The quasi-geostrophic theory of the thermal shallow water equations

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The thermal shallow water equations provide a depth-averaged description of motions in a fluid layer that permits horizontal variations in material properties. They typically arise through an equivalent barotropic approximation of a two-layer system, with a spatially varying density contrast due to an evolving temperature field in the active layer. We formalise a previous derivation of the quasi-geostrophic (QG) theory of these equations, by performing a direct asymptotic expansion for small Rossby number. We then present a second derivation as the small Rossby number limit of a balanced model that projects out high-frequency dynamics due to inertia-gravity waves. This later derivation has wider validity, not being restricted to mid-latitude β-planes. We also derive their local energy conservation equation from the QG limit of a thermal shallow water pseudo-energy conservation equation. This derivation involves the ageostrophic correction to the leading order geostrophic velocity that is eliminated in the usual derivation of a closed evolution equation for the QG potential vorticity. Finally, we derive the non-canonical Hamiltonian structure of the thermal QG equations from a decomposition in Rossby number of a pseudo-energy and Poisson bracket for the thermal shallow water equations.

Key words: shallow water flows, quasi-geostrophic flows, Hamiltonian theory

1. Introduction

Shallow water equations are widely used in geophysical and planetary fluid dynamics as conceptual models for the behaviour of rotating, stratified fluids. They offer the simplest models for interactions between waves and vortical motions, and may be derived as amplitude equations for the vertical normal modes in continuously stratified fluids. While standard shallow water theory applies to one or more layers of homogeneous fluid, thermal shallow water theory permits horizontal variations of the thermodynamic properties of the fluid within each layer. These models were introduced by Lavoie (1972) to describe atmospheric mixed layers over frozen lakes, and later adopted for the tropical oceans (Schopf & Cane 1983; McCreary & Yu 1992), the coastal ocean (McCreary & Kundu 1988; McCreary *et al.* 1991; Røed & Shi 1999; Carbonel & Galeao 2007), and atmospheric currents (Anderson 1984). They are commonly employed for the uppermost layer of predictive numerical ocean models using Lagrangian discretisations in the vertical, such as the Miami Isopycnic Coordinate Ocean Model (MICOM) (Bleck *et al.* 1992), and the Generalized Ocean Layered Model (GOLD) (Adcroft & Hallberg 2006). The thermal shallow water equations also resemble shallow water models of moist convection (Zeitlin 2007; Bouchut *et al.* 2009; Lambaerts *et al.* 2011).

Vortical motions typically evolve on time scales much longer than those characteristic of inertia-gravity waves. Quasi-geostrophic (QG) theory, first introduced by Charney (1948), is the simplest example of an intermediate or balanced model that exploits this separation of time scales to produce a simplified model, while still retaining the time evolution omitted from purely geostrophic flow. QG theory may be applied to both shallow water and three-dimensional continuously stratified equations. It filters out inertia-gravity waves, replacing them with an elliptic relation between a materially conserved scalar, the potential vorticity, and the geostrophic streamfunction. In this respect it closely resembles the derivation of incompressible fluid equations from compressible fluid equations, where sound waves are filtered out in favour of an elliptic equation for the pressure.

QG equations may be derived by posing expansions of the velocity, layer depth, and other variables in a small Rossby number, the dimensionless parameter that characterises the significance of inertia relative to the Coriolis force. It is also necessary to assume small (order Rossby number) displacements of the layer depth, in shallow water theory, or of the isopycnal surfaces, in the three-dimensional Boussinesq equations, from their uniform equilibrium positions for a state of rest (Pedlosky 1987; Salmon 1998; White 2002; Majda 2003). An alternative derivation of QG theory projects the underlying equations onto a basis of the linear wave modes for this unperturbed rest state, and retains nonlinear interactions only between the vortical modes (Leith 1980; Salmon 1998; Majda 2003; Remmel & Smith 2009). This gives an explicit origin to the filtering out of inertia-gravity waves. The usage of the linear wave modes of the unperturbed state as coordinates for the projection is the origin of the restriction of QG theory to small perturbations of the isopycnals, which is logically distinct from the small Rossby number approximation (Leith 1980; Salmon 1988*b*). This projection approach may be used to develop QG theory on a sphere (Kuo 1959; Charney & Stern 1962;

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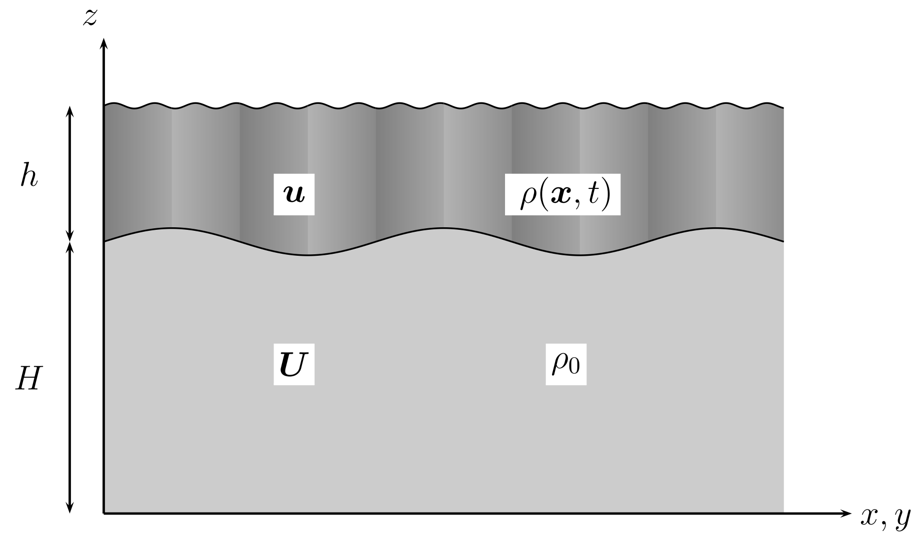

FIGURE 1. A two-layer system with variable density $\rho(\boldsymbol{x},t) < \rho_0$ in the upper layer. The equivalent barotropic approximation holds when $h \ll H$, $|\boldsymbol{U}| \ll |\boldsymbol{u}|$, $\rho_0 - \rho(\boldsymbol{x},t) \ll \rho_0$, and allows the lower layer to be treated as quiescent.

Schubert *et al.* 2009; Verkley 2009), as opposed to the usual restriction to mid-latitude β-planes. Charney & Flierl (1981), Hoskins *et al.* (1985) and White (2002) give broader surveys and reviews of QG theory, while Majda (2003) gives an introduction to the rigorous mathematical theory of the QG limit.

QG theory is attractive for numerical simulation, because the time step is limited only by the maximum velocity of the fluid, rather than by the typically much larger velocity of inertia-gravity waves. The geostrophic streamfunction may be reconstructed from the instantaneous potential vorticity distribution via the inversion of a modified Helmholtz operator, which is diagonalised by a Fourier or spherical harmonic transform. Numerical simulations of multi-layer QG equations have evolved from pioneering experiments in numerical weather prediction (e.g. Charney & Phillips 1953) to large-scale simulations of rotating stratified turbulence (e.g. Reinaud *et al.* 2003). The presence of only quadratic nonlinearities in QG theory also makes them amenable to the techniques of statistical mechanics (e.g. Salmon 1998; Majda & Wang 2006), notably the use of cumulant expansions to study zonally symmetric mean flows in geostrophic turbulence on spheres and β-planes (Farrell & Ioannou 1993, 2003, 2007; Marston *et al.* 2008; Tobias *et al.* 2011; Srinivasan & Young 2012).

Ripa (1996*b*) derived a QG form of the thermal shallow water equations, which coincide with Young's (1994) subinertial mixed layer model in the limit of rapid vertical mixing. In this paper, we provide a formal derivation of these thermal QG equations from an asymptotic expansion for small Rossby number. We then present a second derivation via a nonlinear balanced model that corresponds to a projection of the dynamics onto the slow normal modes for perturbations around a uniform rest state in the the thermal shallow water equations. The thermal QG equations are then the small Rossby number limit of this balanced model. We derive their local energy density and energy flux from the QG limit of a pseudo-energy density and flux for the thermal shallow water equations. This derivation involves the small ageostrophic correction to the leading order geostrophic velocity, the correction that is eliminated in the derivation of the closed QG evolution equations. Finally, we derive the non-canonical Hamiltonian structure of the thermal QG equations (as stated by Ripa 1996*b*) from a small Rossby number decomposition of a pseudo-energy and Poisson bracket for the thermal shallow water equations.

2. Thermal shallow water equations

Shallow water theory describes one or more layers of inviscid fluid. Within each layer the horizontal velocity is supposed to be depth-independent, so the fluid moves in columns. One may relax the homogeneity assumption of standard shallow water theory, and allow the fluid properties within each layer to vary in the horizontal. This scenario arises most naturally in an equivalent barotropic or $1\frac{1}{2}$-layer configuration, as sketched in figure 1. The active upper layer is characterised by a spatially varying, but bounded below, density contrast $\Delta\rho = \rho_0 - \rho(\boldsymbol{x},t)$ from a much deeper, quiescent lower layer with constant density ρ_0. We introduce $\Theta = g\Delta\rho/\rho_0$ to represent this density contrast, in the form commonly called the reduced gravity g' for spatially homogeneous layers. The equivalent barotropic approximation (e.g. Gill 1982) then leads to a single set of shallow water equations for the active upper layer.

The thermal shallow water equations may be written as

$$\frac{\partial h}{\partial t} + \boldsymbol{\nabla}\cdot(h\overline{\boldsymbol{u}}) = 0, \tag{2.1a}$$

$$\frac{\partial \Theta}{\partial t} + (\overline{\boldsymbol{u}}\cdot\boldsymbol{\nabla})\,\Theta = -\kappa\,(h\Theta - H_0\Theta_0)\,, \tag{2.1b}$$

$$\frac{\partial \overline{\boldsymbol{u}}}{\partial t} + (\overline{\boldsymbol{u}}\cdot\boldsymbol{\nabla})\,\overline{\boldsymbol{u}} + f\hat{\boldsymbol{z}}\times\overline{\boldsymbol{u}} = -\boldsymbol{\nabla}\,(\Theta h) + \frac{1}{2}h\boldsymbol{\nabla}\Theta, \tag{2.1c}$$

where h is the depth of the active layer, $\overline{\boldsymbol{u}}$ is the depth-averaged horizontal velocity, and $\boldsymbol{\nabla} = (\partial_x, \partial_y)^{\mathsf{T}}$ is the horizontal gradient operator. These equations are written for a frame rotating with angular velocity $f/2$ about the vertical axis, with $\hat{\boldsymbol{z}}$ being a unit vector in the z-direction. The momentum equation (2.1c) reduces to the standard shallow water momentum equation for a homogeneous layer when the density contrast Θ is constant.

The $-\kappa\,(h\Theta - H_0\Theta_0)$ term on the right hand side of (2.1b) models a Newtonian cooling back to an equilibrium state with $h = H_0$ and $\Theta = \Theta_0$. A similar term was included in the mass conservation equation (2.1a) for the standard shallow water equations by Juckes (1989); Polvani *et al.* (1995); Thuburn & Lagneau (1999) to model radiative cooling in the terrestrial stratosphere, and the resulting transfer of mass between the stratosphere and the troposphere below. The same $-\kappa(h - H_0)$ term in (2.1a) was used by Scott & Polvani (2008) to model the atmospheres of gas giant planets. Making this change while keeping (2.1c) for the velocity unchanged introduces a second source term $-\kappa(h - H_0)\overline{\boldsymbol{u}}$ into the evolution equation for the momentum $h\overline{\boldsymbol{u}}$. Our system (2.1$a$-$c$) incorporates radiative relaxation directly in the temperature equation instead, so that mass and momentum are conserved within the active layer.

2.1. *Derivation*

Shallow water equations such as (2.1) may be derived by rescaling the three-dimensional fluid equations for a small aspect ratio $\delta = H/L \ll 1$ between the vertical length scale H and the horizontal length scale L (e.g. Pedlosky 1987; Salmon 1998; Vallis 2006). The dominant balance in the vertical momentum equation is then the hydrostatic balance $\partial_z p = -g\rho$ that determines the pressure p in terms of the density distribution within each fluid column. We consider the two-layer system sketched in figure 1 and suppose that $\Delta\rho \ll \rho_0$. Deviations in the position of the upper free surface are then much smaller than deviations in the position of the interface between the two layers, so we may approximate the upper free surface by a rigid lid at $z = 0$ that applies a pressure $P(x, y, t)$ (e.g. Vallis 2006). The pressure $p(x, y, z, t)$ in the upper layer is then given by hydrostatic balance as

$$p = P(x, y, t) - g(\rho_0 - \Delta\rho(x, y, t))z. \tag{2.2}$$

We have assumed that the density contrast $\Delta\rho(x, y, t)$ is independent of depth in the upper layer, as sketched in figure 1. Hydrostatic balance, together with continuity of pressure across the interface at $z = -h(x, y, t)$, gives the pressure p_2 in the lower layer as

$$p_2 = P - \rho_0 gz - \Delta\rho gh. \tag{2.3}$$

The equivalent barotropic approximation applies to a lower layer that is deep and quiescent, so we impose a vanishing horizontal pressure gradient $\boldsymbol{\nabla}_{\perp}\, p_2 = \mathbf{0}$. This determines the pressure $P = gh\Delta\rho$ at the rigid lid. The pressure in the upper layer is thus

$$p = \rho_0[-gz + \Theta(z + h)], \tag{2.4}$$

where we have put $\Theta = g\Delta\rho/\rho_0$ as before.

