Hamiltonian and symmetric hyperbolic structures of shallow water magnetohydrodynamics

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Shallow water magnetohydrodynamics is a recently proposed model for a thin layer of incompressible, electrically conducting fluid. The velocity and magnetic field are taken to be nearly two dimensional, with approximate magnetohydrostatic balance in the perpendicular direction. In this paper a Hamiltonian description, with the ubiquitous non-canonical Lie-Poisson bracket for barotropic magnetohydrodynamics, is derived by integrating the three dimensional energy density in the perpendicular direction. Specialization to two dimensions yields an elegant form of the bracket, from which further conserved quantities (Casimirs) of shallow water magnetohydrodynamics are derived. These Casimirs closely resemble the Casimirs of incompressible reduced magnetohydrodynamics, so the stability properties of the two systems may be expected to be similar. The shallow water magnetohydrodynamics system is also cast into symmetric hyperbolic form. The symmetric and Hamiltonian properties become incompatible when the appropriate divergence-free constraint $\nabla \cdot (h\mathbf{B}) = 0$ is relaxed.

I. INTRODUCTION

The shallow water magnetohydrodynamics (SWMHD) equations were recently proposed by Gilman [1] as a model for phenomena in the solar tachocline [2], the thin layer between the outer turbulent convection zone, and the quiescent interior where heat transfer is predominantly radiative. The tachocline also marks a transition between an almost rigidly rotating interior, and an outer region where the angular velocity at fixed latitude is nearly independent of depth. The resulting strong shear across the tachocline may be expected to align any local magnetic field with the azimuthal direction. In fact, the tachocline may well be the site of the solar dynamo. As well as having interesting physical applications in their own right [1, 3, 4, 5, 6], the five-variable SWMHD equations may also shed light on, and aid the development of numerical algorithms for, the much more complicated eight-variable fully compressible non-isentropic gas magnetohydrodynamics (MHD) equations, just as the shallow water equations have played a similar rôle for more complex systems like the meteorological primitive equations in geophysical fluid dynamics [7, 8].

The SWMHD equations describe a thin layer of incompressible, electrically conducting fluid above a rigid, perfectly conducting horizontal boundary. The three dimensional fluid velocity $\mathbf{u}_3$ and magnetic field $\mathbf{B}_3$ are assumed to be predominantly horizontal, and functions of the two horizontal coordinates x and y only. The two solenoidal constraints $\nabla_3\cdot\mathbf{u}_3 = 0$ and $\nabla_3\cdot\mathbf{B}_3 = 0$, and the boundary conditions $u_z = 0$ and $B_z = 0$ on $z = 0$, then imply that $u_z = -z(\nabla\cdot\mathbf{u})$ and $B_z = -z(\nabla\cdot\mathbf{B})$ are both linear in the vertical coordinate z. Here ∇, $\mathbf{u}$, and $\mathbf{B}$ denote the horizontal (x and y) components of the three dimensional vectors ∇_3, $\mathbf{u}_3$, and $\mathbf{B}_3$ respectively. If L is a typical horizontal lengthscale, and h the layer depth, $u_z = O(h/L)|\mathbf{u}|$ and $B_z = O(h/L)|\mathbf{B}|$ are both small. Previously, two dimensional MHD models have been derived in the opposite limit with a strong vertical (more accurately, normal) magnetic field component, as present in fusion experiments in toroidal geometries, leading to approximately incompressible "reduced" MHD equations [9, 10] in the horizontal plane.

A small aspect ratio implies that the vertical acceleration may be neglected for the assumed nearly horizontal motions, since u_z and $\dot{u}_z$ should both be small, so the three-dimensional fluid pressure is given by the magnetohydrostatic approximation [1, 3]

$$\frac{\partial}{\partial z}(p + \frac{1}{2}\mathbf{B}^2) = \rho_0 g, \tag{1}$$

where ρ_0 is the (constant) density of the fluid making up the layer. This integrates to give

$$\int_0^h (p + \frac{1}{2}\mathbf{B}^2)dz = \frac{1}{2}g\rho_0 h^2, \tag{2}$$

where $h(x, y)$ is the layer depth. The SWMHD equations then follow from integrating the three-dimensional incompressible MHD equations in the vertical and absorbing ρ_0 into g [1]. A more compact derivation is given below that integrates the energy density in the vertical to obtain a Hamiltonian. Analogous derivations of the non-magnetic shallow water equations in Lagrangian variables via Hamilton's variational principle may be found in Refs. [7] and [11], and by scaling arguments in Refs. [7] and [8].

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The SWMHD equations may be written in conservative form as the hyperbolic system [3]

$$\partial_t \begin{pmatrix} h\mathbf{u} \\ h \\ h\mathbf{B} \end{pmatrix} + \nabla\cdot \begin{pmatrix} h\mathbf{u}\mathbf{u} - h\mathbf{B}\mathbf{B} + \frac{1}{2}gh^2\mathsf{I} \\ h\mathbf{u} \\ h\mathbf{u}\mathbf{B} - h\mathbf{B}\mathbf{u} \end{pmatrix} = 0, \tag{3}$$

subject to the constraint $\nabla\cdot(h\mathbf{B}) = 0$. Although the unmagnetized shallow water equations (SWE) coincide with the Euler equations for a barotropic fluid with density h and equation of state $p = \frac{1}{2}gh^2$, the SWMHD equations differ from the barotropic fluid MHD equations through the omission of an isotropic magnetic pressure term $\frac{1}{2}\mathbf{B}^2\mathsf{I}$. The magnetic pressure is already included in the $\frac{1}{2}gh^2$ term because the height is determined by the *total* pressure, fluid plus magnetic, balancing gravity in Eq. (1). Moreover, the total horizontal magnetic flux $h\mathbf{B}$ in a fluid column is conserved, rather than the pointwise magnetic field intensity $\mathbf{B}$.

