1) Establish the classical analogue of Ehrenfest’s theorem for observables in the Liouville equation:
\[
\frac{d}{dt} \langle O \rangle = \langle \{O, H\} \rangle.
\]

2) The Hamiltonian for \( n \) identical particles interacting through a pairwise potential \( \phi \) is
\[
H = \sum_{i=1}^{N} \frac{|p_i|^2}{2m} + \sum_{1 \leq i < j \leq N} \phi(|q_i - q_j|),
\]
with expectation \( \langle H \rangle = \int dV_1 \ldots dV_N \rho(p, q, t) H(p, q) \), as defined in lectures.

a) Show that \([\text{with } 1/2 \text{ in second term}]\)
\[
\frac{d}{dt} \langle H \rangle = \int dV_1 \frac{|p_1|^2}{2m} \frac{\partial f_1}{\partial t} + \frac{1}{2} \int dV_1 dV_2 \phi(|q_1 - q_2|) \frac{\partial f_2}{\partial t}.
\]

b) Use the evolution equations for \( f_1 \) and \( f_2 \) from the BBGKY hierarchy to show that the above right hand side vanishes, assuming the usual decay conditions for \( f_1 \) and \( f_2 \) with large arguments.

3) Show that substituting the “mean-field” ansatz
\[
f_2(p_1, q_1, p_2, q_2, t) = f_1(p_1, q_1, t) f_1(p_2, q_2, t)
\]
into the first equation of the BBGKY hierarchy leads to the Vlasov equation \([\text{with sign of RHS corrected}]\)
\[
(\partial_t + (p_1/m) \cdot \nabla) f_1(p_1, q_1, t) = \left( \int dp_2 \int dq_2 f_1(p_2, q_2, t) \frac{\partial \phi(|q_1 - q_2|)}{\partial q_1} \right) \cdot \frac{\partial f_1(p_1, q_1, t)}{\partial p_1},
\]
and interpret the term on the right hand side when \( \phi \) is the Coulomb potential.

Think about which scaling regimes for the size and range of the interaction potential \( \phi \) make sense for deriving the Vlasov and Boltzmann equations in the limit as \( N \to \infty \). Hint, you may want to consider a potential \( \phi(|q_i - q_j|/d) \).

4) Using a suitably normalised velocity, show that the product of a Hermite polynomial and a rest-state Maxwell-Boltzmann distribution is an eigenfunction of the one-dimensional Lénard–Bernstein model collision operator
\[
L f = \frac{\partial}{\partial v} \left( v f + \frac{1}{2} \frac{\partial f}{\partial v} \right).
\]

What are the eigenvalues? Can you find a related collision operator that also conserves momentum and energy?

5) Homogeneous shearing flow

a) Show that a solution of the Boltzmann equation with the linearised collision operator for Maxwell molecules exists for which \( \rho \) is constant, \( \mathbf{u} = \gamma y \hat{x} \) is a steady linear shear at rate \( \gamma \), and the components of the pressure tensor are spatially uniform, and satisfy (where \( I \) is the \( 3 \times 3 \) identity matrix)
\[
P + \tau \left[ \partial_t P + \gamma \begin{pmatrix} 2P_{xy} & P_{yy} & 0 \\ P_{yy} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right] = \frac{1}{3} (P_{xx} + P_{yy} + P_{zz}) I.
\]

b) Show that these equations for the components of \( P \) have solutions proportional to \( \exp(\chi t/\tau) \), where \( \chi \) is a root of
\[
\chi(1 + \chi)^2 = (2/3)(\gamma \tau)^2.
\]

c) Show that if \( \gamma \tau \ll 1 \), the shear stress approaches the Navier–Stokes form \( P_{xy} = -\mu \gamma \) at long times, with dynamic viscosity \( \mu = \tau \rho \theta \).

This is a rare exact solution of the Boltzmann equation. For the BGK collision operator one can reconstruct \( f \) as well. See Cercignani (2000) section 2.2.
6) Consider the following model ODE system for one moment that is conserved by collisions, and one than is not:
\[
\begin{align*}
\partial_t u + im &= 0, \\
\partial_t m &= -(m - u)/(\epsilon \tau).
\end{align*}
\]
Compare the result of a straightforward expansion of \(u\) and \(m\) in \(\epsilon\) with a multiple-scales expansion. Alternatively, expand only \(m\) as a series in \(\epsilon\), and find a closed evolution equation for the unexpanded function \(u(t)\) at zeroth and first order in \(\epsilon\). How do these solutions compare with the exact solution of the system?

7) Fill in the details to derive the first correction \(f^{(1)}\), the Navier–Stokes viscous stress, and Fourier’s law from the Boltzmann–BGKW equation via the Chapman–Enskog expansion.