In standard shallow water theory, with spatially uniform Θ, the horizontal pressure gradient $\boldsymbol{\nabla}_{\perp}\, p = \rho_0\Theta\boldsymbol{\nabla} h$ is independent of z. This is compatible with the assumption that the fluid moves predominantly in columns, $\boldsymbol{u}(x, y, z, t) = \boldsymbol{u}_0(x, y, t) + O(\delta^2)$ for the horizontal velocity. This property is lost when Θ varies horizontally, since now $\boldsymbol{\nabla}_{\perp}\, p = \rho_0\,[\Theta\boldsymbol{\nabla} h + (h + z)\boldsymbol{\nabla}\Theta]$ varies with z. However, one may regard the thermal shallow water equations as a projection onto columnar motions that results from integrating the three-dimensional equations in the vertical. Inserting the above expression for $\boldsymbol{\nabla}_{\perp}\, p$, for the case where Θ varies horizontally, into Wu's (1981) formula for the depth-averaged horizontal momentum equation gives

$$\partial_t(h\overline{\boldsymbol{u}}) + \boldsymbol{\nabla}\cdot(h\overline{\boldsymbol{u}\boldsymbol{u}}) + hf\hat{\boldsymbol{z}}\times\overline{\boldsymbol{u}} = -\int_{-h}^{0} \Theta\boldsymbol{\nabla} h + (h + z)\boldsymbol{\nabla}\Theta\,\mathrm{d}z = -\frac{1}{2}\boldsymbol{\nabla}(h^2\Theta), \tag{2.5}$$

after making the Boussinesq approximation, and dividing through by ρ_0. An overbar denotes a vertical average, such as $\overline{u}(x, y, t) = h^{-1}\int_0^h u(x, y, z, t)\,\mathrm{d}z$. The only questionable step in the derivation of the thermal shallow water momentum equation (2.1c) is the use of the closure relation $\overline{\boldsymbol{u}\boldsymbol{u}} = \overline{\boldsymbol{u}}\,\overline{\boldsymbol{u}} + O(\delta^4)$ that holds for near-columnar motions (Su & Gardner 1969; Camassa *et al.* 1996; Stewart & Dellar 2010).

In Appendix A we derive the thermal shallow water equations (2.1) from Hamilton's principle by projecting the three-dimensional Lagrangian onto columnar fluid motions, following Miles & Salmon's (1985) re-derivation of the Green & Naghdi (1976) equations for weakly non-hydrostatic motions, and Dellar & Salmon's (2005) derivation of shallow water equations with the complete Coriolis force. This derivation guarantees that the projected equations preserve the expected conservation laws and Hamiltonian structure of the three-dimensional equations.

2.2. *Justification for vertical averaging*

McCreary & Kundu (1988), Fukamachi *et al.* (1995) and Eldevik (2002) postulated turbulent eddy stresses to balance the z-dependent part of $\boldsymbol{\nabla}_\perp p$, while Young (1994) included an explicit relaxation of the horizontal velocities towards their vertical averages in the starting point for deriving his three-dimensional subinertial model of the oceanic mixed layer. The large mixing limit of Young's (1994) equations coincides with the thermal QG equations. Fukamachi *et al.* (1995) showed that the thermal shallow water equations capture an analogue of the ageostrophic baroclinic instability studied by Stone (1966) for continuously stratified fluids (see also Barth 1994). In suitable dimensionless variables, the two dispersion relations for the growth rate σ may be written as

$$(1+\kappa^2)\sigma^2 - \kappa^2 l\sigma + \alpha\kappa^2 l^2 = 0, \tag{2.6}$$

where k and l are wavenumbers parallel and perpendicular to the background temperature gradient, and $\kappa^2 = k^2 + l^2$. The constant α takes the value $\alpha = 1/3$ for a continuously stratified fluid, and $\alpha = 1/4$ for the thermal shallow water equations. The two dispersion relations are otherwise identical, showing that the thermal shallow water equations offer a single layer model that captures some aspects of baroclinic instability in continuously stratified fluids. The change in α may be attributed to the vertical mixing implicit in the derivation of the thermal shallow water equations. Numerical simulations of the nonlinear development of these instabilities in the thermal shallow water equations also show reasonable agreement with three dimensional simulations, and with observations of small-scale frontal instabilities in the ocean (McCreary *et al.* 1991; Barth 1994; Fukamachi *et al.* 1995; Shi & Roed 1999).

The assumption of large vertical mixing seems reasonable for applications of the thermal shallow water equations to the upper oceanic mixed layer, or to the cloud decks of gas giant planets, as clouds are indicative of vertical mixing through convection. Gierasch *et al.* (2000) report observations of strong moist convection in Jupiter's atmosphere made by the Galileo spacecraft. The combination of Jupiter's observed surface heat flux with the opacity of its atmosphere at depth indicates substantial convective heat transfer takes place through much of its interior (Guillot 2005). In addition, rotation about a vertical axis will encourage columnar motion through the propagation of internal waves parallel to the rotation axis.

The projection approach was also used by Charney (1949) in his equivalent barotropic model for a continuously stratified atmosphere (see also Eliassen & Kleinschmidt 1957; Holton 1992). The horizontal components of the three-dimensional velocity field are assumed to take the separable form

$$\boldsymbol{u}(x,y,z,t) = A(z)\,\boldsymbol{u}_\perp(x,y,t) \tag{2.7}$$

for some prescribed vertical structure function $A(z)$. Evolution equations for $\boldsymbol{u}_\perp$ are derived by vertically averaging the three-dimensional equations. This approximation was subsequently applied to the ocean by Neumann (1960) and Salby (1989). It has been widely used to model the Antarctic Circumpolar Current (e.g. Krupitsky *et al.* 1996; Ivchenko *et al.* 1999; Killworth & Hughes 2002; LaCasce & Isachsen 2010) following Killworth's (1992) observation that the time-averaged output of the Fine Resolution Antarctic Model (FRAM Group 1991) fits the equivalent barotropic form (2.7).

2.3. *Potential vorticity*

The thermal shallow water equations imply the evolution equation

$$\frac{\partial \mathit{\Pi}}{\partial t} + (\overline{\boldsymbol{u}}\cdot\boldsymbol{\nabla})\,\mathit{\Pi} = \frac{1}{2h}\hat{\boldsymbol{z}}\cdot(\boldsymbol{\nabla}h\times\boldsymbol{\nabla}\Theta) \tag{2.8}$$

for the shallow water potential vorticity $\mathit{\Pi} = h^{-1}(f + \overline{v}_x - \overline{u}_y)$. The source term on the right hand side may be derived from the depth average of the baroclinic torque in a continuously stratified fluid (Ripa 1995).

The curl of the three-dimensional momentum equation gives

$$\frac{\mathrm{D}\boldsymbol{\omega}_\mathrm{a}}{\mathrm{D}t} - (\boldsymbol{\omega}_\mathrm{a}\cdot\boldsymbol{\nabla})\,\boldsymbol{u} = -\boldsymbol{\nabla}\times\left(\frac{\boldsymbol{\nabla}p}{\rho}\right) = -\boldsymbol{\nabla}\times\left(\frac{\boldsymbol{\nabla}[\Theta(h+z)-gz]}{1-\Theta/g}\right) = \boldsymbol{S}, \tag{2.9}$$

where $\boldsymbol{u}$ is the three-dimensional velocity field, and $\boldsymbol{\omega}_\mathrm{a} = f\hat{\boldsymbol{z}} + \boldsymbol{\nabla}\times\boldsymbol{u}$ is the absolute vorticity (Vallis 2006). We have assumed that the buoyancy Θ is an advected scalar so $\boldsymbol{\nabla}\cdot\boldsymbol{u} = 0$, and that f is constant. There is no need to divide by ρ to eliminate the $\boldsymbol{\omega}_\mathrm{a}\boldsymbol{\nabla}\cdot\boldsymbol{u}$ term and obtain a material derivative for $\boldsymbol{\omega}_\mathrm{a}/\rho$. However, it is necessary to take spatial derivatives before applying the Boussinesq approximation to $\boldsymbol{\nabla}\times(\rho^{-1}\boldsymbol{\nabla}p)$. The components of the baroclinic torque vector on the right hand side of (2.9) are

$$S_x = \frac{\partial_y\Theta}{1-\Theta/g},\quad S_y = -\frac{\partial_x\Theta}{1-\Theta/g},\quad S_z = \frac{g\Theta}{(g-\Theta)^2}\hat{\boldsymbol{z}}\cdot(\boldsymbol{\nabla}h\times\boldsymbol{\nabla}\Theta)\,. \tag{2.10}$$

The Boussinesq approximation now corresponds to neglecting $O(\Theta/g)$ terms in the denominators. Moreover, the vertical component S_z is $O(\Theta/g)$ smaller than the two horizontal components.

An evolution equation for a potential vorticity follows from taking the inner product of (2.9) with the gradient of an advected scalar. Normally this scalar would be the buoyancy Θ, to form the Ertel potential vorticity, but in our scenario $\boldsymbol{\nabla}\Theta$ has only horizontal components, while the largest component of vorticity is ω_z. Instead, we use the quantity $c = z/h(x,y,t)$ which becomes a Lagrangian label when the fluid is constrained to move in columns (Miles & Salmon 1985; Ripa 1995; Dellar & Salmon 2005, and §A.2 of the Appendix) and compute the vertical average

$$\frac{1}{h}\int_{-h}^{0} \boldsymbol{S}\cdot\boldsymbol{\nabla} c\,\mathrm{d}z = \frac{1}{2h}\hat{\boldsymbol{z}}\cdot(\boldsymbol{\nabla} h\times\boldsymbol{\nabla}\Theta)\,. \tag{2.11}$$

This is the source term that appears on the right hand side of (2.8). The corresponding vertical average of $\boldsymbol{\omega}_{\mathrm{a}}\cdot\boldsymbol{\nabla} c$ gives the standard shallow water potential vorticity Π (Miles & Salmon 1985; Dellar & Salmon 2005) .

Equation (2.8) becomes easier to interpret if we consider the integral form of Ertel potential vorticity conservation, the Kelvin theorem for conservation of circulation around any material curve lying within an isentropic surface. The analogues of isentropic surfaces in our thermal shallow water equations are isotherms (lines of constant Θ) in the horizontal plane, and from (2.8) we obtain an integral conservation law for the total potential vorticity inside a closed isotherm (Dellar 2003),

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{S(\Theta_*)} h\Pi\,\mathrm{d}\boldsymbol{x} = 0, \tag{2.12}$$

where $S(\Theta_*)$ is the material surface bounded by the $\Theta = \Theta_*$ isotherm. This is the thermal shallow water analog of the potential vorticity impermeability property of isentropic surfaces in a stratified fluid (e.g. Vallis 2006). By Stokes' theorem, (2.12) implies conservation of the circulation of $\overline{\boldsymbol{u}} + \boldsymbol{R}$ around the boundary isotherm of each material surface $S(\Theta_*)$. The vector field $\boldsymbol{R}$ satisfying $\boldsymbol{\nabla}_3\times\boldsymbol{R} = f\hat{\boldsymbol{z}}$ is the vector potential for twice the angular velocity vector, so $\hat{\boldsymbol{z}}\cdot\boldsymbol{\nabla}_3\times(\overline{\boldsymbol{u}}+\boldsymbol{R}) = h\Pi$ is the integrand in (2.12). In the Appendix we relate (2.8) and (2.12) to the particle relabelling symmetry, as modified from standard shallow water theory by the existence of an additional Lagrangian scalar Θ. From the rest of this paper we drop the overbar on the depth-averaged horizontal velocity $\overline{\boldsymbol{u}}$ in the thermal shallow water equations.

3. Non-canonical Hamiltonian structure

Most equations used in geophysical fluid dynamics have a Hamiltonian structure, which enables their conservation laws to be related to symmetries via Noether's theorem (Salmon 1998; Shepherd 1990; Salmon 1988*a*; Morrison 1998). Equations that are derived by making approximations that preserve these symmetries will then inherit equivalent conservation laws. The material in this section also provides the key to deriving a local energy conservation equation for QG theory in §6.

A continuous non-canonical Hamiltonian system determines the evolution of any functional $\mathcal{F}$ using a Poisson bracket $\{\cdot,\cdot\}$ and a Hamiltonian functional $\mathcal{H}$,

$$\frac{\mathrm{d}\mathcal{F}}{\mathrm{d}t} = \{\mathcal{F},\mathcal{H}\}\,. \tag{3.1}$$

The Poisson bracket is a bilinear, antisymmetric map that satisfies the Jacobi identity, $\{\mathcal{F},\{\mathcal{G},\mathcal{K}\}\} + \{\mathcal{G},\{\mathcal{K},\mathcal{F}\}\} + \{\mathcal{K},\{\mathcal{F},\mathcal{G}\}\} = 0$ for any three functionals $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{K}$. This formalism offers a coordinate-independent generalisation of classical Hamiltonian particle mechanics and canonical Hamiltonian field theory (e.g. Goldstein 1980) that allows us to find Hamiltonian descriptions of fluid systems using Eulerian variables. For more details see Morrison (1982, 1998, 2006); Salmon (1988*a*, 1998); Shepherd (1990).

