlier deformation radius R_{d} and the planetary radius.

We seek waves that are harmonic in longitude and time, of the form $\tilde{h}(x, y, t) = \hat{h}(y)\exp(i(kx - \omega t))$ etc. Equations (85) may then be combined into a single ordinary differential equation for $\hat{v}(y)$,

$$\frac{d^2\hat{v}}{dy^2} = (Ay^2 - B)\hat{v}, \quad A = \frac{1}{4 + \delta^2}, \quad B = \frac{\omega^2 - k^2 - (1/2)k/\omega + \delta(k\omega + 1/4)}{1 + \delta^2/4}, \tag{86}$$

the same ordinary differential equation that governs a quantum harmonic oscillator. Solutions that decay as $y \to \pm\infty$ are of the form

$$\hat{v} = H_n(\xi)\exp(-\xi^2/2), \quad \xi = yA^{1/4}, \tag{87}$$

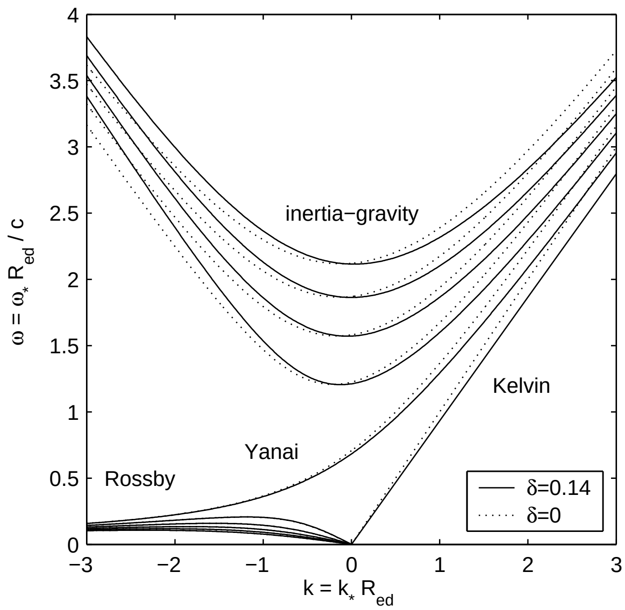

FIG. 6: Dispersion relation for trapped waves on an equatorial β-plane, showing the inertia-gravity, Rossby, Yanai, and Kelvin waves.

where $H_n(\xi)$ is the Hermite polynomial of degree n. These solutions represent trapped waves localised within a few deformation radii of the equator ($y = 0$). The dispersion relation is $A^{-1/2}B = 2n + 1$ for $n = 0, 1, 2, \ldots$, which becomes a cubic equation for ω,

$$\omega^2 - k^2 - \frac{k}{2\omega} + \delta(k\omega + 1/4) = \sqrt{1 + \delta^2/4}\left(n + \frac{1}{2}\right), \text{ for } n = 0, 1, 2, \ldots \tag{88}$$

The only deviation from the standard treatment[29] arises from the ϵ terms in A and B.

Figure 6 shows the dispersion relation (88) for the first few Rossby and inertia-gravity waves, $n = 1, 2, 3, 4$, for both the traditional shallow water equations ($\delta = 0$) and the obliquely rotating shallow water equations in a regime relevant to slow baroclinic ocean waves ($\delta = 0.14$). The east/west asymmetry in the inertia-gravity waves caused by the horizontal component of the rotation vector is visible for this physically plausible value $\delta = 0.14$. By contrast, the Rossby wave branch is hardly affected, as shown by the enlarged view in figure 7, even for the much larger value $\delta = 0.6$ of the reduced Lamb parameter. This agrees with Kasahara's[7] results for normal modes in a stratified atmosphere: that "fast" or short wavelength inertia-gravity waves are more sensitive to the horizontal component of the rotation vector than longer and slower waves.

