1) Establish the classical analogue of Ehrenfest's theorem for observables in the Liouville equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \mathcal{O}\rangle = \langle \{\mathcal{O}, \mathcal{H}\}\rangle.$$

2) The Hamiltonian for n identical particles interacting through a pairwise potential ϕ is

$$\mathcal{H} = \sum_{i=1}^{N} \frac{|\mathbf{p}_i|^2}{2m} + \sum_{1\le i<j\le N} \phi(|\mathbf{q}_i - \mathbf{q}_j|),$$

with expectation $\langle \mathcal{H}\rangle = \int \mathrm{d}V_1 \ldots \mathrm{d}V_N \rho(\mathbf{p}, \mathbf{q}, t)\mathcal{H}(\mathbf{p}, \mathbf{q})$, as defined in lectures.

a) Show that

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \mathcal{H}\rangle = \int \mathrm{d}V_1 \frac{|\mathbf{p}_1|^2}{2m}\frac{\partial f_1}{\partial t} + \frac{1}{2}\int \mathrm{d}V_1 \mathrm{d}V_2\, \phi(|\mathbf{q}_1 - \mathbf{q}_2|)\frac{\partial f_2}{\partial t}.$$

b) Use the evolution equations for f_1 and f_2 from the BBGKY hierarchy to show that the above right hand side vanishes, assuming the usual decay conditions for f_1 and f_2 with large arguments.

3) Show that substituting the "mean-field" ansatz

$$f_2(\mathbf{p}_1, \mathbf{q}_1, \mathbf{p}_2, \mathbf{q}_2, t) = f_1(\mathbf{p}_1, \mathbf{q}_1, t) f_1(\mathbf{p}_2, \mathbf{q}_2, t)$$

into the first equation of the BBGKY hierarchy leads to the Vlasov equation

$$(\partial_t + (\mathbf{p}_1/m)\cdot\boldsymbol{\nabla}) f_1(\mathbf{p}_1, \mathbf{q}_1, t) = \left(\int \mathrm{d}\mathbf{p}_2 \int \mathrm{d}\mathbf{q}_2\, f_1(\mathbf{p}_2, \mathbf{q}_2, t)\frac{\partial\phi(|\mathbf{q}_1 - \mathbf{q}_2|)}{\partial \mathbf{q}_1}\right)\cdot \frac{\partial f_1(\mathbf{p}_1, \mathbf{q}_1, t)}{\partial \mathbf{p}_1},$$

and interpret the term on the right hand side when ϕ is the Coulomb potential.

Think about which scaling regimes for the size and range of the interaction potential ϕ make sense for deriving the Vlasov and Boltzmann equations in the limit as $N \to \infty$. Hint, you may want to consider a potential $\Phi(|\mathbf{q}_i - \mathbf{q}_j|/d)$.

4) Using a suitably normalised velocity, show that the product of a Hermite polynomial and a rest-state Maxwell-Boltzmann distribution is an eigenfunction of the one-dimensional Lénard–Bernstein model collision operator

$$\mathsf{L}f = \frac{\partial}{\partial v}\left(vf + \frac{1}{2}\frac{\partial f}{\partial v}\right).$$

What are the eigenvalues? Can you find a related collision operator that also conserves momentum and energy by adopting coefficients that depend upon integrals of f?

The "physicist's" Hermite polynomials are

$$H_n(v) = (-1)^n\, e^{v^2}\left(\frac{\mathrm{d}}{\mathrm{d}v}\right)^n e^{-v^2},$$

for $n = 0, 1, 2, \ldots$.

5) Homoenergetic shearing flow

a) Show that a solution of the Boltzmann equation with the linearised collision operator for Maxwell molecules exists for which ρ is constant, $\mathbf{u} = \gamma y \hat{\boldsymbol{x}}$ is a steady linear shear at rate γ, and the components of the pressure tensor are spatially uniform, and satisfy (where I is the 3×3 identity matrix)

$$\mathsf{P} + \tau \left[\partial_t \mathsf{P} + \gamma \begin{pmatrix} 2P_{xy} & P_{yy} & 0 \\ P_{yy} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right] = \frac{1}{3}(P_{xx} + P_{yy} + P_{zz})\mathsf{I}.$$

b) Show that these equations for the components of P have solutions proportional to $\exp(\chi t/\tau)$, where χ is a root of

$$\chi(1+\chi)^2 = (2/3)\,(\gamma\tau)^2.$$

c) Show that if $\gamma\tau \ll 1$, the shear stress approaches the Navier–Stokes form $P_{xy} = -\mu\gamma$ at long times, with dynamic viscosity $\mu = \tau\rho\theta$.

This is a rare exact solution of the Boltzmann equation. For the BGK collision operator one can reconstruct f as well. See Cercignani (2000) section 2.2.

6) Consider the following model ODE system for one moment that is conserved by collisions, and one than is not:

$$\partial_t u + im = 0,$$
$$\partial_t m = -(m-u)/(\epsilon\tau).$$

Compare the result of a straightfoward expansion of u and m in ϵ with a multiple-scales expansion. Alternatively, expand only m as a series in ϵ, and find a closed evolution equation for the unexpanded function $u(t)$ at zeroth and first order in ϵ. How do these solutions compare with the exact solution of the system?

7) Fill in the details to derive the first correction $f^{(1)}$, the Navier–Stokes viscous stress, and Fourier's law from the Boltzmann–BGKW equation via the Chapman–Enskog expansion.

As in the construction of the linearised collision operator, it is easiest to find $h = f^{(1)}/f^{(0)}$ by considering the evolution equation for $\log f^{(0)}$, and then write the $\mathbf{w}$-dependence of h using Grad's tensor Hermite polynomials,

$$1, \quad w_i, \quad w_i w_j - \Theta\delta_{ij}, \quad w_i w_j w_k - \Theta\left(w_i\delta_{jk} + w_j\delta_{ki} + w_k\delta_{ij}\right).$$

These polynomials are orthogonal with respect to the weight function $\exp(-|\mathbf{w}|^2/(2\Theta))$, which makes it relatively to find the necessary moments of your expression for $f^{(1)}$.