By introducing the dependent or field variables $\boldsymbol{\eta}$, themselves functions of $\boldsymbol{x}$ and t, we write the Poisson bracket as

$$\{\mathcal{F},\mathcal{G}\} = \left\langle\frac{\delta\mathcal{F}}{\delta\boldsymbol{\eta}}, \mathsf{J}\frac{\delta\mathcal{G}}{\delta\boldsymbol{\eta}}\right\rangle, \tag{3.2}$$

in terms of a Poisson tensor J and an inner product $\langle\cdot,\cdot\rangle$. Variational derivatives are defined by the relation

$$\delta\mathcal{F} = \mathcal{F}(\boldsymbol{\eta}+\delta\boldsymbol{\eta}) - \mathcal{F}(\boldsymbol{\eta}) = \left\langle\frac{\delta\mathcal{F}}{\delta\boldsymbol{\eta}},\delta\boldsymbol{\eta}\right\rangle + O\big(\delta\boldsymbol{\eta}^2\big)\,, \tag{3.3}$$

for arbitrary variations $\delta\boldsymbol{\eta}$. This expression for the Poisson bracket is automatically bilinear, and is also antisymmetric when J is either an antisymmetric matrix or an anti-selfadjoint differential operator. The Jacobi identity imposes constraints on the $\boldsymbol{\eta}$-dependence of the components of J. Considering functionals $\mathcal{F} = \boldsymbol{\eta}(\boldsymbol{x}_0,t)$, for some fixed $\boldsymbol{x}_0$, in (3.1) implies the evolution equation

$$\frac{\partial\boldsymbol{\eta}}{\partial t} = \mathsf{J}\frac{\delta\mathcal{H}}{\delta\boldsymbol{\eta}} \tag{3.4}$$

for the field variables. Non-canonical Hamiltonian systems possess an additional class of conserved quantities, the Casimir functionals $\mathcal{C}$ associated with the kernel of the Poisson tensor J. The Casimir functionals are solutions of (Morrison 1998; Salmon 1988*a*, 1998; Shepherd 1990; Ripa 1993)

$$\mathsf{J}\frac{\delta\mathcal{C}}{\delta\boldsymbol{\eta}} = 0. \tag{3.5}$$

The equivalent condition in terms of the Poisson bracket is

$$\{\mathcal{C},\mathcal{F}\}=0,\quad \text{for all functionals } \mathcal{F}. \tag{3.6}$$

Ripa (1993) described the non-canonical Hamiltonian structure of the undamped ($\kappa=0$) thermal shallow water equations (2.1), but only gave a long "formal proof" of the Jacobi identity. The field variables in this formulation are $\boldsymbol{\eta}=(u,v,h,\Theta)^{\mathsf{T}}$, the Hamiltonian is

$$\mathcal{H}=\frac{1}{2}\int h|\boldsymbol{u}|^2+\Theta h^2\,\mathrm{d}\boldsymbol{x}, \tag{3.7}$$

and the Poisson tensor is

$$\mathsf{J}=-\begin{pmatrix} 0 & -\mathit{\Pi} & \partial_x & -h^{-1}\Theta_x \\ \mathit{\Pi} & 0 & \partial_y & -h^{-1}\Theta_y \\ \partial_x & \partial_y & 0 & 0 \\ h^{-1}\Theta_x & h^{-1}\Theta_y & 0 & 0 \end{pmatrix}, \tag{3.8}$$

where $\Theta_x=\partial_x\Theta$, $\Theta_y=\partial_y\Theta$, and $\mathit{\Pi}=h^{-1}(f+v_x-u_y)$ is the shallow water potential vorticity as in (2.8). The corresponding Poisson bracket is

$$\begin{aligned}\{\mathcal{F},\mathcal{G}\}=\int\ &\mathit{\Pi}\left(\frac{\delta\mathcal{F}}{\delta u}\frac{\delta\mathcal{G}}{\delta v}-\frac{\delta\mathcal{G}}{\delta u}\frac{\delta\mathcal{F}}{\delta v}\right)+\boldsymbol{\nabla}\boldsymbol{\cdot}\left(\frac{\delta\mathcal{F}}{\delta\boldsymbol{u}}\right)\frac{\delta\mathcal{G}}{\delta h}-\boldsymbol{\nabla}\boldsymbol{\cdot}\left(\frac{\delta\mathcal{G}}{\delta\boldsymbol{u}}\right)\frac{\delta\mathcal{F}}{\delta h}\\ &+\frac{\boldsymbol{\nabla}\Theta}{h}\boldsymbol{\cdot}\left(\frac{\delta\mathcal{F}}{\delta\boldsymbol{u}}\frac{\delta\mathcal{G}}{\delta\Theta}-\frac{\delta\mathcal{G}}{\delta\boldsymbol{u}}\frac{\delta\mathcal{F}}{\delta\Theta}\right)\mathrm{d}\boldsymbol{x}.\end{aligned} \tag{3.9}$$

These expressions reduce to the standard shallow water Poisson tensor and Poisson bracket (e.g. Shepherd 1990) on omitting the last row and column containing the $\boldsymbol{\nabla}\Theta$ terms in (3.8), and the last term containing $\boldsymbol{\nabla}\Theta$ in (3.9).

Dellar (2003) later gave a much shorter proof of the Jacobi identity for the Poisson bracket (3.9) by identifying it with the Poisson bracket for the shallow water magnetohydrodynamics equations (Gilman 2000; Dellar 2002) through a change of variables in which Θ plays the rôle of the magnetic vector potential. An even shorter proof is available by observing that (3.9) is the Poisson bracket for non-isentropic gas dynamics written in density, velocity, and entropy variables (e.g. Morrison 2006). The Casimir functionals for (3.8) and (3.9) are (Holm *et al.* 1985; Ripa 1993; Dellar 2003)

$$\mathcal{C}=\int h\mathit{\Pi}F(\Theta)+hG(\Theta)\,\mathrm{d}\boldsymbol{x} \tag{3.10}$$

for arbitrary functions F and G. These results rely upon being able to integrate by parts over the fluid domain $\mathcal{D}$ without acquiring surface terms. The necessary boundary conditions (Ripa 1993) are $\boldsymbol{u}\boldsymbol{\cdot}\boldsymbol{n}=0$ and $\boldsymbol{\nabla}\Theta\times\boldsymbol{n}=\boldsymbol{0}$ on the boundary $\partial\mathcal{D}$, or $|\boldsymbol{u}|\to 0$ and $\Theta\to\Theta_0$ (constant) as $|\boldsymbol{x}|\to\infty$.

4. Derivation of the thermal quasi-geostrophic equations by an expansion in Rossby number

We formalise the derivation of the QG limit of the thermal shallow water equations given in Ripa (1996*b*) by performing a direct asymptotic expansion in a small Rossby number. We begin with the thermal shallow water equations (2.1) formulated on a mid-latitude β-plane in which we treat x, y, z as Cartesian coordinates, while taking $f=f_0+\beta^*y$ to capture the latitude-dependence of the Coriolis parameter in spherical geometry (Pedlosky 1987; Salmon 1998; Vallis 2006; Dellar 2011). Our derivation relies upon the leading order velocity satisfying a leading order geostrophic balance condition with f approximated by f_0. This approximation is only valid outside the equatorial region.

4.1. *Quasi-geostrophic scaling*

Following Salmon (1998), we rescale the variables in (2.1) using

$$\boldsymbol{x}=L\tilde{\boldsymbol{x}},\quad \boldsymbol{u}=U\tilde{\boldsymbol{u}},\quad t=(L/U)\,\tilde{t}, \tag{4.1}$$

where L and U are horizontal length and velocity scales respectively. The superscript tilde denotes a dimensionless variable. We use the advective time scale L/U, so that $\partial\boldsymbol{u}/\partial t$ is comparable to $(\boldsymbol{u}\boldsymbol{\cdot}\boldsymbol{\nabla})\,\boldsymbol{u}$ in the dimensionless momentum equation. We write the layer depth as

$$h=H_0\left(1+\frac{f_0UL}{\Theta_0H_0}\tilde{\eta}\right), \tag{4.2}$$

for small displacements from a mean depth H_0. This scaling for $\tilde{\eta}$ is set by a leading order geostrophic balance between the Coriolis force and the pressure gradient in (2.1*c*). We also set

$$\Theta=\Theta_0\left(1+2\frac{f_0UL}{\Theta_0H_0}\tilde{\theta}\right), \tag{4.3}$$

with a factor of two for future convenience in the definition of the geostrophic streamfunction.

Substituting these scalings into (2.1) gives the dimensionless equations

$$\mathit{Ro}\left(\frac{\partial\tilde{\eta}}{\partial\tilde{t}}+\tilde{\boldsymbol{\nabla}}\cdot(\tilde{\eta}\tilde{\boldsymbol{u}})\right)+\mathit{Bu}\left(\tilde{\boldsymbol{\nabla}}\cdot\tilde{\boldsymbol{u}}\right)=0, \tag{4.4a}$$

$$\frac{\partial\tilde{\theta}}{\partial\tilde{t}}+\left(\tilde{\boldsymbol{u}}\cdot\tilde{\boldsymbol{\nabla}}\right)\tilde{\theta}=-2\lambda\left(\tilde{\theta}+\frac{1}{2}\tilde{\eta}+\frac{\mathit{Ro}}{\mathit{Bu}}\tilde{\eta}\tilde{\theta}\right), \tag{4.4b}$$

$$\mathit{Ro}\left(\frac{\partial\tilde{u}}{\partial\tilde{t}}+\left(\tilde{\boldsymbol{u}}\cdot\tilde{\boldsymbol{\nabla}}\right)\tilde{u}\right)-(1+\tilde{\beta}\tilde{y})\tilde{v}=-\frac{\partial\tilde{\eta}}{\partial\tilde{x}}-\frac{\partial\tilde{\theta}}{\partial\tilde{x}}-\frac{\mathit{Ro}}{\mathit{Bu}}\left(2\tilde{\theta}\frac{\partial\tilde{\eta}}{\partial\tilde{x}}+\tilde{\eta}\frac{\partial\tilde{\theta}}{\partial\tilde{x}}\right), \tag{4.4c}$$

$$\mathit{Ro}\left(\frac{\partial\tilde{v}}{\partial\tilde{t}}+\left(\tilde{\boldsymbol{u}}\cdot\tilde{\boldsymbol{\nabla}}\right)\tilde{v}\right)+(1+\tilde{\beta}\tilde{y})\tilde{u}=-\frac{\partial\tilde{\eta}}{\partial\tilde{y}}-\frac{\partial\tilde{\theta}}{\partial\tilde{y}}-\frac{\mathit{Ro}}{\mathit{Bu}}\left(2\tilde{\theta}\frac{\partial\tilde{\eta}}{\partial\tilde{y}}+\tilde{\eta}\frac{\partial\tilde{\theta}}{\partial\tilde{y}}\right), \tag{4.4d}$$

with the dimensionless parameters

$$\mathit{Ro}=\frac{U}{f_0L},\quad \mathit{Bu}=\left(\frac{L_D}{L}\right)^2,\quad \tilde{\beta}=\frac{\beta^*L}{f_0},\quad \lambda=\frac{\kappa H_0}{2f_0\mathit{Ro}}, \tag{4.5}$$

where *Ro* is the Rossby number, *Bu* is the Burger number, and $L_D=\sqrt{\Theta_0H_0}/f_0$ is the deformation radius, the horizontal length scale on which the Coriolis force becomes comparable to the pressure gradient. We now recognise the scalings in (4.2) and (4.3) as

$$h=H_0\left(1+\frac{\mathit{Ro}}{\mathit{Bu}}\tilde{\eta}\right),\quad \Theta=\Theta_0\left(1+2\frac{\mathit{Ro}}{\mathit{Bu}}\tilde{\theta}\right). \tag{4.6}$$

4.2. *Asymptotic expansion*

We now set $\mathit{Ro}=\epsilon$, where ϵ is a small parameter. We also set $\tilde{\beta}=\epsilon\beta$, where β remains $O(1)$ as $\epsilon\to 0$, recognising that fractional changes in the Coriolis parameter must be small on the horizontal length scale L of the flow (Salmon 1998). We also assume that *Bu* and λ remain $O(1)$ as $\epsilon\to 0$. We drop the tilde notation, with the understanding that all variables are now dimensionless.

We expand the velocity $\boldsymbol{u}$ in powers of ϵ,

$$\boldsymbol{u}=\boldsymbol{u}_0+\epsilon\boldsymbol{u}_1+\cdots, \tag{4.7}$$

while leaving the other variables η and θ unexpanded (as in Warn *et al.* 1995; Muraki *et al.* 1999; White 2002). Following Van Kampen's (1985) general theory, we obtain closed evolution equations for the unexpanded slow variables η and θ that remain valid on long time scales with $t=O(1/\epsilon)$. Pedlosky (1987) and Majda (2003) avoid this step and expand η and θ as well, but they first supplement the continuity, momentum and temperature equations, with the exact evolution equation for the shallow water potential vorticity $\mathit{\Pi}=h^{-1}(f+v_x-u_y)$. This supplementary equation identifies the potential vorticity as a slow variable whose evolution should be captured by the asymptotic description.

Substituting (4.7) into (4.4) and truncating at leading order gives

$$\boldsymbol{\nabla}\cdot\boldsymbol{u}_0=0,\quad v_0=\partial_x(\eta+\theta),\quad u_0=-\partial_y(\eta+\theta), \tag{4.8a}$$

$$\partial_t\theta+(\boldsymbol{u}_0\cdot\boldsymbol{\nabla})\,\theta=-\lambda\,(2\theta+\eta)\,. \tag{4.8b}$$

We thus have a geostrophic flow with streamfunction $\psi=\eta+\theta$. This simple expression justifies the factor of two in (4.3), and in the second of (4.6). At next order we obtain the $O(\epsilon)$ continuity and velocity equations,

$$\frac{\partial\eta}{\partial t}+\boldsymbol{\nabla}\cdot(\eta\boldsymbol{u}_0)+\mathit{Bu}\,(\boldsymbol{\nabla}\cdot\boldsymbol{u}_1)=0, \tag{4.9a}$$

$$\frac{\partial u_0}{\partial t}+(\boldsymbol{u}_0\cdot\boldsymbol{\nabla})\,u_0-v_1-\beta yv_0=-\frac{1}{\mathit{Bu}}\left(2\theta\frac{\partial\eta}{\partial x}+\eta\frac{\partial\theta}{\partial x}\right), \tag{4.9b}$$

$$\frac{\partial v_0}{\partial t}+(\boldsymbol{u}_0\cdot\boldsymbol{\nabla})\,v_0+u_1+\beta yu_0=-\frac{1}{\mathit{Bu}}\left(2\theta\frac{\partial\eta}{\partial y}+\eta\frac{\partial\theta}{\partial y}\right). \tag{4.9c}$$

Substituting (4.8*a*) into (4.9), and eliminating $\boldsymbol{\nabla}\cdot\boldsymbol{u}_1$ between (4.9*a*) and (4.9*b*,*c*), gives

$$\frac{\partial q}{\partial t}+[\psi,q]=\frac{1}{\mathit{Bu}}\,[\eta,\theta]\,, \tag{4.10}$$

where the QG potential vorticity

$$q=\boldsymbol{\nabla}^2\psi-\frac{\eta}{\mathit{Bu}}+\beta y \tag{4.11}$$

is a consistent linear approximation to the shallow water potential vorticity $\mathit{\Pi}$, and $[\cdot,\cdot]$ is the two-dimensional Jacobian defined by $[\psi,q]=\hat{\boldsymbol{z}}\cdot(\boldsymbol{\nabla}\psi\times\boldsymbol{\nabla}q)$.