Figure 6 also shows two special cases. The Yanai wave, corresponding to $n = 0$ in the dispersion relation, joins the Rossby and inertia-gravity wave branches. It resembles a Rossby wave for $k < 0$, and an inertia-gravity wave for $k > 0$. The other special case is the equatorial Kelvin wave, distinguished by having no meridional velocity ($\tilde{v} = 0$). Its frequency is given by $\omega = (\sqrt{1 + \delta^2/4} - \delta/2)k$, which corresponds to $n = -1$ in the dispersion relation (88). The Kelvin wave thus remains nondispersive, as under the traditional approximation, but its wavespeed is shifted by $O(\delta)$ away from the non-rotating gravity wave speed. The geostrophic balance that usually holds in the zonal (x) direction is also modified by the $O(\delta)$ term in (85c). Finally, the Kelvin waves' meridional (y) structure is given by

$$\tilde{h}(y) = \exp\left[-\frac{y^2}{4}\left(\frac{\sqrt{1 + \delta^2/4} - \delta/2}{1 - (\delta/2)\sqrt{1 + \delta^2/4} + \delta^2/4}\right)\right]. \tag{89}$$

This differs slightly from the $\exp[-\frac{1}{4}y^2(1 + \delta^2/4)^{-1/2}]$ structure of the other waves given by (87), although the factors multiplying y^2 agree when expanded to $O(\delta^2)$. This slight difference in meridional structure may affect weakly nonlinear interactions between the Kelvin wave and other waves.

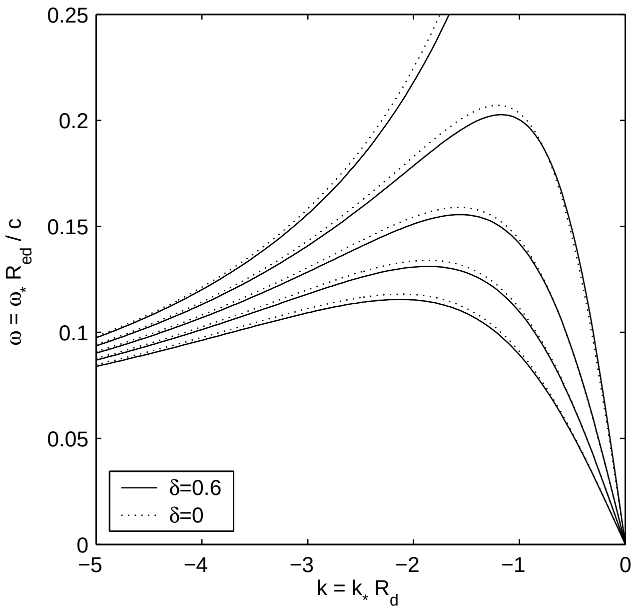

FIG. 7: Enlarged view of the Rossby wave branch in figure 6. The uppermost curve is the Yanai wave.

XIII. CONCLUSION

We have derived an extended set of shallow water equations that describe a thin inviscid fluid layer above fixed topography in a frame rotating about an arbitrary axis. These equations have been derived from a variational principle, as well as from averaging the three dimensional Euler equations, and so share the energy, momentum, and potential vorticity conservation properties of the traditional shallow water equations with a vertical rotation axis. In particular, we have obtained a second topographic term that corrects the equations given previously by Bazdenkov *et al.*[20] to restore the expected conservation properties in the presence of bottom topography.

Our two derivations explicitly integrate out the third dimension by assuming predominantly columnar motion. We have integrated the three dimensional equations of motion directly, and also integrated the three dimensional Lagrangian in Hamilton's principle. The derivation from a Lagrangian introduces a vector potential $\mathbf{R}$ for the rotation vector, which is only determined up to the gradient of an arbitrary scalar gauge. While is is straightforward to show gauge-invariance for three dimensional fluids that either extend to infinity or terminate at rigid boundaries, we have also shown gauge-invariance for flows with free surfaces. Changing the gauge then changes the Lagrangian by an integral over the free surface. We have shown that this integral is an exact time derivative, and thus makes no contribution to the action.