II. HAMILTONIAN STRUCTURE

In the absence of shocks, the SWMHD equations conserve a total energy, or Hamiltonian, given by [1]

$$\mathcal{H} = \frac{1}{2}\int h(|\mathbf{u}|^2 + |\mathbf{B}|^2) + gh^2 dxdy. \tag{4}$$

This Hamiltonian may be derived by integrating the three dimensional energy density $\frac{1}{2}(|\mathbf{u}|^2 + |\mathbf{B}|)^2 + \rho g z$ in the vertical from $z = 0$ to $z = h(x, y)$, and discarding the contributions from u_z^2 and B_z^2 that are $O(h^2/L^2)$ smaller than the horizontal contributions. The pressure makes no contribution to the energy density for an incompressible fluid, since incompressibility in the Hamiltonian formulation is maintained by constraints on $\mathbf{u}$. This is in line with the general philosophy that kinematic constraints belong in the Poisson bracket, whereas dynamics are generated by the Hamiltonian. Conservation of $\mathcal{H}$ is also verified by direct calculation in Eq. (26) below.

In terms of the conserved variables ($\mathbf{m}$,h,$\mathbf{Q}$), where $\mathbf{m} = h\mathbf{u}$ and $\mathbf{Q} = h\mathbf{B}$, the Hamiltonian is

$$\mathcal{H} = \frac{1}{2}\int \frac{1}{h}(|\mathbf{m}|^2 + |\mathbf{Q}|)^2 + gh^2 dxdy, \tag{5}$$

with variational derivatives

$$\frac{\delta\mathcal{H}}{\delta\mathbf{m}} = \mathbf{u}, \quad \frac{\delta\mathcal{H}}{\delta h} = gh - \frac{1}{2}(|\mathbf{u}|^2 + |\mathbf{B}|^2), \quad \frac{\delta\mathcal{H}}{\delta\mathbf{Q}} = \mathbf{B}. \tag{6}$$

We write the Poisson bracket as

$$\{\mathcal{F}, \mathcal{G}\} = \int \left(\frac{\delta\mathcal{F}}{\delta m_i}, \frac{\delta\mathcal{F}}{\delta h}, \frac{\delta\mathcal{F}}{\delta Q_i}\right) \mathsf{J}_{ij} \begin{pmatrix} \delta\mathcal{G}/\delta m_j \\ \delta\mathcal{G}/\delta h \\ \delta\mathcal{G}/\delta Q_j \end{pmatrix} dxdy, \tag{7}$$

in terms of the cosymplectic operator (or Poisson tensor)

$$\mathsf{J}_{ij} = -\begin{pmatrix} m_j\partial_i + \partial_j m_i & h\partial_i & Q_j\partial_i - \partial_k Q_k\delta_{ij} \\ \partial_j h & 0 & 0 \\ \partial_j Q_i - Q_k\partial_k\delta_{ij} & 0 & 0 \end{pmatrix}, \tag{8}$$

where partial derivatives act on everything to their right. This Poisson bracket is manifestly bilinear and antisymmetric (after an integration by parts). Here, and subsequently, the fluid variables are assumed to satisfy suitable boundary conditions, such as decaying sufficiently rapidly at infinity, to justify the neglect of surface terms arising from an integration by parts. The necessary boundary conditions for a finite domain are $\mathbf{u}\cdot\mathbf{n} = 0$ and $\mathbf{B}\cdot\mathbf{n} = 0$, or impermeable and perfectly conducting boundaries [12].

This Poisson bracket was shown in Ref. [13] to satisfy the Jacobi identity $\{\{\mathcal{F}, \mathcal{G}\}, \mathcal{H}\} + \{\{\mathcal{G}, \mathcal{H}\}, \mathcal{F}\} + \{\{\mathcal{H}, \mathcal{F}\}, \mathcal{G}\} = 0$ for all functionals $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{H}$. It is, however, not in canonical form. Instead, each term is linear in one of the conserved variables ($\mathbf{m}, h, \mathbf{Q}$), and contains one spatial derivative. This is typical for hydrodynamic systems expressed in Eulerian variables [12, 14, 15]. In fact, the Poisson bracket with J given by Eq. (8) is the natural non-canonical Lie-Poisson bracket for a fluid system with an advected scalar density h, and an advected magnetic field $\mathbf{Q}$. This advective, or "semi-direct product" structure is responsible for the block of zeros in J outside the first row and first column. The different arrangement of indices in the $\mathbf{Q}$ terms is because the magnetic field is most naturally treated as a "two-form," a geometrical object defining the magnetic flux through surface elements, rather than as a vector like the momentum. The same Poisson bracket arose previously in conventional barotropic fluid magnetohydrodynamics [13, 16], and in special relativistic MHD [17], although these brackets directly involve $\mathbf{B}$ rather than $\mathbf{Q} = h\mathbf{B}$, since $\mathbf{B}$ is the conserved quantity under conventional MHD. Similarly, the non-magnetic part involving only $\mathbf{m}$ and h arose previously in various shallow water models [11, 18, 19].