Eliminating $\eta = \psi - \theta$ gives the closed thermal QG system

$$\frac{\partial q}{\partial t} + [\psi, q] = \frac{1}{Bu}[\psi, \theta], \tag{4.12a}$$
$$\frac{\partial \theta}{\partial t} + [\psi, \theta] = -\lambda(\theta + \psi), \tag{4.12b}$$
$$q = \boldsymbol{\nabla}^2\psi - \frac{(\psi - \theta)}{Bu} + \beta y. \tag{4.12c}$$

The usual QG shallow water potential vorticity is modified by the additional θ/Bu term in (4.12*c*), which acts as an additional source in the modified Helmholtz equation $\boldsymbol{\nabla}^2\psi - \psi/Bu = q - \beta y - \theta/Bu$. Potential vorticity is not materially conserved, due to the additional term on the right hand side of (4.12*a*). However, the total potential vorticity inside a closed isotherm (line of constant θ) is conserved (see 7.6), as for the thermal shallow water equations (c.f. §2).

5. Derivation of the thermal quasi-geostrophic equations by a projection onto vortical modes

We now present an alternative derivation of the thermal QG equations (4.12) that explicitly projects out inertia-gravity waves. The thermal shallow water equations (2.1) support interactions between four different wave modes: the usual two inertia-gravity waves, the vortical or Rossby wave of the standard shallow water equations, and an additional thermal wave due to the extra temperature equation. We first seek a balanced model that describes only the low-frequency dynamics represented by the Rossby and thermal waves by projecting out the high-frequency dynamics associated with the inertia-gravity waves (see Leith 1980; Salmon 1998; White 2002; Majda 2003; Remmel & Smith 2009; Theiss & Mohebalhojeh 2009). Ripa (1996*a*) studied linear waves in the thermal shallow water equations, but we prefer to use a different set of variables that give particularly simple forms for the different wave modes.

Following Viúdez & Dritschel's (2004) treatment of the standard shallow water equations, we reformulate the thermal shallow water equations (2.1) in terms of new variables: the potential vorticity Π, the velocity divergence D, and the acceleration divergence γ. Writing $h = H_0 + \eta$ and $\Theta = \Theta_0 + \theta$, these new variables are given by

$$\Pi = \frac{1}{H_0 + \eta}\left(f_0 + \beta^* y + \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right), \tag{5.1a}$$
$$D = \boldsymbol{\nabla}\cdot\boldsymbol{u}, \tag{5.1b}$$
$$\gamma = \boldsymbol{\nabla}\cdot\left(-(f_0 + \beta^* y)\,\hat{\boldsymbol{z}}\times\boldsymbol{u} - (\Theta_0 + \theta)\boldsymbol{\nabla}\eta - \frac{1}{2}(H_0 + \eta)\boldsymbol{\nabla}\theta\right). \tag{5.1c}$$

We again consider a mid-latitude β-plane on which $f = f_0 + \beta^* y$. In these new variables the thermal shallow water equations become

$$\frac{\partial \Pi}{\partial t} = \left\{-(\boldsymbol{u}\cdot\boldsymbol{\nabla})\,\Pi + \frac{1}{2}\frac{[\eta,\theta]}{(H_0+\eta)}\right\}, \tag{5.2a}$$
$$\frac{\partial \theta}{\partial t} = \left\{-\kappa(H_0\theta + \Theta_0\eta) - \kappa\eta\theta - (\boldsymbol{u}\cdot\boldsymbol{\nabla})\,\theta\right\}, \tag{5.2b}$$
$$\frac{\partial D}{\partial t} = \gamma + \left\{-(\boldsymbol{u}\cdot\boldsymbol{\nabla})\,D - D^2 + 2[u,v]\right\}, \tag{5.2c}$$
$$\begin{aligned}\frac{\partial \gamma}{\partial t} = {}& -f_0^2 D + \Theta_0 H_0\boldsymbol{\nabla}^2 D + \left\{\frac{1}{2}\kappa H_0^2\boldsymbol{\nabla}^2\theta + \frac{1}{2}\kappa H_0\Theta_0\boldsymbol{\nabla}^2\eta\right\} \\ & + \left\{-2f_0\beta^* yD - \beta^{*2}y^2 D - 2\beta^*(f_0 + \beta^* y)\,v + \beta^*\Theta_0\frac{\partial \eta}{\partial x} + \frac{1}{2}\beta^* H_0\frac{\partial \theta}{\partial x}\right\} \\ & + \{\text{nonlinear terms}\},\end{aligned} \tag{5.2d}$$

where $[\cdot,\cdot]$ is the two-dimensional Jacobian as above. We rewrite (5.2) in matrix form as

$$\partial_t\boldsymbol{\eta} = -\mathrm{i}\mathsf{L}\boldsymbol{\eta} + \mathsf{N}(\boldsymbol{\eta}), \tag{5.3}$$

where

$$\boldsymbol{\eta} = \begin{pmatrix}\Pi \\ \theta \\ D \\ \gamma\end{pmatrix}, \quad \mathsf{L} = \begin{pmatrix}0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathrm{i} \\ 0 & 0 & \mathrm{i}\left(\Theta_0 H_0\boldsymbol{\nabla}^2 - f_0^2\right) & 0\end{pmatrix}, \tag{5.4}$$

and $\mathsf{N}(\boldsymbol{\eta})$ is a column vector containing the terms from (5.2) in curly brackets. The linear operator L characterises the wave modes of the undamped f-plane thermal shallow water equations. All coupling between these modes is expressed through $\mathsf{N}(\boldsymbol{\eta})$, both the nonlinear coupling and the linear coupling through terms involving β^* and κ. This choice of decomposition follows the approach of Leith (1980) and Salmon (1998).

The eigenmodes of L for an unbounded or periodic domain are the Fourier modes

$$\boldsymbol{\eta} = \hat{\boldsymbol{\eta}} \exp\left(\mathrm{i}\left(kx + ly - \omega t\right)\right), \tag{5.5}$$

with $\hat{\boldsymbol{\eta}}$ a constant vector. Substituting (5.5) into (5.3) and setting $N = \mathbf{0}$ gives

$$\omega_{R_1} = 0, \quad \omega_{R_2} = 0, \quad \omega_{G_\pm} = \pm\sqrt{f_0^2 + \Theta_0 H_0\left(k^2 + l^2\right)}, \tag{5.6}$$

with the corresponding eigenvectors

$$\boldsymbol{e}_{R_1} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \boldsymbol{e}_{R_2} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \boldsymbol{e}_{G_\pm} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ \mp i\sqrt{f_0^2 + \Theta_0 H_0\left(k^2 + l^2\right)} \end{pmatrix}. \tag{5.7}$$

The thermal mode with eigenvector $\boldsymbol{e}_{R_2}$ was called the force-compensated mode by Ripa (1996*a*). Young & Chen (1995) named the equivalent mode in their subinertial mixed layer equations the buoyancy-compensated mode. The fluctuation in θ is balanced by a compensating fluctuation in η that leaves the pressure, and hence the velocity field, unchanged. This mode has a particularly simple eigenvector $\boldsymbol{e}_{R_2}$ in our variables $\mathit{\Pi}, \theta, D, \gamma$ because we have eliminated the height η in favour of the acceleration divergence γ. The combined contribution to γ from $\boldsymbol{\nabla}\eta$ and $\boldsymbol{\nabla}\theta$ vanishes precisely because this mode is force-compensated.

The pair $\{\omega_{R_1}, \boldsymbol{e}_{R_1}\}$ represent the low-frequency Rossby mode, the pair $\{\omega_{R_2}, \boldsymbol{e}_{R_2}\}$ represent the low-frequency thermal mode, and the pairs $\{\omega_{G_+}, \boldsymbol{e}_{G_+}\}$ and $\{\omega_{G_-}, \boldsymbol{e}_{G_-}\}$ represent the high-frequency inertia-gravity modes. We call the space $\mathcal{R}$ spanned by the Rossby and thermal modes the slow or Rossby manifold, while we call the $\mathcal{G}$ space spanned by the inertia-gravity modes the fast or inertia-gravity manifold. The manifolds $\mathcal{R}$ and $\mathcal{G}$ are orthogonal and span the whole space, so any vector $\boldsymbol{\eta}$ may be decomposed uniquely into two components, $\boldsymbol{\eta} = \boldsymbol{\eta}_R + \boldsymbol{\eta}_G$, with $\boldsymbol{\eta}_R$ in $\mathcal{R}$ and $\boldsymbol{\eta}_G$ in $\mathcal{G}$. This clean separation into fast and slow manifolds motivates the introduction of the $\mathit{\Pi}, \theta, D, \gamma$ variables in (5.1).

We now restrict nonlinear interactions to include only the Rossby and thermal modes. A vector $\hat{\boldsymbol{\eta}}_R$ on the Rossby manifold may be expressed as

$$\hat{\boldsymbol{\eta}}_R = \begin{pmatrix} \hat{\mathit{\Pi}}_R \\ \hat{\theta}_R \\ \hat{D}_R \\ \hat{\gamma}_R \end{pmatrix} = \hat{y}_{R_1}\boldsymbol{e}_{R_1} + \hat{y}_{R_2}\boldsymbol{e}_{R_2} = \begin{pmatrix} \hat{y}_{R_1} \\ \hat{y}_{R_2} \\ 0 \\ 0 \end{pmatrix}, \tag{5.8}$$

where $\hat{y}_{R_1}(k,l,t)$ and $\hat{y}_{R_2}(k,l,t)$ are the coordinates of $\hat{\boldsymbol{\eta}}_R$ on the Rossby manifold, and hats denote a Fourier transform in x and y. Transforming the lower two components of (5.8) back to physical space gives the balance relations

$$D_R = 0 \quad \text{and} \quad \gamma_R = 0. \tag{5.9}$$

We find evolution equations for $\hat{\mathit{\Pi}}_R$ and $\hat{\theta}_R$ by Fourier transforming (5.3) in x and y, and substituting $\hat{\boldsymbol{\eta}} = \hat{\boldsymbol{\eta}}_R$ as given by (5.8), to obtain

$$\partial_t\hat{\mathit{\Pi}}_R\boldsymbol{e}_{R_1} + \partial_t\hat{\theta}_R\boldsymbol{e}_{R_2} = -\mathrm{i}\left(\hat{\mathit{\Pi}}_R\hat{L}\boldsymbol{e}_{R_1} + \hat{\theta}_R\hat{L}\boldsymbol{e}_{R_2}\right) + \hat{N}, \tag{5.10}$$

where $\hat{N}$ and $\hat{L}$ are the Fourier transforms of N and L. We know from (5.6) that $\boldsymbol{e}_{R_1}$ and $\boldsymbol{e}_{R_2}$ are eigenvectors of $\hat{L}$ with zero eigenvalues, so (5.10) reduces to

$$\partial_t\hat{\mathit{\Pi}}_R\boldsymbol{e}_{R_1} + \partial_t\hat{\theta}_R\boldsymbol{e}_{R_2} = \hat{N}. \tag{5.11}$$

Projecting (5.11) onto $\boldsymbol{e}_{R_1}$ and inverting the Fourier transform gives

$$\partial_t\mathit{\Pi}_R + \left(\boldsymbol{u}_R\cdot\boldsymbol{\nabla}\right)\mathit{\Pi}_R = \frac{1}{2}\frac{[\eta_R, \theta_R]}{(H_0 + \eta_R)}, \tag{5.12}$$

while projecting (5.11) onto $\boldsymbol{e}_{R_2}$ and inverting the Fourier transform gives

$$\partial_t\theta_R + \left(\boldsymbol{u}_R\cdot\boldsymbol{\nabla}\right)\theta_R = -\kappa\left(H_0\theta_R + \Theta_0\eta_R + \eta_R\theta_R\right). \tag{5.13}$$

We thus obtain a balanced model consisting of the two evolution equations (5.12) and (5.13), and the two balance relations (5.9). Dropping the R subscripts for convenience, our complete balanced model is

$$\frac{\partial \mathit{\Pi}}{\partial t} + (\boldsymbol{u}\cdot\boldsymbol{\nabla})\,\mathit{\Pi} = \frac{1}{2}\frac{[\eta,\theta]}{(H_0+\eta)}, \tag{5.14a}$$

$$\frac{\partial \theta}{\partial t} + (\boldsymbol{u}\cdot\boldsymbol{\nabla})\,\theta = -\kappa\left(H_0\theta + \Theta_0\eta + \eta\theta\right), \tag{5.14b}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{u} = 0, \tag{5.14c}$$

$$(f_0+\beta^* y)\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) - \beta^* u = \frac{3}{2}\boldsymbol{\nabla}\eta\cdot\boldsymbol{\nabla}\theta + (\Theta_0+\theta)\,\boldsymbol{\nabla}^2\eta + \frac{1}{2}(H_0+\eta)\,\boldsymbol{\nabla}^2\theta, \tag{5.14d}$$

where $\mathit{\Pi}$ is defined by (5.1a). This generalises the δ-γ balanced model of Mohebalhojeh & Dritschel (2001) to the thermal shallow water equations. The derivation is not restricted to mid-latitude β-planes, and provides a route towards developing a spherical version of thermal QG theory (as in Kuo 1959; Charney & Stern 1962; Verkley 2009; Schubert *et al.* 2009).