The derivation from a Lagrangian motivates a choice of gauge for which $\mathbf{R}$ has no vertical component. Using this gauge, the obliquely rotating shallow water equations may be formulated as a noncanonical Hamiltonian system in Eulerian variables using the same Hamiltonian and Lie–Poisson bracket as the nonrotating and vertically rotating (traditional) shallow water equations, but with a modified momentum and potential vorticity. This is the same relation that holds between the noncanonical Hamiltonian formulations of the quasi-hydrostatic equations[16] and the hydrostatic primitive equations.[17] They share a Hamiltonian and Poisson bracket, but require different definitions of the momentum and potential vorticity. Further study of our obliquely rotating shallow water equations may help to illuminate other properties of "deep atmosphere" equations like the quasi-hydrostatic equations or the regional model by Tanguay *et al.*[24] in comparison with the usual "shallow atmosphere" primitive equations.

These obliquely rotating equations should also be useful for studying rotating flow over topography, such as rotating hydraulic control problems. Numerical experiments with nonrotating flows by Nadiga *et al.*[48] compared the one dimensional Green–Naghdi equations with a fully two dimensional solution of the Euler equations with a free surface, and found good agreement for obstacle height to width ratios as large as $1/3$. In a geophysical context of a thin layer on a spherical planet, inclusion of the non-hydrostatic pressure gradient due to vertical acceleration logically requires inclusion of the formally larger contribution from the horizontal part of the rotation vector as well. Our obliquely rotating Green–Naghdi equations capture the first corrections to columnar motion from both effects, and linear stability analyses (both our section XI and Kasahara's treatment[7] of the three-dimensional stratified equations) suggest that both effects may have comparable magnitudes for realistic layer depths, even though one is formally asymptotically smaller than the other.

In fact, the effects arising from non-traditional rotation are all quite subtle, and a full investigation will require numerical experiments analogous to those performed by Polvani *et al.*[30] for the traditional shallow water equations. Since the structure of the two equation sets is very similar, it should be straightforward to modify existing numerical algorithms for the traditional shallow water equations. The Hamiltonian structure and potential vorticity conservation properties are the same, and could be exploited by particle methods, while hyperbolic approaches based primarily on the conservation form of the shallow water equations could be applied to the form (62). The latter form requires the same topographic forcing term as the usual shallow water equations, and just replaces Ω_z by $\Omega_z - \boldsymbol{\Omega}\cdot\nabla B$ in the Coriolis force.

The traditional shallow water equations are just the starting point for families of more complicated models using multiple layers, or including additional effects like horizontal temperature gradients or magnetic fields. Many oceanic phenomena outside the tropics may be captured by two layer shallow water models.[11] Within the tropics thermodynamic effects like solar heating or fresh water forcing become important, but these effects may be included by allowing the fluid density in each layer to vary horizontally.[41,53–56] Since the rotation vector is nearly horizontal in the tropics, we might expect the effects of oblique rotation to be particularly significant for these models. Moreover, thermodynamic forcing has been identified previously as mechanism that may circumvent the usual argument based on the inequality $|\boldsymbol{\Omega}| \ll N$ that supports the use of the traditional approximation in a stratified fluid with buoyancy or Brunt–Väisälä frequency N. Several authors have used the inertialess limit of the traditional shallow water equations to study transport of fluid across the equator.[57–59] They postulated an Ekman friction to balance the along-stream component of the pressure gradient. The effects of non-traditional rotation included in our equations offer an alternative to Ekman friction that is at least comparable in magnitude. Shallow water magnetohydrodynamics (SWMHD) is another extension of the shallow water equations designed to model the solar tachocline.[50,60] The relevant aspect ratio may not be particularly small, especially when based on the deformation radius, so the effects of oblique rotation should be investigated.