Hamilton's evolution equations, $\partial_t \mathcal{F} = \{\mathcal{F}, \mathcal{H}\}$ for all functionals $\mathcal{F}$, then correspond to

$$\frac{\partial}{\partial t}\begin{pmatrix} m_i \\ h \\ Q_i \end{pmatrix} = \mathsf{J}_{ij}\begin{pmatrix} \delta\mathcal{H}/\delta m_j \\ \delta\mathcal{H}/\delta h \\ \delta\mathcal{H}/\delta Q_j \end{pmatrix}, \tag{9}$$

which coincide with the above conservative form of the SWMHD equations, provided $\nabla\cdot(h\mathbf{B}) = 0$. Since $\partial_t \nabla\cdot(h\mathbf{B}) = 0$ under the SWMHD equations, this divergence-free constraint on the magnetic field is usually (by analogy with the conventional gas MHD equations) treated as an initial condition [13, 16]. This Hamiltonian formulation offers a very compact derivation of the SWMHD equations by restricting the ubiquitous Lie-Poisson bracket, Eqs. (7) and (8), to two spatial coordinates, and integrating the three dimensional incompressible fluid energy density in the suppressed vertical coordinate to obtain the Hamiltonian.

III. ALTERNATIVE VARIABLES

The above formulation simplifies slightly when transformed to the variables $(\mathbf{u}, h, \mathbf{Q})$, in which the Poisson bracket becomes [16]

$$\begin{aligned} \{\mathcal{F}, \mathcal{G}\} \;=\; & \int \frac{1}{h}(\nabla\times\mathbf{u})\cdot\left(\frac{\delta\mathcal{F}}{\delta\mathbf{u}}\times\frac{\delta\mathcal{G}}{\delta\mathbf{u}}\right) + \left(\nabla\cdot\frac{\delta\mathcal{F}}{\delta\mathbf{u}}\right)\frac{\delta\mathcal{G}}{\delta h} - \left(\nabla\cdot\frac{\delta\mathcal{G}}{\delta\mathbf{u}}\right)\frac{\delta\mathcal{F}}{\delta h} \\ & + Q_i\frac{1}{h}\left[\frac{\delta\mathcal{G}}{\delta u_j}\partial_j\frac{\delta\mathcal{F}}{\delta Q_i} - \frac{\delta\mathcal{F}}{\delta u_j}\partial_j\frac{\delta\mathcal{G}}{\delta Q_i}\right] + Q_j\left[\frac{\delta\mathcal{F}}{\delta Q_i}\partial_j\left(\frac{1}{h}\frac{\delta\mathcal{G}}{\delta u_i}\right) - \frac{\delta\mathcal{G}}{\delta Q_i}\partial_j\left(\frac{1}{h}\frac{\delta\mathcal{F}}{\delta u_i}\right)\right] dxdy. \end{aligned} \tag{10}$$

This form emphasizes the rôle of the potential vorticity $h^{-1}(\nabla\times\mathbf{u})$, and extends easily to include the Coriolis force in a frame rotating with angular velocity $\mathbf{\Omega}$ by replacing $\nabla\times\mathbf{u}$ with $\nabla\times\mathbf{u} + 2\mathbf{\Omega}$ in the bracket [20]. Alternatively, the unmodified bracket in Eqs. (7) and (8) may be used provided the momentum is taken to be $\mathbf{m} = h(\mathbf{u} + \mathbf{R})$, where $\mathbf{R}$ is any vector potential for the Coriolis force with $\nabla\times\mathbf{R} = 2\mathbf{\Omega}$ [19]. This is equivalent to working with the momentum in an inertial frame, rather than the velocity in the rotating frame as in Eq. (10), and is simply a change of variables. In either case, the Hamiltonian comprises the kinetic energy in the rotating frame, plus the magnetic and gravitational potential energies. The centrifugal force is typically absorbed into the gravity, or may be included as a $\frac{1}{2}|\mathbf{\Omega}\times\mathbf{x}|^2$ potential in the Hamiltonian.

Equation (10) may be simplified slightly when $\nabla\cdot(h\mathbf{B}) = 0$ (or $\nabla\cdot\mathbf{Q} = 0$ since $\mathbf{Q} = h\mathbf{B}$) by rewriting the magnetic terms using double cross products (Eq. (6) of Ref. [16]). However, this simplied form does not satisfy the Jacobi identity unless $\nabla\cdot(h\mathbf{B}) = 0$. Further simplification is possible in two dimensions using the constraint $\nabla\cdot(h\mathbf{B}) = 0$ to write $h\mathbf{B} = \hat{\mathbf{z}}\times\nabla\psi = (-\psi_y, \psi_x, 0)$ in terms of a flux function ψ. The choice of ψ and the sign convention is the usual one in magnetohydrodynamics. The Poisson bracket then becomes

$$\{\mathcal{F}, \mathcal{G}\} = \int \left(\frac{\delta\mathcal{F}}{\delta u_x}, \frac{\delta\mathcal{F}}{\delta u_y}, \frac{\delta\mathcal{F}}{\delta h}, \frac{\delta\mathcal{F}}{\delta\psi}\right) \mathsf{J} \begin{pmatrix} \delta\mathcal{G}/\delta u_x \\ \delta\mathcal{G}/\delta u_y \\ \delta\mathcal{G}/\delta h \\ \delta\mathcal{G}/\delta\psi \end{pmatrix} dxdy, \tag{11}$$

with Poisson tensor

$$\mathsf{J} = -\begin{pmatrix} 0 & -q & \partial_x & -B_y \\ q & 0 & \partial_y & B_x \\ \partial_x & \partial_y & 0 & 0 \\ B_y & -B_x & 0 & 0 \end{pmatrix}, \tag{12}$$

where $q = h^{-1}\hat{\mathbf{z}}\cdot(\nabla\times\mathbf{u}+2\mathbf{\Omega})$ is the (scalar) potential vorticity [7, 8]. The upper left, purely hydrodynamic, 3×3 block appeared previously in Ref. [20]. The variables $(\mathbf{m}, h, \psi)$ used in Ref. [12] gave a rather more complicated structure than Eq. (12).