The earlier thermal QG equations now emerge as the leading order approximation to (5.14) for small Rossby number. We non-dimensionalise (5.1a) and (5.14) using the variables

$$\boldsymbol{x} = L\tilde{\boldsymbol{x}}, \quad \boldsymbol{u} = U\tilde{\boldsymbol{u}}, \quad t = (L/U)\,\tilde{t}, \quad \eta = (f_0 UL/\Theta_0)\,\tilde{\eta}, \quad \theta = (2f_0UL/H_0)\,\tilde{\theta}, \tag{5.15}$$

as in (4.1) - (4.3), to obtain the dimensionless equations

$$\left(\frac{\partial}{\partial\tilde{t}} + \left(\tilde{\boldsymbol{u}}\cdot\tilde{\boldsymbol{\nabla}}\right)\right)\left\{\frac{Ro\,(\tilde{v}_{\tilde{x}} - \tilde{u}_{\tilde{y}}) + 1 + \tilde{\beta}\tilde{y}}{Ro + \left(Ro^2/Bu\right)\tilde{\eta}}\right\} = \frac{1}{Bu}\left(\frac{[\tilde{\eta},\tilde{\theta}]}{1 + (Ro/Bu)\,\tilde{\eta}}\right), \tag{5.16a}$$

$$\frac{\partial\tilde{\theta}}{\partial\tilde{t}} + \left(\tilde{\boldsymbol{u}}\cdot\tilde{\boldsymbol{\nabla}}\right)\tilde{\theta} = -\lambda\left(2\tilde{\theta} + \tilde{\eta} + 2\frac{Ro}{Bu}\tilde{\eta}\tilde{\theta}\right), \tag{5.16b}$$

$$\tilde{\boldsymbol{\nabla}}\cdot\tilde{\boldsymbol{u}} = 0, \tag{5.16c}$$

$$\begin{aligned}\left(1+\tilde{\beta}\tilde{y}\right)\left(\frac{\partial\tilde{v}}{\partial\tilde{x}} - \frac{\partial\tilde{u}}{\partial\tilde{y}}\right) - \tilde{\beta}\tilde{u} &= \tilde{\boldsymbol{\nabla}}^2\tilde{\eta} + \tilde{\boldsymbol{\nabla}}^2\tilde{\theta}\\ &\quad + \frac{Ro}{Bu}\left(3\tilde{\boldsymbol{\nabla}}\tilde{\eta}\cdot\tilde{\boldsymbol{\nabla}}\tilde{\theta} + 2\tilde{\theta}\tilde{\boldsymbol{\nabla}}^2\tilde{\eta} + \tilde{\eta}\tilde{\boldsymbol{\nabla}}^2\tilde{\theta}\right),\end{aligned} \tag{5.16d}$$

with parameters *Ro*, *Bu*, $\tilde{\beta}$, λ as in (4.5). We drop the tilde notation, with the understanding that all variables are now dimensionless in this section. As in §4.2, we write $Ro = \epsilon$ and $\tilde{\beta} = \epsilon\beta$, where ϵ is a small parameter, and assume that β, *Bu* and λ all remain $O(1)$ as $\epsilon \to 0$. Expanding $\boldsymbol{u}$ in powers of ϵ as in (4.7), at leading order we obtain the equation set

$$\left(\frac{\partial}{\partial t} + (\boldsymbol{u}_0\cdot\boldsymbol{\nabla})\right)\left\{\frac{\partial v_0}{\partial x} - \frac{\partial u_0}{\partial y} - \frac{\eta}{Bu} + \beta y\right\} = \frac{1}{Bu}[\eta,\theta], \tag{5.17a}$$

$$\frac{\partial\theta}{\partial t} + (\boldsymbol{u}_0\cdot\boldsymbol{\nabla})\,\theta = -\lambda\,(2\theta + \eta), \tag{5.17b}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{u}_0 = 0, \tag{5.17c}$$

$$\frac{\partial v_0}{\partial x} - \frac{\partial u_0}{\partial y} = \boldsymbol{\nabla}^2\eta + \boldsymbol{\nabla}^2\theta. \tag{5.17d}$$

Equation (5.17c) allows us to introduce a geostrophic streamfunction ψ such that

$$u_0 = -\partial_y\psi, \quad v_0 = \partial_x\psi. \tag{5.18}$$

Equation (5.17d) then implies that $\boldsymbol{\nabla}^2\psi = \boldsymbol{\nabla}^2\eta + \boldsymbol{\nabla}^2\theta$, so we take $\psi = \eta + \theta$. Substituting (5.18) into (5.17a,b), and eliminating η using $\eta = \psi - \theta$, leads to our previous thermal QG equations (4.12).

6. The quasi-geostrophic limit of the thermal shallow water energy equation

The thermal QG equations (4.12) imply the local energy conservation law

$$\partial_t E_{\mathrm{QG}} + \boldsymbol{\nabla}\cdot\boldsymbol{F}_{\mathrm{QG}} = S_{\mathrm{QG}}, \tag{6.1}$$

for the energy density and energy flux

$$E_{\mathrm{QG}} = \frac{1}{2}\left(|\boldsymbol{\nabla}\psi|^2 + \psi^2/Bu\right), \quad \boldsymbol{F}_{\mathrm{QG}} = -\psi\boldsymbol{\nabla}\psi_t + \psi^2\hat{\boldsymbol{z}}\times\boldsymbol{\nabla}\left(q/2 - \theta/Bu\right), \tag{6.2}$$

and a source term due to radiative damping,

$$\mathcal{S}_{QG} = -(\lambda/Bu)\psi\,(\theta+\psi)\,. \tag{6.3}$$

This energy equation may be derived by considering $\psi\partial_t(q - \theta/Bu)$. The expression E_{QG} is identical to the energy density for the the standard QG equations (e.g. Weinstein 1983), but the relation between ψ and the prognostic variables has changed. For standard QG theory, McIntyre & Shepherd (1987) gave the equivalent formulae in the infinite deformation radius ($Bu \to \infty$) limit, while Pedlosky (1987) and Longuet-Higgins (1964) gave related formulae for linear waves in fluids with a finite deformation radius.

However, (6.1) is not the QG limit of the corresponding local energy conservation law

$$\partial_t E + \boldsymbol{\nabla}\cdot\boldsymbol{F} = S, \tag{6.4}$$

for the thermal shallow water equations (2.1). The thermal shallow water energy density and flux are

$$E = \frac{1}{2}\left(h\,|\boldsymbol{u}|^2 + \Theta h^2\right), \quad \boldsymbol{F} = \boldsymbol{u}\left(\frac{1}{2}h\,|\boldsymbol{u}|^2 + \Theta h^2\right), \tag{6.5}$$

and the source term due to radiative damping is

$$S = -\frac{1}{2}h^2\kappa\,(h\Theta - H_0\Theta_0)\,. \tag{6.6}$$

Applying the QG scalings (4.1) - (4.3) to E, and expanding as in §4.2, leads to the expression

$$E(\boldsymbol{u},h,\Theta) = E(\boldsymbol{0},H_0,\Theta_0) + \frac{H_0 f_0^2 L^2}{2}\left[\epsilon^2|\tilde{\boldsymbol{\nabla}}\tilde{\psi}|^2 + 2\epsilon(\tilde{\theta}+\tilde{\eta}) + \epsilon^2\frac{(\tilde{\eta}^2 + 4\tilde{\theta}\tilde{\eta})}{Bu} + O\big(\epsilon^3\big)\right], \tag{6.7}$$

where $\tilde{\psi}$, $\tilde{\eta}$, $\tilde{\theta}$ are the dimensionless variables introduced in §4. The kinetic energy, proportional to $|\tilde{\boldsymbol{\nabla}}\tilde{\psi}|^2$, is $O\big(\epsilon^2\big)$, while the potential energy contains an $O(\epsilon)$ term. The kinetic energy is thus asymptotically smaller than the potential energy, and the potential energy is itself not sign-definite. The energy difference $E(\boldsymbol{u},h,\Theta) - E(\boldsymbol{0},H_0,\Theta_0)$ is thus not a satisfactory measure of the magnitude of deviations from an underlying basic state with $\boldsymbol{u} = \boldsymbol{0}$, $h = H_0$, and $\Theta = \Theta_0$.

6.1. *Available potential energy and pseudo-energy*

This difficulty was first encountered by Lorenz (1955) in defining the potential energy of a disturbance to an ideal gas in hydrostatic balance, and also arises when deriving an acoustic energy equation from the underlying energy equation for a compressible gas (e.g. Lighthill 1978). Lorenz (1955) combined the mass conservation law with the energy conservation law to yield a new conserved quantity, the kinetic plus *available* potential energy (APE) that is quadratic to leading order in the disturbance quantities (for reviews see Winters *et al.* 1995; Tailleux 2013).

Shepherd (1993) gave a general discussion of the construction of an energy equation for disturbances around an equilibrium $\boldsymbol{\eta}_0$ of a Hamiltonian system, which must satisfy

$$J\frac{\delta\mathcal{H}}{\delta\boldsymbol{\eta}} = 0 \quad \text{at} \quad \boldsymbol{\eta} = \boldsymbol{\eta}_0. \tag{6.8}$$

This condition implies $\delta\mathcal{H}/\delta\boldsymbol{\eta} = 0$ for a canonical Hamiltonian system with an invertible J. The Hamiltonian thus has the expected Taylor expansion $\mathcal{H}(\eta_0 + \delta\boldsymbol{\eta}) = \mathcal{H}(\eta_0) + \delta\boldsymbol{\eta}\cdot\mathcal{H}''(\eta_0)\cdot\delta\boldsymbol{\eta} + O\big(|\delta\boldsymbol{\eta}|^3\big)$. However, all we may conclude from (6.8) for a non-canonical system is

$$\frac{\delta\mathcal{H}}{\delta\boldsymbol{\eta}} = \frac{\delta\mathcal{C}}{\delta\boldsymbol{\eta}} \quad \text{at} \quad \boldsymbol{\eta} = \boldsymbol{\eta}_0, \tag{6.9}$$

for some Casimir functional $\mathcal{C}$. Lorenz's (1955) combination of energy conservation with mass conservation thus generalises to finding an additional conservation law, and its associated Casimir $\mathcal{C}$, for which the first variation of the combined functional $\mathcal{H} - \mathcal{C}$, known as the pseudo-energy, vanishes at equilibrium. The same observation underlies the energy-Casimir method for establishing the stability of non-canonical Hamiltonian systems (Arnold 1965*a*,*b*; Holm *et al.* 1985; McIntyre & Shepherd 1987; Shepherd 1990; Arnold & Khesin 1998).

6.2. *Pseudo-energy for the thermal shallow water equations*

We follow Shepherd's (1993) algorithm to construct a pseudo-energy for the thermal shallow water equations from the solution of

$$\frac{\delta\mathcal{C}}{\delta\boldsymbol{u}} = \frac{\delta\mathcal{H}}{\delta\boldsymbol{u}} = \boldsymbol{0}, \quad \frac{\delta\mathcal{C}}{\delta h} = \frac{\delta\mathcal{H}}{\delta h} = H_0\Theta_0, \quad \frac{\delta\mathcal{C}}{\delta\Theta} = \frac{\delta\mathcal{H}}{\delta\Theta} = \frac{1}{2}H_0^2 \tag{6.10}$$

evaluated at the basic state $\boldsymbol{u} = \boldsymbol{0}$, $h = H_0$, $\Theta = \Theta_0$. A solution is given by

$$\mathcal{C}(h,\Theta) = \int \frac{3}{4}H_0\Theta h + \frac{3}{8}\Theta_0 H_0 h - \frac{1}{8}\frac{H_0}{\Theta_0}\Theta^2 h\,\mathrm{d}\boldsymbol{x}. \tag{6.11}$$

We may omit the $\varPi$-containing term in (3.10) because we evaluate the variational derivatives at a rest state with $\boldsymbol{u} = \boldsymbol{0}$.

The corresponding disturbance pseudo-energy is

$$\mathcal{A} = \mathcal{H}(\boldsymbol{u},h,\Theta) - \mathcal{H}(\boldsymbol{0},H_0,\Theta_0) - \mathcal{C}(h,\Theta) + \mathcal{C}(H_0,\Theta_0)\,. \tag{6.12}$$

Expanding $\mathcal{A}$ for small disturbances to the background state, and transforming to the dimensionless variables of (5.15), gives the QG energy at leading order,

$$\mathcal{A} = H_0 f_0^2 L^4 \epsilon^2 \left[\frac{1}{2} \int |\tilde{\boldsymbol{\nabla}}\tilde{\psi}|^2 + \frac{\tilde{\psi}^2}{Bu}\, \mathrm{d}\tilde{\boldsymbol{x}} + O(\epsilon) \right]. \tag{6.13}$$

The kinetic energy and potential energy are now of the same order in ϵ, and the whole expression is positive definite. Moreover, $\mathcal{A}$ is an exact invariant of the undamped ($\kappa = 0$) thermal shallow water equations. Initial disturbances to the background state that are small, in the sense that $\mathcal{A}$ is small, thus remain small disturbances in the sense measured by $\mathcal{A}$. However, $\mathcal{A}$ does not control the amplitude of the force-compensated mode, in which the disturbances $\tilde{\theta}$ and $\tilde{\eta}$ to Θ and h respectively occur in antiphase, and so do not alter $\tilde{\psi} = \tilde{\eta} + \tilde{\theta}$.

A second Casimir satisfying the conditions (6.10) is

$$\mathcal{C}_{\mathrm{R}} = \int \sqrt{\Theta\Theta_0}\, h H_0 \,\mathrm{d}\boldsymbol{x}. \tag{6.14}$$

The general pseudo-energy definition (6.12) then takes the form proposed by Ripa (1995) and Røed (1997),

$$\mathcal{A}_{\mathrm{R}} = \frac{1}{2} \int h|\boldsymbol{u}|^2 + \left(\sqrt{\Theta}\, h - \sqrt{\Theta_0}\, H_0 \right)^2 \mathrm{d}\boldsymbol{x}, \tag{6.15}$$

which combines the kinetic energy with a measure of the disturbance potential energy. Røed (1997) named the latter "available gravitational energy" following the work of Holliday & McIntyre (1981) and Andrews (1981) for continuously stratified fluids. The available gravitational energy may be computed locally for a fluid column, unlike Lorenz's (1955) available potential energy that is defined relative to the potential energy minimising rearrangement of the fluid in the whole domain. Expanding $\mathcal{A}_{\mathrm{R}}$ for small deviations in h and Θ, and transforming to dimensionless variables gives an expression that coincides with our pseudo-energy (6.13) at leading order, but this approach has not previously been used in the derivation of a QG theory. The pseudo-energy (6.15) is a sum of squares, while the pseudo-energy defined using the Casimir (6.11) only has this property for small disturbances to the background state. In the following we use $\mathcal{A}_{\mathrm{R}}$ instead of the $\mathcal{A}$ constructed from the Casimir in (6.11), because $\mathcal{A}_{\mathrm{R}}$ gives simpler intermediate expressions. Our final results, which depend only upon expansions truncated at $O(\epsilon^2)$ for small disturbances, are identical whether one uses the Casimir from (6.11) or (6.14) to construct a pseudo-energy.