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APPENDIX A: PARTICLE RELABELING SYMMETRY

In this appendix we extend the usual particle relabeling symmetry arguments to derive a general expression for the conserved potential vorticity in terms of the canonical momentum obtained from a Lagrangian. These ideas have a long history[11–13,37,61] as summarised by Padhye and Morrison[62] and Müller[28]. Our approach and notation follows section 7.2 of Salmon[11], and applies to all Lagrangians in which the particle labels $\mathbf{a} = (a, b)$ only appear through the height field h defined by

$$h(x, y, t) = \frac{\partial(a, b)}{\partial(x, y)}. \tag{A1}$$

According to Hamilton's principle, the evolution equations are such as to make the action stationary. In particular, the action must be stationary under infinitesimal relabelings of the form

$$\mathbf{a} \to \mathbf{a}' = \mathbf{a} + \delta\mathbf{a}, \tag{A2}$$

provided $\delta\mathbf{a}$ vanishes at the endpoints of the integration in τ. We consider relabelings that leave the height h unchanged. Using the chain rule for Jacobians,

$$\frac{\partial(a', b')}{\partial(a, b)} = \frac{\partial(a', b')}{\partial(x, y)}\frac{\partial(x, y)}{\partial(a, b)} = 1, \tag{A3}$$

we find that

$$\frac{\partial \delta a}{\partial a} + \frac{\partial \delta b}{\partial b} = 0, \tag{A4}$$

on neglecting terms of $O(\delta\mathbf{a}^2)$. Height-preserving infinitesimal relabelings are thus of the form

$$\delta\mathbf{a} = \nabla^{\perp}\delta\psi = (-\partial_b, \partial_a)\delta\psi, \tag{A5}$$

for some scalar $\delta\psi(a, b, \tau)$ analogous to a streamfunction.

Putting $\mathbf{a} = \mathbf{a}' - \delta\mathbf{a}$ in the change of variables formula

$$\left.\frac{\partial\mathbf{x}}{\partial\tau}\right|_{\mathbf{a}'} = \left.\frac{\partial\mathbf{x}}{\partial\tau}\right|_{\mathbf{a}} + \frac{\partial\mathbf{x}}{\partial a}\left.\frac{\partial a}{\partial\tau}\right|_{\mathbf{a}'} + \frac{\partial\mathbf{x}}{\partial b}\left.\frac{\partial b}{\partial\tau}\right|_{\mathbf{a}'}, \tag{A6}$$

the variation in $\dot{\mathbf{x}} = \frac{\partial\mathbf{x}}{\partial\tau}$ due to relabeling is

$$\delta\dot{\mathbf{x}} = \left.\frac{\partial\mathbf{x}}{\partial\tau}\right|_{\mathbf{a}'} - \left.\frac{\partial\mathbf{x}}{\partial\tau}\right|_{\mathbf{a}} = -\left(\frac{\partial\mathbf{x}}{\partial a}\frac{\partial\delta a}{\partial\tau} + \frac{\partial\mathbf{x}}{\partial b}\frac{\partial\delta b}{\partial\tau}\right). \tag{A7}$$

The corresponding variation in the Lagrangian is purely kinetic, since h is unchanged,

$$\delta\mathcal{L} = \int dadb\, p_i \delta\dot{x}_i = -\int dadb\, p_i \frac{\partial x_i}{\partial a_j}\frac{\partial \delta a_j}{\partial \tau}, \text{ where } p_i = \frac{\delta \mathcal{L}}{\delta \dot{x}_i}. \tag{A8}$$

The variational derivative defining p_i is taken using the natural mass-weighted inner product for integrals with respect to $dadb$ instead of $dxdy$. Integrating by parts on τ and using (A5), we obtain

$$\delta\mathcal{L} = -\int dadb\, \frac{\partial}{\partial\tau}\left(p_i \frac{\partial x_i}{\partial a}\right)\frac{\partial\delta\psi}{\partial b} - \frac{\partial}{\partial\tau}\left(p_i\frac{\partial x_i}{\partial b}\right)\frac{\partial \delta\psi}{\partial a} = -\int dadb\, \frac{\partial q}{\partial\tau}\delta\psi. \tag{A9}$$