In the variables $(\mathbf{u}, h, \psi)$ the Hamiltonian is

$$\mathcal{H} = \frac{1}{2}\int h|\mathbf{u}|^2 + \frac{1}{h}|\nabla\psi|^2 + gh^2 dxdy, \tag{13}$$

with variational derivatives

$$\frac{\delta\mathcal{H}}{\delta\mathbf{u}} = h\mathbf{u}, \quad \frac{\delta\mathcal{H}}{\delta\psi} = -\nabla\cdot\left(\frac{1}{h}\nabla\psi\right) = -\hat{\boldsymbol{z}}\cdot(\nabla\times\mathbf{B}) = \frac{\partial B_x}{\partial y} - \frac{\partial B_y}{\partial x}, \quad \frac{\delta\mathcal{H}}{\delta h} = gh + \frac{1}{2}(|\mathbf{u}|^2 - |\mathbf{B}|^2). \tag{14}$$

Hamilton's equations for Eqs. (11), (12) and (13) then yield the SWMHD equations in primitive variable form,

$$\begin{aligned} \partial_t h + \nabla\cdot(\mathbf{u}h) &= 0, \quad \partial_t\psi + \mathbf{u}\cdot\nabla\psi = 0, \\ \partial_t\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + 2\boldsymbol{\Omega}\times\mathbf{u} &= -g\nabla h + \mathbf{B}\cdot\nabla\mathbf{B}, \end{aligned} \tag{15}$$

that includes the Coriolis force in a frame rotating with angular velocity $\boldsymbol{\Omega}$ as in Ref. [4].

IV. CONSERVED QUANTITIES

The unmagnetized shallow water equations (SWE) materially conserve the potential vorticity q, $Dq/Dt = \partial_q + \mathbf{u}\cdot\nabla q = 0$. This conservation law strongly constrains the qualitative behavior of the SWE, and other systems with the same property such as the meteorological primitive equations [7, 8]. While energy and momentum may be transported by waves, the potential vorticity q must remain tied to material fluid elements. In the Hamiltonian formalism this conservation property is expressed as the existence of a set of Casimir functionals [14, 19, 20]

$$\mathcal{C}_{\text{SW}} = \int hc(q)dxdy, \quad \text{for any function } c(q), \tag{16}$$

that are annihilated by the hydrodynamic ($\mathbf{B} = 0$) part of the Poisson bracket in Eq. (11). Since $\{\mathcal{C},\mathcal{F}\} = 0$ for all functionals $\mathcal{F}$, Hamilton's evolution equation then implies $\partial_t\mathcal{C} = \{\mathcal{C},\mathcal{H}\} = 0$. Thus Casimir functionals are conserved under evolution for all possible Hamiltonians.

The SWMHD equations do not materially conserve potential vorticity, as observed in Ref. [1]. Equivalently, Eq. (16) no longer defines a Casimir because the magnetic part of J acting on $\delta\mathcal{C}$ does not vanish in general. Gilman [1] left open the possibility of other conservation properties of the SWMHD equations, a question that we address by computing the Casimirs of the above Poisson bracket [14, 15, 21]. By solving a system of four first order partial differential equations, $\mathsf{J}\,\delta\mathcal{C} = 0$, the general Casimir of the SWMHD bracket is found to be (as in Ref. [12] for the non-rotating case)

$$\mathcal{C} = \int hf(\psi) + hqg(\psi)dxdy, \quad \text{for any functions } f(\psi), g(\psi). \tag{17}$$

These Casimirs are very similar to the analogous Casimirs for (two dimensional, incompressible) reduced MHD [10, 22]

$$\mathcal{C}_{\text{RMHD}} = \int f(\psi) + \omega g(\psi)dxdy, \quad \text{for any functions } f(\psi), g(\psi), \tag{18}$$

where $\omega = \hat{\boldsymbol{z}}\cdot(\nabla\times\mathbf{u})$ is the vorticity associated with an incompressible velocity field $\mathbf{u}$. The reduced MHD Poisson tensor is

$$\mathsf{J}_{\text{RMHD}} = -\begin{pmatrix} [\omega,\cdot] & [\psi,\cdot] \\ [\psi,\cdot] & 0 \end{pmatrix}, \tag{19}$$

in (ω,ψ) variables, where $[f,g] = (\partial_x f)(\partial_y g) - (\partial_y f)(\partial_x g)$ is the Jacobian of two functions.