6.3. *The quasi-geostrophic limit of the pseudo-energy flux for the thermal shallow water equations*

The pseudo-energy (6.15) may be written as $\mathcal{A}_{\mathrm{R}} = \int E_P(\boldsymbol{x},t) d\boldsymbol{x}$. The pseudo-energy density

$$E_P = \frac{1}{2} h|\boldsymbol{u}|^2 + \frac{1}{2}\left(\sqrt{\Theta} h - \sqrt{\Theta_0} H_0 \right)^2, \tag{6.16}$$

satisfies the local conservation equation

$$\partial_t E_P + \boldsymbol{\nabla}\cdot\boldsymbol{F}_P = S_P, \tag{6.17}$$

with pseudo-energy flux

$$\boldsymbol{F}_P = \boldsymbol{u}\left(\frac{1}{2} h|\boldsymbol{u}|^2 + \Theta h^2 - \sqrt{\Theta\Theta_0}\, h H_0 \right), \tag{6.18}$$

and a source term due to radiative damping,

$$S_P = \frac{1}{2}\kappa h \left(h\Theta - H_0\Theta_0 \right) \left(\sqrt{\Theta_0/\Theta} H_0 - h \right). \tag{6.19}$$

We now apply the QG scaling to E_P, $\boldsymbol{F}_P$, and S_P. We non-dimensionalise using (4.1) - (4.3), and then drop the tilde notation with the understanding that all variables in this section are now dimensionless. Expanding as in §4.2 with $u_0 = -\partial\psi/\partial y$, $v_0 = \partial\psi/\partial x$, and $\eta = \psi - \theta$, and dividing by U/L gives

$$\frac{\partial E_P}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{F}_P = S_P, \tag{6.20}$$

where

$$E_P = \frac{1}{2} H_0 f_0^2 L^2 \epsilon^2 \left(|\boldsymbol{\nabla}\psi|^2 + \frac{\psi^2}{Bu} + O(\epsilon) \right), \tag{6.21a}$$

$$S_P = -H_0 f_0^2 L^2 \epsilon^2 \left(\frac{\lambda}{Bu} \psi \left(\theta + \psi \right) + O(\epsilon) \right), \tag{6.21b}$$

and

$$\boldsymbol{F}_P = H_0 f_0^2 L^2 \epsilon^2 \left(\boldsymbol{u}_0 \left(\frac{1}{2} |\boldsymbol{\nabla}\psi|^2 + \frac{\psi}{\epsilon} + \frac{1}{Bu}\left(\psi^2 + \psi\theta - \frac{3}{2}\theta^2 \right) \right) + \boldsymbol{u}_1 \psi + O(\epsilon) \right). \tag{6.21c}$$

The term proportional to $\psi\boldsymbol{u}_0 = -\boldsymbol{\nabla}\times(\hat{\boldsymbol{z}}\,\psi^2/2)$ has zero divergence, so may be omitted from $\boldsymbol{F}_P$. The vanishing of this otherwise asymptotically large term is essential for the rigorous mathematical theory of the QG limit (Majda

2003). Eliminating $\boldsymbol{u}_1$ using the thermal shallow water momentum equations (4.9*b*,*c*) gives

$$\boldsymbol{F}_P = H_0 f_0^2 L^2 \epsilon^2 \left(-\psi \boldsymbol{\nabla} \psi_t + \psi^2 \hat{\boldsymbol{z}} \times \boldsymbol{\nabla} \left(\frac{q}{2} - \frac{\theta}{Bu} \right) + O(\epsilon) \right), \tag{6.22}$$

which explains the origin of the time derivative term $\psi \boldsymbol{\nabla} \psi_t$. We thus recover the local thermal QG energy conservation equation

$$\frac{\partial}{\partial t} \left(\frac{|\boldsymbol{\nabla}\psi|^2}{2} + \frac{\psi^2}{2Bu} \right) + \boldsymbol{\nabla} \boldsymbol{\cdot} \left(-\psi \boldsymbol{\nabla} \psi_t + \psi^2 \hat{\boldsymbol{z}} \times \boldsymbol{\nabla} \left(\frac{q}{2} - \frac{\theta}{Bu} \right) \right) = -\frac{\lambda}{Bu} \psi \left(\theta + \psi \right) \tag{6.23}$$

at leading order in ϵ by applying the QG scalings to the local thermal shallow water pseudo-energy conservation equation (6.17). Although the derivation of the thermal QG equations (4.12) eliminates the ageostrophic velocity $\boldsymbol{u}_1$, we must calculate $\boldsymbol{u}_1$ to obtain the thermal QG energy equation as a limit of the corresponding thermal shallow water pseudo-energy equation. Setting $\theta = 0$ and $\lambda = 0$ in (6.23) gives the local energy conservation law for the QG form of the standard shallow water equations (e.g. Pedlosky 1987).

7. Non-canonical Hamiltonian structure of the thermal quasi-geostrophic equations

The undamped thermal QG equations (4.12) with $\lambda = 0$ have a non-canonical Hamiltonian structure, as given by Ripa (1996*b*). The field variables are $\boldsymbol{\eta} = (q, \theta)^{\mathsf{T}}$, and the Hamiltonian for an infinite domain is

$$\mathcal{H}_{\mathrm{QG}} = \frac{1}{2} \int |\boldsymbol{\nabla}\psi|^2 + \frac{\psi^2}{Bu} \, \mathrm{d}\boldsymbol{x}. \tag{7.1}$$

The Hamiltonian for a bounded domain contains additional terms (e.g. Holm 1986; Ripa 1993, 1996*b*) that enforce the constraints derived by McWilliams (1977) for the circulations around each connected component of the boundary. The variational derivatives of $\mathcal{H}_{\mathrm{QG}}$ are

$$\frac{\delta \mathcal{H}_{\mathrm{QG}}}{\delta q} = -\psi, \quad \frac{\delta \mathcal{H}_{\mathrm{QG}}}{\delta \theta} = \frac{\psi}{Bu}. \tag{7.2}$$

The evolution equations for q and θ may thus be written in matrix notation as

$$\frac{\partial \boldsymbol{\eta}}{\partial t} = J_{QG} \frac{\delta \mathcal{H}_{\mathrm{QG}}}{\delta \boldsymbol{\eta}}, \quad \text{with} \quad J_{QG} = -\begin{pmatrix} [q, \cdot] & [\theta, \cdot] \\ [\theta, \cdot] & 0 \end{pmatrix}. \tag{7.3}$$

The corresponding Poisson bracket is

$$\{\mathcal{F}, \mathcal{G}\} = \int q \left[\frac{\delta \mathcal{F}}{\delta q}, \frac{\delta \mathcal{G}}{\delta q} \right] + \theta \left(\left[\frac{\delta \mathcal{F}}{\delta q}, \frac{\delta \mathcal{G}}{\delta \theta} \right] + \left[\frac{\delta \mathcal{F}}{\delta \theta}, \frac{\delta \mathcal{G}}{\delta q} \right] \right) \mathrm{d}\boldsymbol{x}. \tag{7.4}$$

Morrison & Hazeltine (1984) derived this Poisson bracket for two-dimensional reduced magnetohydrodynamics, in which θ is the magnetic flux function, verified that it satisfies the Jacobi identity, and found its Casimir functionals

$$\mathcal{C}_{QG} = \int qF(\theta) + G(\theta) \, \mathrm{d}\boldsymbol{x}, \tag{7.5}$$

for arbitrary functions $F(\theta)$ and $G(\theta)$. Taking $G(\theta) = 0$, and $F(\theta) = H(\theta - \theta_*)$ to be the step function that is unity for $\theta \geqslant \theta_*$ and zero otherwise, we find from (7.5) that the total potential vorticity inside any contour of constant θ is conserved, in the sense that

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{S(\theta_*)} q \, \mathrm{d}\boldsymbol{x} = 0, \tag{7.6}$$

where $S(\theta_*)$ is the material surface bounded by the $\theta = \theta_*$ contour.

7.1. *Derivation of the thermal quasi-geostrophic Hamiltonian and Poisson bracket*

We now derive this Hamiltonian structure from the pseudo-energy (6.15) and Poisson bracket (3.9) of the undamped thermal shallow water equations. Bokhove (2002) gave such a derivation for the standard shallow water equations using variables similar to those in §5, but we prefer to follow the simpler change of variables used by Tassi *et al.* (2009) for an analogous derivation of the Charney–Hasegawa–Mima equation in plasma physics.

We first make a change of variables $(\boldsymbol{u}, h, \Theta) \mapsto \left(\mathit{\Pi}, D, \check{h}, \check{\Theta}\right)$ defined by

$$\mathit{\Pi} = \frac{1}{h} \left(f_0 + \beta^* y + \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right), \quad D = \boldsymbol{\nabla} \boldsymbol{\cdot} \boldsymbol{u}, \quad \check{h} = h, \quad \check{\Theta} = \Theta, \tag{7.7}$$

which corresponds to a Helmholtz decomposition of $\boldsymbol{u}$ with potentials $h\mathit{\Pi}$ and D. This decomposition has the unique inverse

$$\boldsymbol{u} = \hat{\boldsymbol{z}} \times \boldsymbol{\nabla} \Delta^{-1} (\mathit{\Pi} \check{h} - f_0 - \beta^* y) + \boldsymbol{\nabla} \Delta^{-1} D, \quad h = \check{h}, \quad \Theta = \check{\Theta}, \tag{7.8}$$

where the Green's function Δ^{-1} for the two-dimensional Laplacian is defined by

$$\Delta^{-1}F = -\frac{1}{2\pi}\int \ln|\boldsymbol{x}-\boldsymbol{x}'|F(\boldsymbol{x}')\,\mathrm{d}\boldsymbol{x}'. \tag{7.9}$$

We introduce the hat notation for $\check{h}$ and $\check{\Theta}$ so that $\mathcal{F}_{\check{h}} = \delta\mathcal{F}/\delta\check{h}$ denotes the variational derivative of $\mathcal{F}$ with respect to $\check{h}$ keeping $\mathit{\Pi}$, D and $\check{\Theta}$ fixed, while $\mathcal{F}_h$ denotes the variational derivative keeping $\boldsymbol{u}$ and Θ fixed.

In terms of these new variables, the pseudo-energy (6.15) becomes

$$\begin{aligned}\mathcal{A}_{\mathrm{R}} = \int\ & \frac{1}{2}\check{h}\left(|\boldsymbol{\nabla}\Delta^{-1}(\mathit{\Pi}\check{h}-f_0-\beta^* y)|^2 + 2\left[\Delta^{-1}(\mathit{\Pi}\check{h}-f_0-\beta^* y),\Delta^{-1}D\right] + |\boldsymbol{\nabla}\Delta^{-1}D|^2\right)\\ &+\ \frac{1}{2}\check{\Theta}\check{h}^2 - \sqrt{\check{\Theta}\Theta_0}\check{h}H_0 + \frac{1}{2}\Theta_0 H_0^2\,\mathrm{d}\boldsymbol{x},\end{aligned} \tag{7.10}$$

using the result $|\hat{\boldsymbol{z}}\times\boldsymbol{\nabla}a + \boldsymbol{\nabla}b|^2 = |\boldsymbol{\nabla}a|^2 + 2\,[a,b] + |\boldsymbol{\nabla}b|^2$ for all scalar fields a and b. The Poisson bracket becomes

$$\begin{aligned}\{\mathcal{F},\mathcal{G}\} = \int & \mathit{\Pi}\left\{[\mathcal{F}_D,\mathcal{G}_D] + \left[\mathcal{F}_{\mathit{\Pi}}/\check{h},\mathcal{G}_{\mathit{\Pi}}/\check{h}\right] + \boldsymbol{\nabla}\mathcal{F}_D\cdot\boldsymbol{\nabla}\left(\mathcal{G}_{\mathit{\Pi}}/\check{h}\right) - \boldsymbol{\nabla}\mathcal{G}_D\cdot\boldsymbol{\nabla}\left(\mathcal{F}_{\mathit{\Pi}}/\check{h}\right)\right\}\\ &+\mathcal{F}_{\check{h}}\boldsymbol{\nabla}^2\mathcal{G}_D - \mathcal{G}_{\check{h}}\boldsymbol{\nabla}^2\mathcal{F}_D\\ &+\frac{1}{\check{h}}\left\{\mathcal{G}_{\check{\Theta}}\left[\check{\Theta},\mathcal{F}_{\mathit{\Pi}}/\check{h}\right] - \mathcal{F}_{\check{\Theta}}\left[\check{\Theta},\mathcal{G}_{\mathit{\Pi}}/\check{h}\right] + \mathcal{F}_{\check{\Theta}}\boldsymbol{\nabla}\check{\Theta}\cdot\boldsymbol{\nabla}\mathcal{G}_D - \mathcal{G}_{\check{\Theta}}\boldsymbol{\nabla}\check{\Theta}\cdot\boldsymbol{\nabla}\mathcal{F}_D\right\}\mathrm{d}\boldsymbol{x},\end{aligned} \tag{7.11}$$

where $\mathcal{F}_D = \delta\mathcal{F}/\delta D$ etc. We now apply the QG scalings to (7.10) and (7.11) using (4.1), scaling h with H_0 and Θ with Θ_0. This equates to scaling $\mathit{\Pi}$ with f_0/H_0, D with U/L, $\check{h}$ with H_0, and $\check{\Theta}$ with Θ_0. We drop the tilde notation, with the understanding that all variables are now dimensionless, and decompose the dimensionless variables as

$$\boldsymbol{u} = \boldsymbol{u}_0 + \epsilon\boldsymbol{u}_1, \quad h = 1 + \frac{\epsilon}{Bu}\eta, \quad \Theta = 1 + 2\frac{\epsilon}{Bu}\theta, \tag{7.12}$$

where $\epsilon = Ro$, $u_0 = -\partial\psi/\partial y$, $v_0 = \partial\psi/\partial x$, $\eta = \psi - \theta$, and Ro and Bu are defined in (4.5). This corresponds to the following decomposition in our dimensionless new variables

$$\mathit{\Pi} = 1 + \epsilon q, \quad D = \epsilon D_1, \quad \check{h} = 1 + \frac{\epsilon}{Bu}\eta, \quad \check{\Theta} = 1 + 2\frac{\epsilon}{Bu}\theta, \tag{7.13}$$

where $q = \boldsymbol{\nabla}^2\psi - \eta/Bu + \beta y$, $D_1 = \boldsymbol{\nabla}\cdot\boldsymbol{u}_1$, and $\beta = \tilde{\beta}/\epsilon = \beta^* L^2/U$. Then the pseudo-energy (7.10) becomes

