Last Time:

Cor 32: W_J is the Cox gp of type M_J

Cor 33: $C(W)$ is a simplicial complex

Thm 38: $\begin{cases} \ell(wr_i) > \ell(w) \Rightarrow wr_i > 0 \\ \ell(wr_i) < \ell(w) \Rightarrow we_i < 0 \end{cases}$

Consequences: (σ, V) is faithful

& W finite, V is irreducible, $(\cdot,\cdot)$ is Eucl.

We define H_i, $H_i^{+/-}$, $i \in I$ spaces in V
and $C = \bigcap_i H_i^+$

Prop 42 (a) $Wv \cap \overline{C} \neq \emptyset$, $\forall v$

(b) $w(u) = v$, $u, v \in \overline{C} \Rightarrow u = v$.

Pf: assume $\ell(w) > 0$. $\exists i$ s.t. $\ell(wr_i) < \ell(w)$

$\Rightarrow 0 \geq (we_i, v) = (e_i, u) \geq 0$

$\Rightarrow s_i(u) = u$: $\ell(wr_i) < \ell(w)$ and $\sigma(wr_i)(u) = v$ $\Rightarrow u = v$ □

Cor 43: Let $C_J = \bigcap_{j \in J} H_j \cap \bigcap_{i \notin J} H_i^+$

Then $\mathrm{Stab}_W(C_J) = W_J$.

Pf: Note $\bigcup_{J \subset I} C_J = \overline{C}$.

Suppose $w(v) = v$, $\forall v \in C_J$. $\exists j$ s.t. $\ell(wr_j) < \ell(w)$

$\Rightarrow 0 \geq (we_j, v) = (e_j, v) \geq 0$

$\Rightarrow s_j(v) = v \Rightarrow$ $\begin{cases} \ell(wr_j) < \ell(w) \\ \sigma(wr_j)(v) = v \end{cases}$ $\Rightarrow wr_j \in W_J$

$\therefore j \in J$ $\Rightarrow w \in W_J$ □

Thm 44 if W finite, $C(W) \cong S^{n-1}$

Pf:

$$\begin{tikzcd}
\bigcup_{\substack{J \subseteq I \\ w \in W^J}} \sigma = wW_J \arrow[r] \arrow[dr] & \bigcup_{\substack{J \subset I \\ w \in W^J}} w\overline{C}_J \cap S^{n-1} \arrow[d, phantom, "\shortparallel"] \\
& C(W) \arrow[u, "\cong"]
\end{tikzcd}$$

S^{n-1}

$\square$

§3. Buildings

- Def. of buildings: Let us use W, M, I of last section

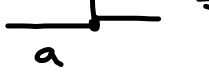
Def 1: A building of type M is a chamber system Δ over I s.t. each panel of Δ lies in at least 2 chambers and together with $\delta: \Delta \times \Delta \longrightarrow W$ s.t.

if p reduced, $\delta(x,y) = r_p \iff \exists\, \gamma = (x, \dots, y)$ of type p.

Example 2. $C(W)$ is a building with $\delta_W(x,y) = x^{-1}y$.

Remark suppose (a,b,c) of type ii in Δ

if $c = a \Rightarrow$ then (a) is a gall. conn. a, r.

if $c \neq a$ 

$\Rightarrow$ can substitute by (a,c) of type i.

Prop 3. Conseq. of definition

(a) Δ is conn., δ is onto, $\delta(x,y) = \delta(y,x)^{-1}$

(b) $\delta(x,y) = r_i$ iff $x \sim_i y$

(c) i-, j-adjacency is mutually exclusive if $i \neq j$

(d) if $\exists \gamma$ of type p and $q \simeq p$ then exists γ' of type q.

(e) γ is of type p, p red iff γ minimal

(f) p reduced, a gallery of type p is unique

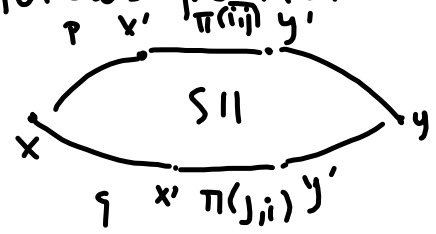
(g) If $\delta(y,x) = w$, $z \sim_i y \Rightarrow \delta(z,x) \in \{r_i w, w\}$

Pf.: (a) let $w = r_p$, $\ell(w) > 0$ p is reduced.
say $p = qi$, q reduced. Then $\ell(wr_i) < \ell(w)$
and $wr_i = r_q$ $\Rightarrow \exists$ $(x, \dots, y)$ of type q.
if $y' \sim_i y$ $\Rightarrow$ extend $(x, \dots, y, y')$ and $\delta(x, y') = w$.