The first term in Eq. (17) confirms that ψ is a materially conserved quantity, like the potential vorticity q in the non-magnetic shallow water equations. It also implies conservation of the total fluid mass $\int hdxdy$ through the special case $f(\psi) = 1$ and $g(\psi) = 0$. The second term in Eq. (17) may be rewritten as

$$\mathcal{C} = \int hqg(\psi)dxdy = \int\left(\frac{\partial\overline{v}}{\partial x} - \frac{\partial\overline{u}}{\partial y}\right)g(\psi)dxdy = -\int(\overline{v}\frac{\partial\psi}{\partial x} - \overline{u}\frac{\partial\psi}{\partial y})\frac{dg}{d\psi}dxdy = -\int h\overline{\mathbf{u}}\cdot\mathbf{B}\frac{dg}{d\psi}dxdy, \tag{20}$$

where $\overline{\mathbf{u}} = \mathbf{u} + \mathbf{R}$ is the velocity with respect to an inertial frame, so that $\nabla\times\overline{\mathbf{u}} = hq\hat{\boldsymbol{z}}$. For non-rotating systems the distinction between $\mathbf{u}$ and $\overline{\mathbf{u}}$ disappears. Equation (17) includes the global cross-helicity invariant $\int h\overline{\mathbf{u}}\cdot\mathbf{B}dxdy$ as the special case $g(\psi) = -\psi$. This cross-helicity invariant survives from the three dimensional barotropic case [21]. In fact, since Eq. (17) holds for arbitrary functions $g(\psi)$, the total cross-helicity inside each closed magnetic fieldline (line $\psi = cst$) is also conserved. Similarly, although the potential vorticity q is no longer materially conserved, the total potential vorticity inside each closed magnetic fieldline is conserved. Thus the Lorentz force due to the magnetic field only serves to redistribute potential vorticity inside annuli defined by neighboring closed magnetic fieldlines.

In three dimensional barotropic fluids there are other Casimirs involving the specific magnetic helicity $h^{-1}\mathbf{A}\cdot\mathbf{B}$, and its successive gradients along $\mathbf{B}$-lines [21]. These all vanish in two dimensions where the vector potential $\mathbf{A} = -\psi\hat{\boldsymbol{z}}$ is perpendicular to $\mathbf{B} = h^{-1}(-\psi_y,\psi_x,0)$. Similarly, non-barotropic fluids admit further Casimirs involving the directional entropy gradient $\mathbf{B}\cdot\nabla s$ [15, 21], the specific entropy s providing another frozen scalar field in non-barotropic fluids.

V. EXTENSION TO $\nabla\cdot(h\mathbf{B}) \neq 0$

Hyperbolic systems like the SWMHD equations are often solved numerically using Godunov-type methods [23]. These use the solution of the Riemann problem, the result of one-dimensional initial conditions with a single jump discontinuity, as a building block. Unfortunately, the one-dimensional SWMHD equations are degenerate [3], like the gas MHD equations [24, 25, 26], due to the constraint $\nabla\cdot(h\mathbf{B}) = 0$. If the variables $(\mathbf{m}, h, h\mathbf{B})$ are functions of one coordinate x only, the SWMHD equations imply $\partial_t(hB_x) = 0$ so there is no propagating Riemann wave associated with hB_x. There is therefore considerable interest in relaxing the constraint $\nabla\cdot(h\mathbf{B}) = 0$ so that conventional Riemann solver based methods may be used in magnetohydrodynamics. The expectation is that $\nabla\cdot(h\mathbf{B}) \approx 0$ will be maintained to numerical truncation error in a multidimensional sense, though the normal component hB_n will typically jump by an $O(1)$ amount across individual computational cell boundaries [25, 26].

The Poisson bracket defined by Eq. (7) satisfies the Jacobi identity irrespective of whether $\nabla\cdot(h\mathbf{B}) = 0$ [13], unlike the Poisson bracket originally proposed in Ref. [16]. If the assumption $\nabla\cdot(h\mathbf{B}) = 0$ is relaxed, the unmodified Hamiltonian structure given by Eqs. (7) and (8), with the Hamiltonian from Eq. (4), gives the equations

$$\partial_t \begin{pmatrix} h \\ h\mathbf{u} \\ h\mathbf{B} \end{pmatrix} + \nabla\cdot \begin{pmatrix} h\mathbf{u} \\ h\mathbf{uu} - h\mathbf{BB} + \frac{1}{2}gh^2\mathsf{I} \\ h\mathbf{uB} - h\mathbf{Bu} \end{pmatrix} = -\nabla\cdot(h\mathbf{B}) \begin{pmatrix} 0 \\ 0 \\ \mathbf{u} \end{pmatrix}, \tag{21}$$

with a source term proportional to $\nabla\cdot(h\mathbf{B})$ in the induction equation. (The expression "source term" is common in this context, even though the term actually involves spatial derivatives.) Analogous equations for Hamiltonian barotropic gas MHD with $\nabla\cdot\mathbf{B} \neq 0$ were given in Ref. [13]. This modification also serves to restore Galilean invariance when $\nabla\cdot(h\mathbf{B}) \neq 0$. In fact, the Poisson bracket given by Eqs. (7) and (8) was shown in Ref. [27] to generate the complete ten-parameter group of Galilean transformations.