Hamilton's principle, $\delta\mathcal{L} = 0$ under relabeling, thus implies material conservation, $\partial q/\partial\tau = 0$, of the quantity q given by

$$q = \frac{\partial}{\partial a}\left(p_i \frac{\partial x_i}{\partial b}\right) - \frac{\partial}{\partial b}\left(p_i\frac{\partial x_i}{\partial a}\right). \tag{A10}$$

This expression may be rewritten as

$$q = \frac{\partial p_i}{\partial a}\frac{\partial x_i}{\partial b} - \frac{\partial p_i}{\partial b}\frac{\partial x_i}{\partial a} = \frac{\partial(p_y, y)}{\partial(a,b)} + \frac{\partial(p_x, x)}{\partial(a,b)}. \tag{A11}$$

Using the chain rule for Jacobians, we finally obtain

$$q = \frac{\partial(x,y)}{\partial(a,b)}\left(\frac{\partial(p_y,y)}{\partial(x,y)} + \frac{\partial(p_x,x)}{\partial(x,y)}\right) = \frac{1}{h}\left(\frac{\partial p_y}{\partial x} - \frac{\partial p_x}{\partial y}\right). \tag{A12}$$

This is a general expression for the conserved potential vorticity in terms of the Eulerian spatial derivatives of the canonical momenta p_x and p_y (defined as variational derivatives of the Lagrangian using the mass-weighted inner product).

APPENDIX B: DISPERSION RELATION FOR A LAYER OF FINITE DEPTH

Most treatments of linear waves on water of finite depth assume that the flow remains irrotational, because the only source of vorticity is a viscous boundary layer at a rigid lower boundary. However, initially irrotational flow in a rotating frame generally does not remain irrotational because the Coriolis term provides another source of vorticity, $\nabla\times(2\mathbf{\Omega}\times\mathbf{u}) = -4\mathbf{\Omega}\cdot\nabla\mathbf{u} \neq 0$. We therefore work with the velocity vector $\mathbf{u} = (u, v, w)$ and modified pressure $p^\star = p + gz$ instead of a velocity potential.

The algebra becomes far simpler in axes aligned with the wavevector. We therefore seek solutions of the form $u(x, y, z, t) = U(z)\exp(i(\kappa x - \omega t))$ with no y dependence, and similarly for the other variables v, w, and $p^\star$. We must then allow arbitrary orientations of the rotation vector $\mathbf{\Omega}$ to avoid loss of generality. The relevant linearized form of (6) is thus

$$\partial_t \mathbf{u} + 2\mathbf{\Omega}\times\mathbf{u} + \nabla p^\star = 0, \quad \nabla\cdot\mathbf{u} = 0. \tag{B1}$$

The horizontal momentum and continuity equations may be solved for

$$\begin{aligned} U(z) &= \frac{i}{\kappa}\frac{dW}{dz}, \quad V(z) = \frac{2}{\omega}\left(i\Omega_x W(z) + \frac{\Omega_z}{\kappa}\frac{dW}{dz}\right), \\ P(z) &= \frac{i}{\kappa^2}\left(\omega - \frac{4\Omega^2}{\omega}\right)\frac{dW}{dz} + \frac{2}{\kappa}\left(i\Omega_y + \frac{2}{\omega}\Omega_x\Omega_z\right)W(z), \end{aligned} \tag{B2}$$

leaving the vertical momentum equation to determine the vertical structure,

$$W(z) = \exp\left(i\kappa z\frac{4\Omega_x\Omega_z}{\omega^2 - 4\Omega_z^2}\right)\sinh\left(\kappa z\omega\,\frac{\sqrt{\omega^2 - 4(\Omega_x^2 + \Omega_z^2)}}{\omega^2 - 4\Omega_z^2}\right). \tag{B3}$$

An arbitrary constant has been chosen to satisfy the lower boundary condition that $w = 0$ on $z = 0$. The resulting velocity field is rotational, with $\nabla\times\mathbf{u} \neq 0$ as expected.