(b) Clear

(c) follows from (b)

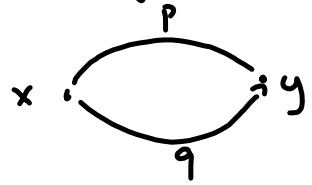
(d)



(e) $\gamma = (x_1 \cdots, y)$ of type p:
if p not reduced $\Rightarrow p \simeq p_1 \, i \, p_2$

$\Rightarrow \gamma$ is not min.

if γ is ~~minimal~~ not minimal, let q be reduced and the type of a min. gal from $x \to y$



We just saw that q is reduced

if p also reduced then

$r_p = \delta(x,y) = r_q \Rightarrow p \simeq q$

$\Rightarrow \gamma$ min., contradiction

(f) let

$\gamma_1 = (\overbrace{x, \dots, y_1}^{q}, y)$ of type $p = qi$.

$\gamma_2 = (\underbrace{x, \dots, y_2}_{q}, y)$

From these gal's, $y_1 \underset{i}{\sim} y \underset{i}{\sim} y_2$

if $y_1 \neq y_2$ we would have galleries of reduced types q and qi from x, y_2, contr.

(g) $\delta(x, y) = w = r_p$, p reduced

ip is reduced $\Rightarrow (z, x, \dots, y) \Rightarrow \delta(z, y) = r_i w$

ip is not reduced $\Rightarrow p \simeq iq \Rightarrow (\overset{i}{z}, \overset{i}{x}, x_1, \dots, y)$

$\Rightarrow \delta(z, y) = w.$

Def 4. Δ is called thick if every panel is the panel of at least 3 chambers.

· Residues and apt. If $J \subset I$, M_J is the subdiag.

Thm 5: Every J-residue of Δ is a building of type M_J.

Pf: Suffices to show that if $x, y \in R$, there is a J-gallery connecting x, y.

Let $\gamma = (x, \dots, y)$ be a gallery of type $p \in F(J)$ of min. length. We claim p is reduced

If not, $p \simeq p_1 \, J \, p_2$, $J \in \mathcal{J}$

Using 3 $\Rightarrow \exists \gamma'$ a J-gallery with $\ell(\gamma') < \ell(\gamma)$, contrad., so p reduced $\square$

Def 6: $X \subseteq W$ and $\alpha : X \to \Delta$

we say that α is an isometry if

$$\delta(\alpha(x), \alpha(y)) = \delta_W(x, y) \quad \forall x, y \in X$$

Lemma 7. Isometries are injective chamber system morphisms, and comp. of isometries is an isometry.

Pf (omit)

Def 8: an apartment A is the isometric image of C(W). By a root or wall we mean the isometric image.

Lemma 9 If $\phi : C \to D$ is a morphism of ch. syst. over the same I, then $d(\phi(x), \phi(y)) \leq d(x,y)$.

Pf: $d(x,y) = \min\{\ell(\gamma) \mid \gamma = (x, \dots, y)\}$.

Given γ, then $\phi(\gamma) = (\phi(x), \dots, \phi(y))$.

then: $d(x,y) = \ell(\gamma) - \ell(\phi(\gamma)) \geq d(\phi(x), \phi(y))$

γ min

□

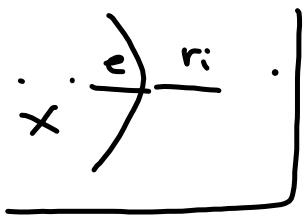
Thm 10. Any isometry of $X \subseteq W$ into Δ extends to an isometry $W \longrightarrow \Delta$.

Pf: It suffices to extend $\alpha : X \to \Delta$ to a domain $X \subsetneq X' \subseteq W$.

Suppose $\emptyset \neq X$ and $x_0 \in X$, $x_0 r_i \notin X$, precomposing with x_0^{-1}, can assume $e \in X$, $r_i \notin X$.

WLOG need to define $\alpha(r_i)$.