By contrast, the extended set of gas MHD equations proposed by Powell [26] for non-isentropic gas MHD in the presence of magnetic monopoles ($\nabla\cdot\mathbf{B} \neq 0$) contain source terms in the momentum and energy equations as well (see Eq. (A1) in Appendix A). The analogous form for the SWMHD equations, as proposed by De Sterck [3], is

$$\partial_t \begin{pmatrix} h \\ h\mathbf{u} \\ h\mathbf{B} \end{pmatrix} + \nabla\cdot \begin{pmatrix} h\mathbf{u} \\ h\mathbf{uu} - h\mathbf{BB} + \frac{1}{2}gh^2\mathsf{I} \\ h\mathbf{uB} - h\mathbf{Bu} \end{pmatrix} = -\nabla\cdot(h\mathbf{B}) \begin{pmatrix} 0 \\ \mathbf{B} \\ \mathbf{u} \end{pmatrix}. \tag{22}$$

These equations are also Galilean invariant, but (like Powell's equations) do not conserve momentum due to the source term. Powell's equations for gas MHD were criticized by Janhunen [24] for not ensuring positivity of the Riemann problem. The solution of the Riemann problem for Powell's equations, for initial left and right states with positive gas pressures, may contain an intermediate state with an unphysical negative gas pressure. Positivity was restored (see Ref. [24]) by a system like Eq. (21) with a single source term in the induction equation (Eq. (A3) in Appendix A). In fact, the momentum (and energy) conserving form for non-isentropic gas MHD with $\nabla\cdot\mathbf{B} \neq 0$ follows naturally from the unmodified energy-momentum conservation equation of special relativistic MHD [25], as well as from the Hamiltonian structure. Conversely, arguments in favor of Powell's version of non-barotropic gas MHD are that the linear Riemann waves have a particularly simple structure, and that the system may be transformed into symmetric hyperbolic form [28]. The possible existence of an analogous symmetric form of Eq. (22) was left open in Ref. [3].

VI. SYMMETRIC STRUCTURE

The Hamiltonian density $\Phi = \frac{1}{2}h(|\mathbf{u}|^2 + |\mathbf{B}|^2) + \frac{1}{2}gh^2$ also serves as an entropy (or free energy) for the hyperbolic system in the sense of Harten [29], at least when $\nabla\cdot(h\mathbf{B}) = 0$. Thus the hyperbolic system (3) may be augmented by another conservation law,

$$\frac{\partial\Phi}{\partial t} + \nabla\cdot\mathbf{\Psi} \leq 0, \tag{23}$$

in the sense of distributions, with equality for smooth solutions (not shocks) [23, 29]. The entropy Φ must be a strongly convex function of the conserved variables $(\mathbf{m}, h, \mathbf{Q})$. The entropy flux $\mathbf{\Psi}$ for SWMHD is given in Eq. (26) below. Equation (23) is the generalisation of the rule that specific entropy s must increase across shocks in gas dynamics, except mathematicians prefer to work with convex entropy functions and so reverse the sign [23, 29]. Hyperbolic systems with this entropy property may be written in symmetric form by using new independent variables, the partial derivatives of the entropy with respect to the conserved variables. For the SWMHD equations these variables are $\boldsymbol{\xi} = (u_x, u_y, p, B_x, B_y)^{\mathsf{T}}$, where $p = gh - \frac{1}{2}(|\mathbf{u}|^2 + |\mathbf{B}|^2)$, coinciding with the variational derivatives of the Hamiltonian with respect to the conserved variables as given in Eq. (6).

Equation (22) may be rewritten symbolically as

$$\frac{\partial \boldsymbol{\eta}}{\partial t} + \frac{\partial}{\partial x}\mathbf{F}(\boldsymbol{\eta}) + \frac{\partial}{\partial y}\mathbf{G}(\boldsymbol{\eta}) = \mathsf{S}^{(x)}\frac{\partial \boldsymbol{\eta}}{\partial x} + \mathsf{S}^{(y)}\frac{\partial \boldsymbol{\eta}}{\partial y}, \tag{24}$$

where $\boldsymbol{\eta} = (m_x, m_y, h, Q_x, Q_y)^{\mathsf{T}}$ is the vector of conserved variables. The x and y-directional fluxes are $\mathbf{F}(\boldsymbol{\eta})$ and $\mathbf{G}(\boldsymbol{\eta})$ respectively, and $\mathsf{S}^{(x)}$ and $\mathsf{S}^{(y)}$ denote source terms involving $\nabla\cdot(h\mathbf{B})$ that cannot be written in conservative form. As found by Godunov [28] for the gas MHD equations, the original system (3) cannot be written in symmetric form without the addition of the particular set of source terms proportional to $\nabla\cdot(h\mathbf{B})$ that appear in Eq. (22). Symmetric forms were generalized to nonconservative systems by Le Floch [30]. In terms of the entropy variables $\boldsymbol{\xi} = (u_x, u_y, p, B_x, B_y)^{\mathsf{T}}$, Eq. (22) becomes

$$\frac{\partial \boldsymbol{\eta}}{\partial \boldsymbol{\xi}}\frac{\partial \boldsymbol{\xi}}{\partial t} + \left(\frac{\partial \mathbf{F}}{\partial \boldsymbol{\xi}} - \mathsf{S}^{(x)}\frac{\partial \boldsymbol{\eta}}{\partial \boldsymbol{\xi}}\right)\frac{\partial \boldsymbol{\xi}}{\partial x} + \left(\frac{\partial \mathbf{G}}{\partial \boldsymbol{\xi}} - \mathsf{S}^{(y)}\frac{\partial \boldsymbol{\eta}}{\partial \boldsymbol{\xi}}\right)\frac{\partial \boldsymbol{\xi}}{\partial y} = 0, \tag{25}$$

