1) Establish the classical analogue of Ehrenfest's theorem for observables in the Liouville equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \mathcal{O}\rangle = \langle\{\mathcal{O},\mathcal{H}\}\rangle.$$

2) The Hamiltonian for n identical particles interacting through a pairwise potential ϕ is

$$\mathcal{H} = \sum_{i=1}^{N}\frac{|\mathbf{p}_i|^2}{2m} + \sum_{1\le i<j\le N}\phi(|\mathbf{q}_i-\mathbf{q}_j|),$$

with expectation $\langle\mathcal{H}\rangle = \int \mathrm{d}V_1\ldots\mathrm{d}V_N\rho(\mathbf{p},\mathbf{q},t)\mathcal{H}(\mathbf{p},\mathbf{q})$, as defined in lectures.

a) Show that *[with* $1/2$ *in second term]*

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\mathcal{H}\rangle = \int \mathrm{d}V_1\frac{|\mathbf{p}_1|^2}{2m}\frac{\partial f_1}{\partial t} + \frac{1}{2}\int \mathrm{d}V_1\mathrm{d}V_2\,\phi(|\mathbf{q}_1-\mathbf{q}_2|)\frac{\partial f_2}{\partial t}.$$

b) Use the evolution equations for f_1 and f_2 from the BBGKY hierarchy to show that the above right hand side vanishes, assuming the usual decay conditions for f_1 and f_2 with large arguments.

3) Show that substituting the "mean-field" ansatz

$$f_2(\mathbf{p}_1,\mathbf{q}_1,\mathbf{p}_2,\mathbf{q}_2,t) = f_1(\mathbf{p}_1,\mathbf{q}_1,t)f_1(\mathbf{p}_2,\mathbf{q}_2,t)$$

into the first equation of the BBGKY hierarchy leads to the Vlasov equation *[with sign of RHS corrected]*

$$(\partial_t + (\mathbf{p}_1/m)\cdot\boldsymbol{\nabla})f_1(\mathbf{p}_1,\mathbf{q}_1,t) = \left(\int \mathrm{d}\mathbf{p}_2\int \mathrm{d}\mathbf{q}_2\, f_1(\mathbf{p}_2,\mathbf{q}_2,t)\frac{\partial\phi(|\mathbf{q}_1-\mathbf{q}_2|)}{\partial\mathbf{q}_1}\right)\cdot\frac{\partial f_1(\mathbf{p}_1,\mathbf{q}_1,t)}{\partial\mathbf{p}_1},$$

and interpret the term on the right hand side when ϕ is the Coulomb potential.

Think about which scaling regimes for the size and range of the interaction potential ϕ make sense for deriving the Vlasov and Boltzmann equations in the limit as $N\to\infty$. Hint, you may want to consider a potential $\Phi(|\mathbf{q}_i-\mathbf{q}_j|/d)$.

4) Using a suitably normalised velocity, show that the product of a Hermite polynomial and a rest-state Maxwell-Boltzmann distribution is an eigenfunction of the one-dimensional Lénard–Bernstein model collision operator

$$\mathsf{L}f = \frac{\partial}{\partial v}\left(vf + \frac{1}{2}\frac{\partial f}{\partial v}\right).$$

What are the eigenvalues? Can you find a related collision operator that also conserves momentum and energy?

5) Homoenergetic shearing flow

a) Show that a solution of the Boltzmann equation with the linearised collision operator for Maxwell molecules exists for which ρ is constant, $\mathbf{u} = \gamma y\hat{\boldsymbol{x}}$ is a steady linear shear at rate γ, and the components of the pressure tensor are spatially uniform, and satisfy (where I is the 3×3 identity matrix)

$$\mathsf{P} + \tau\left[\partial_t\mathsf{P} + \gamma\begin{pmatrix}2P_{xy} & P_{yy} & 0\\ P_{yy} & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\right] = -\frac{1}{3}\tau(P_{xx}+P_{yy}+P_{zz})\mathsf{I}.$$

b) Show that these equations for the components of P have solutions proportional to $\exp(\chi t/\tau)$, where χ is a root of

$$\chi(1+\chi)^2 = (2/3)\,(\gamma\tau)^2.$$

c) Show that if $\gamma\tau\ll 1$, the shear stress approaches the Navier–Stokes form $P_{xy} = -\mu\gamma$ at long times, with dynamic viscosity $\mu = \tau\rho\theta$.

This is a rare exact solution of the Boltzmann equation. For the BGK collision operator one can reconstruct f as well. See Cercignani (2000) section 2.2.

6) Consider the following model ODE system for one moment that is conserved by collisions, and one than is not:

$$\partial_t u + im = 0,$$
$$\partial_t m = -(m-u)/(\epsilon\tau).$$

Compare the result of a straightfoward expansion of u and m in ϵ with a multiple-scales expansion. Alternatively, expand only m as a series in ϵ, and find a closed evolution equation for the unexpanded function $u(t)$ at zeroth and first order in ϵ. How do these solutions compare with the exact solution of the system?

7) Fill in the details to derive the first correction $f^{(1)}$, the Navier–Stokes viscous stress, and Fourier's law from the Boltzmann–BGKW equation via the Chapman–Enskog expansion.