$$\mathcal{A}_{\mathrm{R}} = H_0 f_0^2 L^4\epsilon^2\left[\frac{1}{2}\int|\boldsymbol{\nabla}\Delta^{-1}(q+\eta/Bu-\beta y)|^2 + \frac{\psi^2}{Bu}\,\mathrm{d}\boldsymbol{x} + O(\epsilon)\right], \tag{7.14}$$

since $\int[a,b]\,\mathrm{d}\boldsymbol{x} = 0$ for all scalar fields a and b for which the integral over the boundary vanishes. Similarly the thermal shallow water Poisson bracket (7.11) becomes

$$\begin{aligned}\{\mathcal{F},\mathcal{G}\} = \frac{1}{H_0 f_0 L^4}\Bigg[&\frac{1}{\epsilon^3}\int q\,[\mathcal{F}_{D_1},\mathcal{G}_{D_1}] + \boldsymbol{\nabla}\mathcal{F}_{D_1}\cdot\boldsymbol{\nabla}\mathcal{G}_q - \boldsymbol{\nabla}\mathcal{G}_{D_1}\cdot\boldsymbol{\nabla}\mathcal{F}_q\\ &+Bu\mathcal{F}_\eta\boldsymbol{\nabla}^2\mathcal{G}_{D_1} - Bu\mathcal{G}_\eta\boldsymbol{\nabla}^2\mathcal{F}_{D_1}\,\mathrm{d}\boldsymbol{x} + \frac{1}{\epsilon^2}\int q\boldsymbol{\nabla}\mathcal{F}_{D_1}\cdot\boldsymbol{\nabla}\mathcal{G}_q\\ &-q\boldsymbol{\nabla}\mathcal{G}_{D_1}\cdot\boldsymbol{\nabla}\mathcal{F}_q + \frac{1}{Bu}\boldsymbol{\nabla}\mathcal{G}_{D_1}\cdot\boldsymbol{\nabla}\left(\eta\mathcal{F}_q\right) - \frac{1}{Bu}\boldsymbol{\nabla}\mathcal{F}_{D_1}\cdot\boldsymbol{\nabla}\left(\eta\mathcal{G}_q\right)\\ &+\mathcal{F}_\theta\boldsymbol{\nabla}\theta\cdot\boldsymbol{\nabla}\mathcal{G}_{D_1} - \mathcal{G}_\theta\boldsymbol{\nabla}\theta\cdot\boldsymbol{\nabla}\mathcal{F}_{D_1}\,\mathrm{d}\boldsymbol{x} + \frac{1}{\epsilon}\int q\,[\mathcal{F}_q,\mathcal{G}_q] + \mathcal{G}_\theta\,[\theta,\mathcal{F}_q]\\ &-\mathcal{F}_\theta\,[\theta,\mathcal{G}_q] + \frac{q}{Bu}\boldsymbol{\nabla}\mathcal{G}_{D_1}\cdot\boldsymbol{\nabla}\left(\eta\mathcal{F}_q\right) - \frac{q}{Bu}\boldsymbol{\nabla}\mathcal{F}_{D_1}\cdot\boldsymbol{\nabla}\left(\eta\mathcal{G}_q\right)\\ &+\frac{\eta}{Bu}\mathcal{G}_\theta\boldsymbol{\nabla}\theta\cdot\boldsymbol{\nabla}\mathcal{F}_{D_1} - \frac{\eta}{Bu}\mathcal{F}_\theta\boldsymbol{\nabla}\theta\cdot\boldsymbol{\nabla}\mathcal{G}_{D_1}\,\mathrm{d}\boldsymbol{x} + O(1)\Bigg],\end{aligned} \tag{7.15}$$

using the transformations

$$\frac{\delta\mathcal{F}}{\delta\mathit{\Pi}} = \frac{H_0}{\epsilon f_0 L^2}\frac{\delta\mathcal{F}}{\delta q}, \quad \frac{\delta\mathcal{F}}{\delta D} = \frac{1}{\epsilon UL}\frac{\delta\mathcal{F}}{\delta D_1}, \quad \frac{\delta\mathcal{F}}{\delta\check{h}} = \frac{Bu}{\epsilon H_0 L^2}\frac{\delta\mathcal{F}}{\delta\eta}, \quad \frac{\delta\mathcal{F}}{\delta\check{\Theta}} = \frac{Bu}{2\epsilon\Theta_0 L^2}\frac{\delta\mathcal{F}}{\delta\theta}. \tag{7.16}$$

The evolution of any functional $\mathcal{F}$ is given by $\mathrm{d}\mathcal{F}/\mathrm{d}t = \{\mathcal{F},\mathcal{H}\} = \{\mathcal{F},\mathcal{A}_{\mathrm{R}}\}$, since $\mathcal{A}_{\mathrm{R}}$ only differs from the Hamiltonian $\mathcal{H}$ by the addition of a Casimir. Non-dimensionalising as above, and dropping the tildes, we obtain the dimensionless equation

$$\frac{\mathrm{d}\mathcal{F}}{\mathrm{d}t_{\mathrm{nd}}} = \{\mathcal{F},\mathcal{A}_{\mathrm{nd}}\}_{\mathrm{nd}}, \tag{7.17}$$

where $\mathcal{A}_{\mathrm{nd}} = \mathcal{A}_{\mathrm{R}}/(H_0 f_0^2 L^4\epsilon^2)$ is the dimensionless pseudo-energy, and $\{\cdot,\cdot\}_{\mathrm{nd}} = \epsilon H_0 f_0 L^4\{\cdot,\cdot\}$ is the dimensionless Poisson bracket. Retaining only the leading order term in $\mathcal{A}_{\mathrm{nd}}$ gives the dimensionless QG Hamiltonian,

$$\mathcal{A}_{\mathrm{nd}} = \frac{1}{2}\int|\boldsymbol{\nabla}\Delta^{-1}(q+\eta/Bu-\beta y)|^2 + \frac{\psi^2}{Bu}\,\mathrm{d}\boldsymbol{x} + O(\epsilon) = \mathcal{H}_{QG} + O(\epsilon), \tag{7.18}$$

since $q + \eta/Bu - \beta y = \boldsymbol{\nabla}^2\psi$. Since $\delta\mathcal{A}_{\rm nd}/\delta D_1 = 0$, we may neglect all terms in $\{\cdot,\cdot\}_{\rm nd}$ that contain variational derivatives with respect to D_1. This removes many otherwise formally large terms at $O(\epsilon^{-2})$ and $O(\epsilon^{-1})$ from the dimensionless Poisson bracket $\{\cdot,\cdot\}_{\rm nd}$. Retaining only the largest surviving term gives

$$\{\mathcal{F},\mathcal{G}\}_{QG} = \int q\,[\mathcal{F}_q,\mathcal{G}_q] + \mathcal{G}_\theta\,[\theta,\mathcal{F}_q] - \mathcal{F}_\theta\,[\theta,\mathcal{G}_q]\;{\rm d}\boldsymbol{x}. \tag{7.19}$$

This becomes synonymous with the Poisson bracket (7.4) for the undamped thermal QG equations after an integration by parts, since $\int a\,[b,c]\;{\rm d}\boldsymbol{x} = \int b\,[c,a]\;{\rm d}\boldsymbol{x}$ for all scalar fields a, b, c for which the integral over the boundary vanishes.

8. Conclusion

Thermal shallow water theory extends the ubiquitous workhorse of geophysical and planetary fluid dynamics by allowing horizontal variations of thermodynamic properties, such as potential temperature, within each layer. It offers single active layer descriptions of phenomena arising from differential heating and cooling in winds over lakes (Lavoie 1972), tropical oceans (Schopf & Cane 1983; McCreary & Yu 1992), oceanic mixed layers (Young 1994; Young & Chen 1995; Bleck *et al.* 1992), and coastal or boundary currents and fronts in the atmosphere (Anderson 1984) and oceans (McCreary & Kundu 1988; McCreary *et al.* 1991; Røed & Shi 1999; Eldevik 2002; Carbonel & Galeao 2007). Thermal shallow water theory has close connections with shallow water models of moist convection (Zeitlin 2007; Bouchut *et al.* 2009; Lambaerts *et al.* 2011), and also with shallow water magnetohydrodynamics (Gilman 2000) in which the advected scalar becomes the flux function for a horizontal magnetic field (Dellar 2002, 2003).

We have developed the theory of the QG limit of the thermal shallow water equations, as originally derived by Ripa (1996*b*). The thermal QG equations possess a baroclinic instability (Fukamachi *et al.* 1995; Ripa 1999) and thus capture at least some of the properties of a continuously stratified fluid within a single active layer. The corresponding instability only exists for two or more layers in the standard shallow water and QG theories. We have provided a more formal derivation of the QG equations as an expansion in a small Rossby number, an expansion that retains additional terms needed for the interpretation of the QG energy equation (see below). We have also derived a more accurate nonlinear balanced model by projecting onto the slow normal modes of the thermal shallow water equations. This balanced model reduces to the QG theory for small disturbances, and offers a route towards formulating a thermal QG theory on a sphere.

We have derived the QG local energy conservation equation from the QG limit of the local conservation of *pseudo*-energy in the thermal shallow water equations. The thermal QG evolution equations (4.12) form a closed system that does not involve the ageostrophic velocity $\boldsymbol{u}_1$. However, we needed to calculate $\boldsymbol{u}_1$ to interpret the QG energy flux as the limit of a pseudo-energy flux for the thermal shallow water equations. Finally, we derived a Hamiltonian and Poisson bracket for the thermal QG equations using a small Rossby number decomposition of a pseudo-energy and Poisson bracket for the thermal shallow water equations. The QG theory shares the non-canonical Hamiltonian structure of two-dimensional reduced magnetohydrodynamics (Morrison & Hazeltine 1984) and Gilman's (1967) QG magnetohydrodynamics, further cementing the parallels between the thermal and magnetohydrodynamic extensions of the shallow water equations (Dellar 2003).

In future work we will explore the rôle of the temperature field in the self-organisation of geostrophic turbulence on β-planes and spheres. We will use these equations to model the formation and persistence of zonal jets in the atmospheres of gas giant planets, and in terrestrial oceans.

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Appendix A. Variational derivation of the thermal shallow water equations

We derive the undamped thermal shallow water equations (2.1) from Hamilton's principle, that the evolution of a mechanical system makes the action

$$\mathcal{S} = \int \mathcal{L}\,{\rm d}\tau \tag{A 1}$$

stationary over all variations that vanish at the end-points of the integration in τ. The Lagrangian for a three-dimensional Boussinesq fluid in a frame rotating with constant angular velocity $\boldsymbol{\Omega}_3 = \Omega\hat{\boldsymbol{z}}$ is (Eckart 1960; Salmon 1982, 1983, 1988*a*, 1998)

$$\mathcal{L} = \int \frac{1}{2}\left|\frac{\partial \boldsymbol{x}_3}{\partial\tau} + \boldsymbol{\Omega}_3\times\boldsymbol{x}_3\right|^2 - \frac{1}{2}\left|\boldsymbol{\Omega}_3\times\boldsymbol{x}_3\right|^2 - \Theta(\boldsymbol{a}_3)z + p\,(\boldsymbol{a}_3,\tau)\left[\frac{\partial\,(x,y,z)}{\partial\,(a,b,c)} - \frac{1}{\rho_0}\right]{\rm d}\boldsymbol{a}_3. \tag{A 2}$$

Fluid elements are described by their positions $\boldsymbol{x}_3\,(\boldsymbol{a}_3,\tau)$ as functions of the Lagrangian labels $\boldsymbol{a}_3 = (a,b,c)$ and time τ. The subscript three denotes a three-component vector, e.g. $\boldsymbol{x}_3 = (x,y,z)$, while un-subscripted vectors such as $\boldsymbol{x} = (x,y)$ are purely horizontal. We use τ to emphasise that partial time derivatives $\partial/\partial\tau$ are taken at fixed particle labels $\boldsymbol{a}_3$, as opposed to fixed positions $\boldsymbol{x}_3$. Thus $\partial/\partial\tau = \partial/\partial t + (\boldsymbol{u}_3 \cdot \boldsymbol{\nabla}_3)$ corresponds to a material derivative, and $\boldsymbol{u}_3 = \partial\boldsymbol{x}_3/\partial\tau$ is the Eulerian fluid velocity seen in the rotating frame. We use a coordinate system in which the geopotential gradient $\boldsymbol{\nabla}_3\Phi = -g\hat{\boldsymbol{z}}$ is vertically upward, and the horizontal coordinates x and y are tangent to the surfaces of constant geopotential (e.g. Dellar 2011).

The first term in (A 2) is the kinetic energy seen in an inertial frame. The second term subtracts the contribution to the kinetic energy that gives rise to the centrifugal force, since this is conventionally included in the gravitational potential energy appearing in the third term. We assign the labels $\boldsymbol{a}_3$ so that the density ρ is given by the reciprocal of the Jacobian of the label to particle map,

$$\frac{1}{\rho} = \frac{\partial\,(x,y,z)}{\partial\,(a,b,c)}. \tag{A 3}$$

The fourth term thus enforces incompressibility by imposing $\rho = \rho_0$ is constant using the pressure $p\,(\boldsymbol{a}_3,\tau)$ as a Lagrange multiplier (Eckart 1960). The buoyancy Θ is an advected scalar, $\partial\Theta/\partial\tau = 0$, so we write $\Theta = \Theta(\boldsymbol{a}_3)$ with no τ dependence, as in Seliger & Whitham's (1968) treatment of the entropy for a compressible fluid. We absorb the gravitational acceleration g into Θ, as in the main text. Taking variations of the action, defined by (A 1) and (A 2), separately with respect to p and $\boldsymbol{x}_3$ yields the three-dimensional Boussinesq equations.

A.1. *Shallow water scaling*

We consider a thin layer of fluid between a rigid surface at $z = 0$, and a free surface at $z = h(x,y,t)$. This is configuration with the free surface at the top, because it is the one used in previous Lagrangian derivations of the shallow water equations (Salmon 1983, 1988*a*; Miles & Salmon 1985). Our expression (A 2)for the Lagrangian implicitly imposes the correct free-surface boundary condition that the external fluid above $z = h(x,y,t)$ does no work on the layer (Miles & Salmon 1985; Stewart & Dellar 2010).