which comprises a symmetric hyperbolic system because $\mathsf{A}^{(0)} = \partial\boldsymbol{\eta}/\partial\boldsymbol{\xi}$ is symmetric positive definite, and the two matrices $\mathsf{A}^{(x)} = \partial\mathbf{F}/\partial\boldsymbol{\xi} - \mathsf{S}^{(x)}\mathsf{A}^{(0)}$ and $\mathsf{A}^{(y)} = \partial\mathbf{G}/\partial\boldsymbol{\xi} - \mathsf{S}^{(y)}\mathsf{A}^{(0)}$ are each symmetric. These matrices are listed in Appendix B. The inverse matrix ${\mathsf{A}^{(0)}}^{-1}$ is the Hessian matrix of Φ with respect to the conserved variables $(h, \mathbf{m}, \mathbf{Q})$, so $\mathsf{A}^{(0)}$ being positive definite implies that the entropy Φ is indeed a strongly convex function of these variables. The Jacobian matrices of the fluxes with respect to the entropy variables are not themselves symmetric, which is why the original hyperbolic system (3) could not be written in symmetric form without the addition of source terms proportional to $\nabla\cdot(h\mathbf{B})$.

However, it is disconcerting to note that the supposed entropy Φ is in fact not conserved by smooth solutions of the system (22) unless $\nabla\cdot(h\mathbf{B}) = 0$, because

$$\partial_t\left[\frac{1}{2}h(|\mathbf{u}|^2 + |\mathbf{B}|^2) + \frac{1}{2}gh^2\right] + \nabla\cdot\left[\frac{1}{2}h(|\mathbf{u}|^2 + |\mathbf{B}|^2)\mathbf{u} + gh^2\mathbf{u} - h\mathbf{u}\cdot\mathbf{B}\mathbf{B}\right] = -\mathbf{u}\cdot\mathbf{B}\nabla\cdot(h\mathbf{B}). \tag{26}$$

The source term proportional to $\nabla\cdot(h\mathbf{B})$ is due to the extra source term in the momentum equations in (22). Since the alternative system (21) lacks this term, and *a fortori* is Hamiltonian, it does conserve the total energy density even when $\nabla\cdot(h\mathbf{B}) \neq 0$. Physically, the system (22) exerts no force on magnetic monopoles, whereas in the system (21) they experience a force $\mathbf{B}\nabla\cdot(h\mathbf{B})$ analogous to the electrostatic force on a point electric charge [24]. The source term on the right hand side of Eq. (26) represents the missing work that would have been done by this force. Thus while Eq. (22) is the only form of the SWMHD equations that may be symmetrised using Φ, in fact Φ is only an entropy for the alternative Hamiltonian form given in Eq. (21). The shallow water equations are somewhat special, with the entropy coinciding with the energy density. This is not true in in general, and in particular is not true for non-barotropic gas dynamics or gas MHD. It remains to be seen whether the same restriction on when the symmetric equations actually conserves entropy carries over to non-barotropic gas MHD.

VII. CONCLUSION

The shallow water magnetohydrodynamics (SWMHD) equations have a Hamiltonian structure in terms of the ubiquitous noncanonical Lie-Poisson bracket describing barotropic fluids with magnetic fields in Eulerian variables. The Hamiltonian results from integrating the three-dimensional energy density in the suppressed vertical coordinate and discarding terms involving the small aspect ratio. The Coriolis force may be included via a minor modification to the bracket. The bracket takes a particularly elegant form in two dimensions and $(\mathbf{u}, h, \psi)$ variables, conveniently identifying quantities that are conserved under SWMHD evolution. These are the magnetic flux function ψ, which is a conserved scalar field, and the total vorticity $\int hq dx dy$ inside each closed magnetic field line $\psi = cst$. However, the potential vorticity q itself is not conserved, unlike the shallow water equations. Equivalent quantities are conserved by the two-dimensional, incompressible reduced MHD equations [10, 22], so the stability properties of shallow water MHD may be expected to resemble those of reduced MHD. Some stability results for two dimensional barotropic MHD, with the same Poisson bracket as SWMHD but a different Hamiltonian, were given in Ref. [12].

The SWMHD equations also have a symmetric hyperbolic form, one that differs from the Hamiltonian form unless the constraint $\nabla\cdot(h\mathbf{B})$=0 is satisfied. The two forms may be more or less useful for numerical simulation purposes in different situations, as are various forms of the $\mathbf{u}\cdot\nabla\mathbf{u}$ term in the incompressible Navier-Stokes equations [31]. The Hamiltonian form is strictly only valid in the absence of shocks, since a correct treatment of shocks requires dissipation. Shock formation is generally suppressed, if not entirely eliminated, in a rotating system through the dispersive effect of the Coriolis force. In particular, the Coriolis force was found in Ref. [4] to inhibit steepening of magnetogravitic waves in SWMHD. Thus the Hamiltonian form is likely to be beneficial, as in geophysical fluid dynamics. On the other hand, the symmetric form with its closer connection to an entropy, and thus a consistent form of dissipation, may be preferable when shocks are expected to form. The symmetric form has been used to advantage in finite element discretizations of the non-rotating shallow water equations [32]. However, the entropy associated with the symmetric form of the SWMHD equations is not actually conserved unless $\nabla\cdot(h\mathbf{B}) = 0$, so the implied numerical stability properties may be lost.