Case 1 $\ell(r_i x) > \ell(x), \ \forall x \in X$



Choose $y \overset{i}{\sim} \alpha(e)$ in Δ

define $\alpha(r_i) = y$. We must show

$\delta(y, \alpha(x)) = r_i x$, $\forall x$.

Let q be reduced and $x = r_q \in X$, then

q is reduced $\overset{\alpha \text{ isom.}}{\overset{q \text{ red}}{\Longrightarrow}}$ $\exists\ \gamma = (y, \alpha(e), \ldots, \alpha(x))$

and $\delta(y, \alpha(x)) = r_i x$

Case 2. $\ell(r_i x_1) < \ell(x_1)$, $\exists\, x_1 \in X$

$\exists\ (e, r_i, \dots, x_1)$ of red type

$p = iq$ in W.

$\overset{(3)}{\Rightarrow} \exists!\ (\alpha(e), \overset{i}{y}, \dots, \alpha(x_1))$ in Δ

Let $y = \alpha(r_i)$. Must show $\delta(y, \alpha(x)) = r_i x$, $\forall x$.

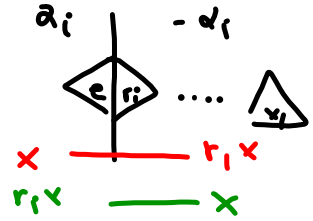
Let $\beta : X \to W$, $\beta(x) = r_i\, \delta(y, \alpha(x))$.

$y \sim_i \alpha(e) \overset{3g1}{\Rightarrow} \delta(y, \alpha(x)) \in \{x, r_i x\}$.

$\Rightarrow \beta(x) \in \{r_i x, x\}$.

$\beta = r_i \circ \delta(y, -) \circ \alpha$, $\beta(e) = e$, $\beta(x_1) = x_1$

Let α_i be the root with $e \in \alpha_i$, $r_i \notin \alpha_i$, $x_1 \notin \alpha_i$:



Suppose $\beta(x) = r_i x$

$x \in \alpha_i \Rightarrow r_i x \notin \alpha_i \Rightarrow d(e, \beta(x)) > d(e, x)$

$x \notin \alpha_i \Rightarrow r_i x \in \alpha_i \Rightarrow d(x_1, \beta(x)) > d(x_1, x)$

Thus, $\beta(x) = x$ $\square$

Cor 11. Any two chambers lie in an apt.

Lem 12. $\alpha : W \to \Delta$ isometry is uniquely defined by $A = \alpha(w)$ and $c = \alpha(e)$.

Pf: if $\alpha' : W \to \Delta$ is another, then

$\alpha^{-1}\alpha' : W \longrightarrow W$ that fixes

Thus: $\alpha^{-1}\alpha'(w) = \delta(e, \alpha^{-1}\alpha'(w))$

$= \delta_w(\alpha^{-1}\alpha'(e), \alpha^{-1}\alpha'(w))$

$= \delta_w(e, w) = w.$

Def 13. Fix A apt and $c \in A$.
Define the retraction of Δ onto A with center c

$\rho_{c,A} : \Delta \to A$, by

$$\rho_{c,A}(x) = \alpha(\delta(x,c))$$

α s.t. $\alpha(e) = c$

Def 14. Given simplices σ, τ in Δ a gallery from σ to τ, is a gallery $(c, \dots, d)$ with σ a face of c, τ a face of d.

Thm 15 A is an apt containing c and σ

Then, any minimal gall from c to σ lies in A

Pf: $\gamma = (c = c_0, c_1, \dots, c_k = d)$, σ a face of d.

If $\gamma \not\subset A$, for some t, we have $c_{t-1} \in A$ but $c_t \notin A$. Let $b \neq c_{t-1}$ be if adj to c_{t-1}, in A.

Thus $\rho_{b,A}(c_{t-1}) = \rho_{b,A}(c_t) = c_{t-1}$

$\Rightarrow \rho_{b,A}(\gamma)$ has smaller length, contr. $\square$

Cor 16 Intersection of apartments is convex

Cor 17 If σ is a simplex and c a chamber then there is unique chamber (nearest to c) having σ as a face

Pf Omit

Def 18. We call the chamber of Cor 17

$$\mathrm{proj}_{\mathrm{St}(\sigma)}(c), \quad (\mathrm{st}(\sigma) = \mathrm{star}(\sigma) = \bigcup c \text{ s.t. } \sigma \text{ is a face of } c)$$