We introduce the non-dimensionalisation

$$\begin{aligned} &\boldsymbol{x}_3 = L\,(\tilde{x},\tilde{y},\delta\tilde{z})\,, \quad \boldsymbol{u}_3 = U\,(\tilde{u},\tilde{v},\delta\tilde{w})\,, \quad \tau = (L/U)\,\tilde{\tau}, \quad \boldsymbol{\Omega}_3 = \Omega\hat{\boldsymbol{z}}, \quad \Theta = \Theta_0\tilde{\Theta}, \\ &p = 2\Omega UL\rho_0\tilde{p}, \quad \boldsymbol{a}_3 = \rho_0^{1/3}L\left(\tilde{a},\tilde{b},\delta\tilde{c}\right), \quad h = \delta L\tilde{h}, \quad \mathcal{L} = 2\delta\Omega UL^4\rho_0\tilde{\mathcal{L}}. \end{aligned} \tag{A 4}$$

Here L is the horizontal length scale, H is the vertical length scale, and their ratio $\delta = H/L \ll 1$ is the small parameter that characterises the shallowness of the layer. The magnitude of the angular velocity is denoted by Ω, Θ_0 is a typical value of Θ, U is a horizontal velocity scale, and tildes denote dimensionless variables. The incompressibility condition becomes

$$\frac{\partial\,(\tilde{x},\tilde{y},\tilde{z})}{\partial(\tilde{a},\tilde{b},\tilde{c})} = 1, \tag{A 5}$$

and the dimensionless Lagrangian is

$$\begin{aligned} \tilde{\mathcal{L}} = \int \frac{1}{2}\textit{Ro}\left\{\left(\frac{\partial\tilde{x}}{\partial\tilde{\tau}}\right)^2 + \left(\frac{\partial\tilde{y}}{\partial\tilde{\tau}}\right)^2 + \delta^2\left(\frac{\partial\tilde{z}}{\partial\tilde{\tau}}\right)^2\right\} + \frac{1}{2}\left(\frac{\partial\tilde{y}}{\partial\tilde{\tau}}\tilde{x} - \frac{\partial\tilde{x}}{\partial\tilde{\tau}}\tilde{y}\right) \\ -\frac{\textit{Bu}}{\textit{Ro}}\tilde{\Theta}\,(\tilde{\boldsymbol{a}}_3)\,\tilde{z} + \tilde{p}\,(\tilde{\boldsymbol{a}}_3,\tilde{\tau})\left[\frac{\partial\,(\tilde{x},\tilde{y},\tilde{z})}{\partial(\tilde{a},\tilde{b},\tilde{c})} - 1\right]\mathrm{d}\tilde{\boldsymbol{a}}_3, \end{aligned} \tag{A 6}$$

where $\textit{Ro} = U/\left(2\Omega L\right)$ and $\textit{Bu} = \Theta_0 H/\left(4\Omega^2L^2\right)$. The dimensionless rotation rate becomes $\tilde{\Omega} = 1$ under these scalings. We now drop the $O\left(\delta^2\right)$ term, and assume that *Ro* and *Bu* remain $O(1)$ as $\delta \to 0$. We also drop the tilde notation, with the understanding that all variables in this appendix are now dimensionless.

A.2. *Restriction to columnar motion*

We follow Salmon (1983, 1988*a*); Miles & Salmon (1985) and restrict the fluid to columnar motion by writing the horizontal particle positions and the buoyancy as $x = x\,(a,b,\tau)\,, y = y\,(a,b,\tau)\,, \Theta = \Theta\,(a,b)$ with no dependence on the vertical Lagrangian label c. Equation (A 5) then simplifies to

$$\frac{\partial\,(x,y)}{\partial\,(a,b)}\frac{\partial z}{\partial c} = 1. \tag{A 7}$$

We assign the labels c so that $c = 0$ on $z = 0$ and $c = 1$ on the free surface $z = h$. Equation (A 7) then gives

$$z = h(a,b,\tau)c, \text{ where } h\,(a,b,\tau) = \frac{\partial\,(a,b)}{\partial\,(x,y)}. \tag{A 8}$$

We substitute (A 8) into the Lagrangian (A 6), having dropped the $O\left(\delta^2\right)$ term, and we discard the term multiplied by the pressure, since the incompressibility constraint is now automatically satisfied. Performing the c integration gives the two-dimensional thermal shallow water Lagrangian

$$\mathcal{L}_{SW} = \frac{1}{2}\int \textit{Ro}\left\{\left(\frac{\partial x}{\partial\tau}\right)^2 + \left(\frac{\partial y}{\partial\tau}\right)^2\right\} + \left(\frac{\partial y}{\partial\tau}x - \frac{\partial x}{\partial\tau}y\right) - \frac{\textit{Bu}}{\textit{Ro}}\Theta\,(a,b)\,h\;\mathrm{d}\boldsymbol{a}. \tag{A 9}$$

From now on $\boldsymbol{u} = (u, v) = (\partial x/\partial\tau, \partial y/\partial\tau)$, where $\partial/\partial\tau$ is now defined by $\partial/\partial\tau = \partial/\partial t + (\boldsymbol{u}\cdot\boldsymbol{\nabla})$.

A.3. *Derivation of the mass and momentum equations*

Differentiating the definition of $h(a, b, \tau)$ in (A 8) with respect to τ gives

$$-\frac{1}{h}\frac{\partial h}{\partial\tau} = \frac{\partial\,(a,b)}{\partial\,(x,y)}\left(\frac{\partial\,(u,y)}{\partial\,(a,b)} + \frac{\partial\,(x,v)}{\partial\,(a,b)}\right) = \boldsymbol{\nabla}\cdot\boldsymbol{u}, \tag{A 10}$$

using the chain rule for Jacobians. The definition of $\partial/\partial\tau$ as a Lagrangian derivative then gives the thermal shallow water continuity equation (2.1a). The temperature equation (2.1b), with $\kappa = 0$, follows directly from the assumption that $\Theta = \Theta\,(a, b)$ with no explicit τ-dependence.

The thermal shallow water Lagrangian (A 9) depends on the time-dependent map $\boldsymbol{a} \to \boldsymbol{x}$ not only explicitly, but also implicitly via the Jacobian of this map that appears in h. Miles & Salmon (1985) derived the formula

$$\int F\delta h\,\mathrm{d}\boldsymbol{a} = \int \frac{1}{h}\boldsymbol{\nabla}\left(h^2F\right)\cdot\delta\boldsymbol{x}\,\mathrm{d}\boldsymbol{a}. \tag{A 11}$$

Using this formula to take variations of (A 9) with respect to $\boldsymbol{x}$, and integrating by parts, gives

$$\begin{aligned}\delta\int\mathcal{L}_{SW}\,\mathrm{d}\tau = \int\int\Big\{ &-Ro\frac{\partial^2x}{\partial\tau^2}\delta x - Ro\frac{\partial^2y}{\partial\tau^2}\delta y + \left(\frac{\partial y}{\partial\tau}\delta x - \frac{\partial x}{\partial\tau}\delta y\right)\\ &- \frac{1}{2}\frac{Bu}{Ro}\frac{1}{h}\boldsymbol{\nabla}\left(h^2\Theta\,(a,b)\right)\cdot\delta\boldsymbol{x}\Big\}\,\mathrm{d}\boldsymbol{a}\,\mathrm{d}\tau.\end{aligned} \tag{A 12}$$

Invoking Hamilton's principle, we obtain the equation

$$Ro\frac{\partial\boldsymbol{u}}{\partial\tau} + \hat{\boldsymbol{z}}\times\boldsymbol{u} = -\frac{Bu}{Ro}\left(\frac{1}{2h}\boldsymbol{\nabla}\left(h^2\Theta\right)\right), \tag{A 13}$$

which coincides with (2.1c) after recalling the definition of $\partial/\partial\tau$ as a material time derivative, re-dimensionalising using (A 4), and replacing 2Ω by f.

A.4. *The thermal shallow water potential vorticity equation*

We now follow Salmon (1983, 1998) in deriving a potential vorticity equation from the particle relabelling symmetry of the Lagrangian (A 9). According to Hamilton's principle, the action must be stationary under infinitesimal relabellings of the form $\boldsymbol{a} \to \boldsymbol{a}' = \boldsymbol{a} + \delta\boldsymbol{a}\,(a, b, \tau)$, provided that $\delta\boldsymbol{a}$ vanishes at the end-points of the τ integration. We consider labellings that leave the layer depth h, as defined by (A 8), unchanged. The chain rule for Jacobians then implies

$$\frac{\partial\,(a',b')}{\partial\,(a,b)} = \frac{\partial\,(a',b')}{\partial\,(x,y)}\frac{\partial\,(x,y)}{\partial\,(a,b)} = 1, \quad \text{so} \quad \frac{\partial\,(\delta a)}{\partial a} + \frac{\partial\,(\delta b)}{\partial b} = 0. \tag{A 14}$$

We may thus introduce a streamfunction $\delta\Psi\,(a, b, \tau)$ such that

$$\delta a = -\frac{\partial\,(\delta\Psi)}{\partial b}, \quad \delta b = \frac{\partial\,(\delta\Psi)}{\partial a}. \tag{A 15}$$

Putting $\boldsymbol{a} = \boldsymbol{a}' - \delta\boldsymbol{a}$ in the change of variables formula gives

$$\left.\frac{\partial\boldsymbol{x}}{\partial\tau}\right|_{\boldsymbol{a}'} = \left.\frac{\partial\boldsymbol{x}}{\partial\tau}\right|_{\boldsymbol{a}} + \frac{\partial\boldsymbol{x}}{\partial a}\left.\frac{\partial a}{\partial\tau}\right|_{\boldsymbol{a}'} + \frac{\partial\boldsymbol{x}}{\partial b}\left.\frac{\partial b}{\partial\tau}\right|_{\boldsymbol{a}'}, \tag{A 16}$$

so the variations in $\dot{\boldsymbol{x}} = \partial\boldsymbol{x}/\partial\tau$ and Θ due to relabelling are

$$\delta\dot{\boldsymbol{x}} = \left.\frac{\partial\boldsymbol{x}}{\partial\tau}\right|_{\boldsymbol{a}'} - \left.\frac{\partial\boldsymbol{x}}{\partial\tau}\right|_{\boldsymbol{a}} = -\left\{\frac{\partial\boldsymbol{x}}{\partial a}\frac{\partial\,(\delta a)}{\partial\tau} + \frac{\partial\boldsymbol{x}}{\partial b}\frac{\partial\,(\delta b)}{\partial\tau}\right\}, \tag{A 17a}$$

$$\delta\Theta = \Theta|_{\boldsymbol{a}'} - \Theta|_{\boldsymbol{a}} = -\frac{\partial\,(\delta\Psi)}{\partial b}\frac{\partial\Theta}{\partial a} + \frac{\partial\,(\delta\Psi)}{\partial a}\frac{\partial\Theta}{\partial b} = \frac{\partial\,(\delta\Psi,\Theta)}{\partial\,(a,b)}. \tag{A 17b}$$

We now consider variations of the two-dimensional thermal shallow water Lagrangian (A 9) due to this relabelling. Noting that h is unchanged by relabelling, we find

$$\delta\int\mathcal{L}_{SW}\,\mathrm{d}\tau = \int\left\{\int\int p_x\delta\dot{x} + p_y\delta\dot{y} - \frac{1}{2}\frac{Bu}{Ro}h\frac{\partial\,(\delta\Psi,\Theta)}{\partial\,(a,b)}\,\mathrm{d}\boldsymbol{a}\right\}\mathrm{d}\tau, \tag{A 18}$$

where the canonical momenta are

$$p_x = \frac{\delta\mathcal{L}_{SW}}{\delta\dot{x}} = Ro\frac{\partial x}{\partial\tau} - \frac{1}{2}y, \text{ and } p_y = \frac{\delta\mathcal{L}_{SW}}{\delta\dot{y}} = Ro\frac{\partial y}{\partial\tau} + \frac{1}{2}x. \tag{A 19}$$

Substituting for $\delta\dot{\boldsymbol{x}}$ using (A 17a), integrating by parts with respect to τ and $\boldsymbol{a}$, substituting for $\delta\boldsymbol{a}$ using (A 15), and integrating by parts again with respect to $\boldsymbol{a}$, we obtain

$$\delta \int \mathcal{L}_{SW}\, \mathrm{d}\tau = -\int \left\{ \int \left[\frac{\partial}{\partial \tau} \left\{ \frac{\partial}{\partial a} \left(p_x \frac{\partial x}{\partial b} + p_y \frac{\partial y}{\partial b} \right) - \frac{\partial}{\partial b} \left(p_x \frac{\partial x}{\partial a} + p_y \frac{\partial y}{\partial a} \right) \right\} \right.\right. \\ \left.\left. - \frac{1}{2} \frac{Bu}{Ro} \frac{\partial (h, \Theta)}{\partial (a, b)} \right] \delta \Psi \, \mathrm{d}\boldsymbol{a} \right\} \mathrm{d}\tau, \tag{A 20}$$

where we assume that $\delta \boldsymbol{a}$ vanishes at the end-points of the τ integration, and that $\delta\Psi$ vanishes at the end-points of the $\boldsymbol{a}$ integration. Hamilton's principle thus implies the evolution equation

$$\frac{\partial \Pi}{\partial \tau} = \frac{1}{2} \frac{Bu}{Ro} \frac{\partial (h, \Theta)}{\partial (a, b)} \tag{A 21}$$

for the dimensionless potential vorticity

$$\Pi = \frac{\partial (p_x, x)}{\partial (a, b)} + \frac{\partial (p_y, y)}{\partial (a, b)} = \frac{1}{h} \left(1 + Ro \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \right). \tag{A 22}$$

Rewriting (A 21) in Eulerian variables gives the dimensionless form of (2.8),

$$\frac{\partial \Pi}{\partial t} + (\boldsymbol{u} \cdot \boldsymbol{\nabla})\, \Pi = \frac{1}{2} \frac{Bu}{Ro} \frac{1}{h} \frac{\partial (h, \Theta)}{\partial (x, y)}. \tag{A 23}$$

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