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APPENDIX A: THE GAS MHD EQUATIONS WITH $\nabla\cdot\mathbf{B}\neq 0$

As the gas magnetohydrodynamics equations are degenerate in one spatial dimension, admitting only seven Riemann waves for the eight dependent variables, Powell [26] proposed the modified equations

$$\frac{\partial}{\partial t}\begin{pmatrix}\rho\\ \rho\mathbf{u}\\ \mathbf{B}\\ U\end{pmatrix}+\nabla\cdot\begin{pmatrix}\rho\mathbf{u}\\ \rho\mathbf{u}\mathbf{u}+(p+\frac{1}{2}B^2)\mathsf{I}-\mathbf{B}\mathbf{B}\\ \mathbf{u}\mathbf{B}-\mathbf{B}\mathbf{u}\\ (U+p+\frac{1}{2}B^2)\mathbf{u}-(\mathbf{u}\cdot\mathbf{B})\mathbf{B}\end{pmatrix}=-\nabla\cdot\mathbf{B}\begin{pmatrix}0\\ \mathbf{B}\\ \mathbf{u}\\ \mathbf{u}\cdot\mathbf{B}\end{pmatrix}, \tag{A1}$$

that should hold even when $\nabla\cdot\mathbf{B}\neq 0$. Here ρ is the gas density (analogous to h in shallow water), U the total energy density, and the pressure p is determined from

$$p=(\gamma-1)\left(U-\frac{1}{2}\rho u^2-\frac{1}{2}B^2\right), \tag{A2}$$

where γ is the ratio of specific heats. By contrast, the alternative form

$$\frac{\partial}{\partial t}\begin{pmatrix}\rho\\ \rho\mathbf{u}\\ \mathbf{B}\\ U\end{pmatrix}+\nabla\cdot\begin{pmatrix}\rho\mathbf{u}\\ \rho\mathbf{u}\mathbf{u}+(p+\frac{1}{2}B^2)\mathsf{I}-\mathbf{B}\mathbf{B}\\ \mathbf{u}\mathbf{B}-\mathbf{B}\mathbf{u}\\ (U+p+\frac{1}{2}B^2)\mathbf{u}-(\mathbf{u}\cdot\mathbf{B})\mathbf{B}\end{pmatrix}=-\nabla\cdot\mathbf{B}\begin{pmatrix}0\\ 0\\ \mathbf{u}\\ 0\end{pmatrix}, \tag{A3}$$

was proposed by Janhunen [24], because the solution of the Riemann problem for Eq. (A1) may contain an unphysical intermediate state with negative gas pressure. This second form restores positivity, momentum and energy conservation, and arose from imposing electromagnetic duality invariance of the Lorentz force. It was subsequently shown by Dellar [25] to follow naturally from special relativistic magnetohydrodynamics. The analogous form for barotropic fluids in fact appeared previously in Ref. [13].

APPENDIX B: THE SYMMETRIZING MATRICES

The matrices giving the symmetrized form of the SWMHD equations in Sec. VI are

$$\mathsf{S}^{(x)} = -\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & B_x & 0 \\ 0 & 0 & 0 & B_y & 0 \\ 0 & 0 & 0 & u_x & 0 \\ 0 & 0 & 0 & u_y & 0 \end{pmatrix}, \quad \mathsf{S}^{(y)} = -\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & B_x \\ 0 & 0 & 0 & 0 & B_y \\ 0 & 0 & 0 & 0 & u_x \\ 0 & 0 & 0 & 0 & u_y \end{pmatrix}, \tag{B1}$$

$$\mathsf{A}^{(0)} = \frac{1}{g}\begin{pmatrix} 1 & u_x & u_y & B_x & B_y \\ u_x & u_x^2+gh & u_x\,u_y & B_x\,u_x & u_x\,B_y \\ u_y & u_x\,u_y & u_y^2+gh & B_x\,u_y & B_y\,u_y \\ B_x & B_x\,u_x & B_x\,u_y & B_x^2+gh & B_x\,B_y \\ B_y & u_x\,B_y & B_y\,u_y & B_x\,B_y & B_y^2+gh \end{pmatrix}, \tag{B2}$$

$$\mathsf{A}^{(x)} = \frac{1}{g}\begin{pmatrix} u_x & u_x^2+gh & u_xu_y & B_xu_x & u_xB_y \\ u_x^2+gh & u_x(u_x^2+3gh) & u_y(u_x^2+gh) & B_xu_x^2 & B_y(u_x^2+gh) \\ u_xu_y & u_y(u_x^2+gh) & u_x(u_y^2+gh) & B_xu_xu_y & B_yu_xu_y-B_xgh \\ B_xu_x & B_xu_x^2 & B_xu_xu_y & u_x(B_x^2+gh) & u_xB_xB_y \\ u_xB_y & B_y(u_x^2+gh) & B_yu_xu_y-B_xgh & u_xB_xB_y & u_x(B_y^2+gh) \end{pmatrix}, \tag{B3}$$

$$\mathsf{A}^{(y)} = \frac{1}{g}\begin{pmatrix} u_y & u_xu_y & u_y^2+gh & B_xu_y & B_yu_y \\ u_xu_y & u_y(u_x^2+gh) & u_x(u_y^2+gh) & B_xu_xu_y-B_ygh & B_yu_xu_y \\ u_y^2+gh & u_x(u_y^2+gh) & u_y(u_y^2+3gh) & B_x(u_y^2+gh) & B_yu_y^2 \\ B_xu_y & B_xu_xu_y-B_ygh & B_x(u_y^2+gh) & u_y(B_x^2+gh) & u_yB_xB_y \\ B_yu_y & B_yu_xu_y & B_yu_y^2 & u_yB_xB_y & u_y(B_y^2+gh) \end{pmatrix}. \tag{B4}$$

The top left 3×3 blocks of the three A matrices coincide with those found by Hauke [32] for the symmetric form of the non-magnetic shallow water equations.

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