The linearized boundary conditions at the free surface are $h_t = w$ and $p^\star = gh$ (corresponding to the true pressure $p = 0$), and they may be applied at the unperturbed free surface $z = H$ to sufficient accuracy. For time harmonic waves they simplify to $-i\omega p^\star = gw$ on $z = H$. The resulting eigenvalue problem for $W(z)$ gives the dispersion relation in implicit form as

$$\omega\sqrt{\omega^2 - 4\left(\Omega_x^2 + \Omega_z^2\right)} = (g\kappa - 2\,\omega\,\Omega_y)\tanh\left(\kappa H\omega\,\frac{\sqrt{\omega^2 - 4\left(\Omega_x^2 + \Omega_z^2\right)}}{\omega^2 - 4\,\Omega_z^2}\right). \tag{B4}$$

The easiest way to transform this dispersion relation back into the standard GFD axes is to rewrite it in terms of the vectors $\mathbf{\Omega}_2 = (\Omega_x, \Omega_y)$ and $\mathbf{k} = (\kappa, 0)$. Multiplying (B4) through by $|\mathbf{k}|$, we obtain

$$\omega\sqrt{(\omega^2 - 4\Omega_z^2)|\mathbf{k}|^2 - 4(\mathbf{\Omega}_2\cdot\mathbf{k})^2} = \big(g|\mathbf{k}|^2 - 2\,\omega\,|\mathbf{\Omega}_2{\times}\mathbf{k}|\big)\tanh\left(H\omega\,\frac{\sqrt{(\omega^2 - 4\Omega_z^2)|\mathbf{k}|^2 - 4(\mathbf{\Omega}_2\cdot\mathbf{k})^2}}{\omega^2 - 4\,\Omega_z^2}\right), \tag{B5}$$

in terms of the invariant quantities $|\mathbf{\Omega}_2|^2$, $|\mathbf{k}|^2$, $\mathbf{k}\cdot\mathbf{\Omega}_2$, and $|\mathbf{k}{\times}\mathbf{\Omega}_2|$. This becomes

$$\begin{aligned} &\omega\sqrt{(\omega^2 - 4\Omega^2)(k^2+l^2) + 4\Omega^2 k^2\cos^2\phi} = \\ &\qquad \big(g(k^2+l^2) - 2\omega\Omega k\cos\phi\big)\tanh\left(H\omega\,\frac{\sqrt{(\omega^2 - 4\Omega^2)(k^2+l^2) + 4\Omega^2 k^2\cos^2\phi}}{\omega^2 - 4\,\Omega^2\sin^2\phi}\right), \end{aligned} \tag{B6}$$

in the standard GFD axes with $\Omega_x = 0$, $\Omega_y = \Omega\cos\phi$, $\Omega_z = \Omega\sin\phi$, and horizontal wavevector $\mathbf{k} = (k, l)$. This expression further simplifies in the dimensionless variables used in Section XI with $K = kR_\mathrm{d}$, $L = lR_\mathrm{d}$, where R_d is the Rossby deformation radius, $\tilde\omega = \omega/(2\Omega)$, and $\delta = H/R_\mathrm{d}$,

$$\tilde\omega\delta\sqrt{(\tilde\omega^2 - 1)|\mathbf{K}|^2 + K^2\cos^2\phi} = \big(|\mathbf{K}|^2 - K\tilde\omega\delta\cos\phi\big)\tanh\left(\frac{\delta\tilde\omega\sqrt{(\tilde\omega^2-1)|\mathbf{K}|^2 + K^2\cos^2\phi}}{\tilde\omega^2 - \sin^2\phi}\right). \tag{B7}$$

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